REFLEXIVITY OF COMMUTATIVE SUBSPACE LATTICES

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ABSTRACT. A short proof is given of Arveson’s reflexivity theorem for strongly closed commutative subspace lattices.

When \( \mathcal{L} \) is a lattice of commuting self-adjoint projections on a Hilbert space \( \mathcal{H} \), one can form the algebra \( \text{Alg}\mathcal{L} \) of all \( T \in \mathcal{B}(\mathcal{H}) \) for which \( TP = PTP \) for all \( P \in \mathcal{L} \). Given a subalgebra \( \mathcal{A} \) of \( \mathcal{B}(\mathcal{H}) \) one can form the lattice \( \text{Lat}\mathcal{A} \) consisting of all self-adjoint projections \( P \) such that \( TP = PTP \) whenever \( T \in \mathcal{A} \). It is a theorem of W. B. Arveson [1] that if \( \mathcal{L} \) is closed in the strong operator topology, then \( \mathcal{L} \) is reflexive, that is to say that \( \mathcal{L} = \text{Lat}\text{Alg}\mathcal{L} \). In this note we shall give a short proof of this result. Our approach avoids topological measure theory and disintegration of measures, though we do use, in a different guise, the class \( \mathcal{A} \) of “pseudo-integral” operators that is the key to Arveson’s original proof. Other proofs of the theorem have been given by K. R. Davidson [2] and by V. S. Shul’mant [3].

Let \((\Omega, \mathcal{F}, \mu)\) be a finite measure space. We shall write \( A(\mu) \) for the algebra of all linear operators \( T : L^2(\mu) \to L^2(\mu) \) which are bounded from the \( L^1 \)-norm to the \( L^1 \)-norm and from the \( L^\infty \)-norm to the \( L^\infty \)-norm. For \( T \in A(\mu) \) we define \( \|T\|_A = \max\{\|T : L^1 \to L^1\|, \|T : L^\infty \to L^\infty\|\} \). By interpolation, the norm of \( T \) in \( \mathcal{B}(L^2(\mu)) \) is less than or equal to \( \|T\|_A \), so that the unit ball of \( \| \cdot \|_A \) is a subset of the unit ball of \( \mathcal{B}(L^2(\mu)) \) for the operator norm. In fact, it a closed subset for the weak operator topology (and hence compact for that topology) since \( \|T\|_A \leq 1 \) if and only if \( (Tf | g) \leq 1 \) whenever the elements \( f, g \) of \( L^2(\mu) \) satisfy \( \|f\|_1 \leq 1 \) and \( \|g\|_\infty \leq 1 \) or \( \|f\|_\infty \leq 1 \) and \( \|g\|_1 \leq 1 \).

When \( \mathcal{L} \) is a sublattice of the \( \sigma \)-algebra \( \mathcal{F} \) the projections \( P_L : f \mapsto f1_L \) \((L \in \mathcal{L})\) form a lattice in \( \mathcal{B}(L^2(\mu)) \). We abuse notation by writing \( \text{Alg}\mathcal{L} \) for \( \text{Alg}\{P_L : L \in \mathcal{L}\} \).

**Theorem.** Let \((\Omega, \mathcal{F}, \mu)\) be a finite measure space and let \( \mathcal{L} \) be a sublattice of \( \mathcal{F} \). For any \( F \in \mathcal{F} \)

\[
\inf_{L \in \mathcal{L}} \mu(F \triangle L) = \max\{\langle T1_F | 1_{\Omega \setminus F} \rangle : T \in A(\mu) \cap \text{Alg}\mathcal{L}, \|T\|_A = 1\}.
\]

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Proof. If \( L \in \mathcal{L} \) and \( T \in \mathcal{A}(\mu) \cap \text{Alg}\mathcal{L} \), then
\[
(T|_{\mathcal{L}|_{\Omega \setminus F}} \leq (T|_{\mathcal{L} \cap L} \mid 1_{\Omega \setminus (L \cup F)}) + (T|_{\mathcal{L} \setminus L} \mid 1_{\Omega \setminus F})
\]
\[
\leq 0 + \|T : L^\infty \rightarrow L^\infty\| \mu(L \setminus F) + \|T : L^1 \rightarrow L^1\| \mu(F \setminus L)
\]
so that one inequality (\( \geq \)) is easily established. To establish the other, we start with the case where \( \mathcal{L} \) is a finite lattice.

**Lemma.** Let \((\Omega, \mathcal{F}, \mu)\) be a finite measure space and let \( \mathcal{L} \) be a finite sublattice of \( \mathcal{F} \). For any \( F \in \mathcal{F} \) there exist \( L \in \mathcal{L} \) and \( T \in \text{Alg}\mathcal{L} \) with \( \|T\|_A = 1 \) such that
\[
\mu(F \triangle L) = (T|_{\mathcal{L}|_{\Omega \setminus F}}).
\]

**Proof.** Let \( \mathcal{S} \) be the algebra generated by \( \mathcal{L} \cup \{F\} \) and let \( A_1, \ldots, A_m \) be the atoms of \( \mathcal{S} \) that are contained in \( F \), \( B_1, \ldots, B_n \) the atoms that are disjoint from \( F \). Let \( G \) be the set of pairs \((i, j)\) such that there is no \( L \in \mathcal{S} \) with \( A_i \subseteq L, B_j \cap L = \emptyset \). If \( x = (x_{i,j})_{(i,j) \in G} \) is a family of positive real numbers then we may define an operator \( T_x \) in \( \text{Alg}\mathcal{L} \) by
\[
T_x f = \sum_{(i,j) \in G} \frac{x_{i,j}}{\mu(A_i)\mu(B_j)} (f \mid 1_{A_i}) 1_{B_j}.
\]
We easily calculate the norms
\[
\|T_x : L^1 \rightarrow L^1\| = \max_i \sum_j \frac{x_{i,j}}{\mu(A_i)},
\]
\[
\|T_x : L^\infty \rightarrow L^\infty\| = \max_j \sum_i \frac{x_{i,j}}{\mu(B_j)}
\]
as well as the quantity
\[
(T_x|_{\mathcal{L}|_{\Omega \setminus F}} = \sum_{(i,j) \in G} x_{i,j},
\]

Let \( \delta \) be the maximum of this quantity for \( x \) as above and \( \|T_x\|_A \leq 1 \). We shall have proved the lemma if we find an element \( L \in \mathcal{L} \) with \( \mu(F \triangle L) \leq \delta \).

