K-THEORETICAL INDEX THEOREMS FOR GOOD ORBIFOLDS

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Abstract. In this note we study index theory for general and good orbifolds. We prove a $K$-theoretical index theorem for good orbifolds, and from this we deduce as a corollary a numerical index formula.

Let $D$ be a pseudodifferential elliptic operator on the closed orbifold $Q$. In §1 we give an index formula involving a certain class $[D]$ associated to $D$. In §2 we prove a $K$-theoretical index theorem (in analogy with the main theorem in [9]) for good orbifolds (a good orbifold is an orbifold that can be covered by a smooth manifold), which relates the class $[D]$ with the class of its symbol. It is also natural to consider in this case, besides the usual (analytical) index of $D$, its Atiyah-Singer index [2, 12]. We then recover from the $K$-theoretical index theorem the main theorem in [2]. In §3 we relate the analytical and the Atiyah-Singer indices of $D$.

1. AN ORBIFOLD INDEX THEOREM

We first give an index theorem for general orbifolds $Q$. We can suppose throughout this paper that $Q$ is even-dimensional, since the case in where the dimension of $Q$ is odd can be obtained from this by crossing with $T$. All the orbifolds are also assumed to be orientable and closed. Note that any nonorientable orbifold is always finitely covered by an orientable orbifold.

Every orbifold $Q$ arises as a quotient $Q = P/G$, where $G$ is a compact group acting locally freely on the smooth manifold $P$ [11]. In our case we can choose $G = SO(q)$, where $q$ is the dimension of $Q$ and $P$ is the orthonormal frame bundle of $Q$.

Let $\eta(0)$ and $\eta(1)$ be two orbifold vector bundles over $Q$. We say that $D$ is an elliptic pseudodifferential operator on $Q$ acting from the $L^2$-sections $\Gamma(\eta(0))$ of the bundle $\eta(0)$ to the $L^2$-sections $\Gamma(\eta(1))$ of $\eta(1)$ if on each orbifold chart $U_i \approx \mathbb{R}^q/G_i$, the lift of $D$ to $\mathbb{R}^q$ is a pseudodifferential elliptic operator. We assume throughout this paper that $D$ has order 0. (We can always reduce to this case.) In analogy with the manifold case, every section of $\eta(0)$ that is in $\text{Ker}(D)$ is $C^\infty$ and so is every section of $\eta(1)$ in $\text{Ker}(D^*)$. This is because we only use local properties of $D$. Also $\text{Ker}(D)$ and $\text{Coker}(D)$ are finite-dimensional, so
that \([12]\)
\[ \text{Ind}_a(D) = \text{Dim}(\text{Ker}(D)) - \text{Dim}(\text{Coker}(D)). \]

We define \(C(P) \times G\) to be the orbifold \(C^\ast\)-algebra \(C^\ast(Q)\) [6]. The element \([\tilde{D}] \in KK(C^\ast(Q), C)\), defined in [6], coincides with the element associated to the lift \(\tilde{D}\) of \(D\) to \(P\). Therefore we can consider the image of \([\tilde{D}]\) in the cyclic cohomology group \(HC^\text{ev}(C^\infty(P \times G))\) via \(ch^\ast\). As remarked by Connes at the end of §8 in [5], \(C^\infty(G)\) embeds in \(C^\infty(P \times G) : C^\infty(G) \hookrightarrow C^\infty(P \times G)\) and the restriction \(r^*ch^\ast([\tilde{D}])\) of \(ch^\ast([\tilde{D}])\) to \(C^\infty(G)\) is given exactly by \(r^*ch^\ast([\tilde{D}]) = S^q\chi, q = 2l = \text{dim} Q\), where \(\chi\) is the distributional index character of \(\tilde{D}\) defined by Atiyah in [1], i.e., \(\chi \in HC^0(C^\infty(G))\),

\[ \chi(f) = \text{Tr} \left( \begin{array}{c} \text{action of} \\ f \in \ker \tilde{D} \end{array} \right) - \text{Tr} \left( \begin{array}{c} \text{action of} \\ f \in \ker \tilde{D}^* \end{array} \right) \]

and \(S\) is an operator defined by Connes in [5]. Therefore,

**Theorem 1.** Let \(D\) be a pseudodifferential elliptic operator on the spin\(^c\) orbifold \(Q\). Then

\[ \text{Ind}_a(D) = \langle ch^\ast([\tilde{D}]), r_\ast(1) \rangle, \]

where \(r : C^\infty(G) \hookrightarrow C^\infty(P \times G)\) is the canonical embedding, \(1 \in HC^\text{ev}(C^\infty(G))\) is the element corresponding to the constant function \(1\), and \(r_\ast : HC^\text{ev}(C^\infty(G)) \rightarrow HC^\text{ev}(C^\infty(P \times G))\) is the induced homomorphism.

**Proof.** \(\text{Ind}_a(D) = \langle \chi, 1 \rangle, \chi \in HC^0(C^\infty(G)), 1 \in HC^\text{ev}(C^\infty(G))\) by [11]. Note that \(1\) corresponds to the function \(1 \in C^\infty(G) \equiv HC^\text{ev}(C^\infty(G))\) by [4]. Because \(\langle \cdot, \cdot \rangle\) is \(S\)-invariant [5],

\[ \text{Ind}_a(D) = \langle r^*ch^\ast([\tilde{D}]), 1 \rangle = \langle ch^\ast([\tilde{D}]), r_\ast(1) \rangle. \]

When \(G\) acts freely on \(P\) this is the Atiyah-Singer index theorem [3] (c.f. [5, §6, Theorem 5]).

### 2. Good Orbifolds

In the case of a good orbifold (i.e., its universal cover is a smooth manifold) another definition of index for a pseudodifferential elliptic operator \(D\) is possible (see [2, 12]). In fact, let \(\tilde{Q}\) be the universal cover of \(Q\), \(\pi_1^{\text{ORB}}(Q) = \Gamma\) be the fundamental group of \(Q\), and \(\tilde{D}\) be the lift of \(D\) to \(\tilde{Q}\). Then \(\tilde{Q}/\Gamma = Q\) and \(\Gamma\) acts on \(\tilde{Q}\) properly. The Atiyah-Singer index of \(D\), AS-index, is defined as follows (c.f. [2]). On \(\tilde{Q}\) we consider a \(\Gamma\)-invariant positive measure \(d\tilde{\mu}\) (the lift of a positive measure \(d\mu\) on \(Q\)). Let \(\tilde{\eta}(i), i = 0, 1\), be the bundle over \(\tilde{Q}\) lift of the bundle \(\eta(i), i = 0, 1\) over \(Q\). Note also that \(L^2(\tilde{\eta}(0) \oplus \tilde{\eta}(1)) \cong L^2(\tilde{Q}) \times \text{End}(C^n), n = \text{Dim}(\tilde{\eta}(0)) + \text{Dim}(\tilde{\eta}(1))\), with the action of \(\Gamma\) trivial on \(\text{End}(C^n)\). The bounded operators on \(L^2(\tilde{\eta}(0) \oplus \tilde{\eta}(1))\) that commute with the action of \(\Gamma\) form a von-Neumann algebra \(A(\tilde{\eta})\) that has a natural trace function denoted by \(\text{Tr}_\Gamma\). In particular if \(P \in A(\tilde{\eta})\) is an orthogonal projection onto a subspace \(H\) of \(L^2(\tilde{\eta}(0) \oplus \tilde{\eta}(1))\), so that \(H\) is a \(\Gamma\) module, we define

\[ \text{Dim}_\Gamma(H) = \text{Tr}_\Gamma(P) \in \mathbb{R}. \]
Applying this to \( \text{Ker}(\hat{D}) \) and \( \text{Coker}(\hat{D}) \) we get a finite real-valued index
\[
\text{AS-ind}(\hat{D}) = \text{Dim}_\Gamma(\text{Ker}(\hat{D})) - \text{Dim}_\Gamma(\text{Coker}(\hat{D})).
\]

Next we will define the \( K \)-theoretical \( \Gamma \)-index of \( D \). Firstly we will rewrite the orbifold, \( C^* \)-algebra \( C^*(Q) \), up to Morita equivalence.

**Proposition 2.** Let \( Q \) be a good orbifold. Then \( C_0(Q) \times \Gamma \) and \( C^*(Q) \) are Morita equivalent.

**Proof.** Let \( \hat{P} \) be the orthogonal frame bundle of \( \hat{Q} \). The following diagram
\[
\begin{array}{ccc}
\hat{P} & \xrightarrow{\text{SO}(q)} & \hat{Q} \\
\Gamma & \downarrow & \Gamma \\
P & \xrightarrow{\text{SO}(q)} & Q
\end{array}
\]

commutes. Since \( \Gamma \) and \( \text{SO}(q) \) act freely on \( \hat{P} \) and their actions commute, by a theorem of P. Green (see [15]), \( C^*(\hat{P}/\Gamma, \text{SO}(q)) \) is Morita equivalent to \( C^*(\hat{P}/\text{SO}(q), \Gamma) \). \( \square \)

Hence an elliptic pseudodifferential elliptic operator \( D \) on \( Q \) determines a class
\[
[D] \in KK(C_0(Q) \times \Gamma, C) \cong KK_{\Gamma}(C_0(Q), C).
\]

To define the \( K \)-theoretical \( \Gamma \)-index \( \text{IND}_{\Gamma}(\hat{D}) \) of \( D \), which is an element of \( K_0(C^*(\Gamma)) \), we first observe that since \( \Gamma \) acts on \( L^2(\eta^{(i)}) \), so also \( C^*(\Gamma) \) does, in a canonical way. Now \( \hat{D} \) is a Fredholm operator between \( L^2(\eta^{(0)}) \) and \( L^2(\eta^{(1)}) \), and so it determines an element of \( K_0(C^*(\Gamma)) \), which we call \( \text{IND}_{\Gamma}(\hat{D}) \) (c.f. [10, §4]). \( \text{IND}_{\Gamma}(\hat{D}) \) is represented by the projections of \( L^2(\eta^{(0)}) \) onto \( \text{Ker}(\hat{D}) \) and of \( L^2(\eta^{(1)}) \) onto \( \text{Ker}(\hat{D}^*) \) The following theorem can be recovered from a theorem of Kasparov [9].

**Theorem 3.** Let \( D \) be a pseudodifferential elliptic operator on the good orbifold \( Q \), \( D : L^2(\eta^{(0)}) \rightarrow L^2(\eta^{(1)}) \). Let \( \hat{Q} \) be the universal cover of \( Q \), \( \Gamma = \pi_1^{\text{orb}}(Q) \), and \( \hat{D} \) be the operator \( D \) lifted to \( \hat{Q} \). Then,
\[
\text{IND}_{\Gamma}(\hat{D}) = \text{IND}_{\Gamma}(\hat{D}) \quad \text{in } KK(C, C^*(\Gamma)),
\]

with
\[
\text{IND}_{\Gamma}(\hat{D}) \overset{\text{def}}{=} [C] \otimes j_\Gamma([\hat{D}]),
\]
where \( [\hat{D}] \in KK_{\Gamma}(C_0(\hat{Q}), C) \), \( j_\Gamma : KK_0(A, B) \rightarrow KK(A \times \Gamma, B \times \Gamma) \) is the canonical homomorphism, and \( [C] \in K_0(C_0(\hat{Q}) \times \Gamma) \) is determined by the projection \( p(x, g) = \sqrt{c(x)c(g^{-1}x)} \), where \( c \in C_c(\hat{Q}) \) is such that \( \int_{\Gamma} c(xg) \, dg = 1 \) and where \( c \geq 0 \).

Note that we could also have an index theorem with coefficients in a \( C^* \)-bundle rather than in a vector bundle in the spirit of [14].

As a corollary to Theorem 3 we obtain.
Theorem 4. Let $D$, $Q$, $\hat{Q}$, $\Gamma$, and $\hat{D}$ be as in Theorem 3. Then,

$$\tau(\text{IND}_a(\hat{D})) = \text{AS-ind}(D),$$

where $\tau$ is the canonical trace on $K_0(C^*(\Gamma)) = (\text{Idempotents of } C^*(\Gamma) \otimes \mathcal{H})/\sim$.

Proof. $\tau$ is given by $\tau(a \otimes A) = \tau^1(a) \otimes T(A)$ where $a \in C^*(\Gamma)$, $a = \sum_{g \in \Gamma} \lambda_g [g]$, $\lambda_g \in \mathbb{R}$, $\tau^1(a) = \lambda_e$, $e = \text{unit of } \Gamma$, $A \in \mathcal{H} = \mathcal{H}(L^2(Q))$, $T$ = canonical trace on $\mathcal{H}(L^2(Q)) \subseteq \mathcal{B}(L^2(Q))$. This trace coincides with the trace in [2, p. 57]. $\square$

3. Relations between indices

As we have seen in §1 and in §2, if $Q$ is a good orbifold and $D$ is a pseudodifferential elliptic operator on $Q$, then we can define the two indices $\text{Ind}_a(D)$ and $\text{AS-ind}(D)$. The first one is necessarily an integer, while the second one is a rational number. In general they do not coincide, but there is an interesting relation between them, which is a corollary of the main theorems in [13] and [2] (c.f. also [12, III]). In fact Atiyah's argument applies also to the case in where the action is not free.

Theorem 5. Let $D$, $Q$, $\hat{Q}$, $\Gamma$, and $\hat{D}$ be as in Theorem 3. Then,

$$\text{Ind}_a(D) = \text{AS-ind}(D) + R,$$

where (with the notation as in the introduction in [13]),

$$R = \sum_{i=1}^c \int \frac{1}{m_i} (\text{ch}^\Sigma(D) \mathcal{F}^\Sigma(Q), [\Sigma_i]),$$

with $\Sigma_i$ running over the strata of $Q$.

For example, if $\mathcal{E}$ is the Euler operator on $Q$, then $\text{Ind}_a(\mathcal{E})$ is equal to the Euler characteristic of $Q$ as a vector space (c.f. [11, PROPOSITION]) and $\text{AS-ind}(\mathcal{E}) = X_S(Q)$, where $X_S(Q)$ is the Euler-Satake characteristic of $Q$, by the Gauss-Bonnet theorem for orbifolds of Satake [16] and the general formula in [12]. Since $R$ depends only on the singular structure of $Q$, it follows that $R$ is 0 if $Q$ is a smooth manifold, and so in that case we recover the main theorem in [2] from Theorem 5, $\text{Ind}_a(D) = \text{AS-ind}(D)$.

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References


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