Now \( \delta \) is thus the solution of the following linear programming problem: Maximize \( \sum_{(i,j) \in G} x_{i,j} \) subject to \( x_{i,j} \geq 0, \sum_j x_{i,j} \leq \alpha_i \) for all \( i \), and \( \sum_i x_{i,j} \leq \beta_j \) for all \( j \), where \( \alpha_i = \mu(A_i) \) and \( \beta_j = \mu(B_j) \). This may be regarded as a network-flow problem: we consider a directed graph whose nodes are \( A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_n \) together with a “source” \( S \) and a “sink” \( T \). For each \( i \) there is a channel from \( S \) to \( A_i \) with maximum capacity \( \alpha_i \), for each \( j \) there is a channel from \( B_j \) to \( T \) with capacity \( \beta_j \), and there is a channel of infinite capacity from \( A_i \) to \( B_j \) whenever \( (i, j) \in G \). Our problem is to find the maximal flow through this network. By the Min-Cut Max-Flow Theorem this maximal flow equals
\[
\min_C \sum_{c \in C} \text{capacity of } c,
\]
where the minimum is taken over sets \( C \) of channels such that \( S \) is separated from \( T \) if all channels in \( C \) are removed from the network. Evidently, we
shall not achieve this minimum if we remove a channel of infinite capacity so that the minimizing \( C \) will consist of the channels \( SA_i \) for \( i \) in a certain set \( I \), and the \( B_jT \) for \( j \) in some \( J \).

We have established the existence of \( I \) and \( J \) such that

\[
\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j = \delta
\]

and such that for every \((i, j) \in G\) at least one of \( i \in I, j \in J \) is true. By the definition of \( G \) there exists, whenever \( i \notin I \) and \( j \notin J \), an element \( L_{i,j} \) of \( \mathcal{L} \) with \( A_i \subseteq L_{i,j} \) and \( B_j \cap L_{i,j} = \emptyset \). Since \( \mathcal{L} \) is a lattice the set

\[
L = \bigcap_{j \notin J} \bigcup_{i \notin I} L_{i,j}
\]

is in \( \mathcal{L} \). Also \( A_i \subseteq L \) whenever \( i \notin I \) and \( B_j \cap L = \emptyset \) whenever \( j \notin J \), so that

\[
\mu(F \Delta L) \leq \sum_{i \in I} \mu(A_i) + \sum_{j \in J} \mu(B_j) = \delta.
\]

We can now resume the proof of the theorem. Let \( \delta = \inf\{\mu(F \Delta L) : L \in \mathcal{L}\} \) and let \( K \) be the set of all \( T \in \mathbf{A}(\mu) \) such that \( \|T\|_A \leq 1 \) and \( \langle T1_F, 1_{\Omega \setminus F} \rangle \geq \delta \). This set is compact in the weak operator topology. If \( \mathcal{L} \) is a finite sublattice of \( \mathcal{L} \), then

\[
K \cap \text{Alg} \mathcal{L}' = \{T \in K : (Tf | g) = 0 \text{ whenever there exists } L \in \mathcal{L} \text{ such that supp } f \subseteq L \text{ and supp } g \subseteq \Omega \setminus L\}
\]

is closed in the weak operator topology, and is nonempty by the lemma. By compactness we deduce that \( K \cap \text{Alg} \mathcal{L} \) is nonempty, which is what we wanted to prove.

**Corollary.** Let \( \mathcal{H} \) be a Hilbert space and let \( \mathcal{L} \) be a strongly closed lattice of commuting selfadjoint projections on \( \mathcal{H} \). Then \( \text{Lat Alg} \mathcal{L} = \mathcal{L} \).

**Proof.** The reduction of the problem on a general Hilbert space to the measure-theoretic version that we have just been looking at is rather standard. Let \( Q \) be a selfadjoint projection that is not in \( \mathcal{L} \); we have to show that \( Q \) is not in \( \text{Lat Alg} \mathcal{L} \). Let \( \mathcal{M} \) be a maximal abelian selfadjoint subalgebra of \( \mathcal{B}(_{\mathcal{H}}) \) containing the lattice \( \mathcal{L} \); then \( \text{Alg} \mathcal{L} \supseteq \mathcal{M} \), so that \( \text{Lat Alg} \mathcal{L} \subseteq \text{Lat} \mathcal{M} \). It is known that \( \text{Lat} \mathcal{M} \subseteq \mathcal{M} \) so that we can certainly assume that \( Q \in \mathcal{M} \). Since \( \mathcal{L} \) is strongly closed, there exist \( f_1, \ldots, f_m \in \mathcal{H} \) such that

\[
\max_{i \leq m} \|(P - Q)f_i\| \geq 1
\]

for all \( P \in \mathcal{L} \). Let \( \mathcal{H}_0 \) be the closed subspace generated by \( \mathcal{M}\{f_1, \ldots, f_m\} \).

The orthogonal projection \( P_0 \) of \( \mathcal{H} \) onto \( \mathcal{H}_0 \) is in \( \mathcal{M} \) and \( \mathcal{M} = \mathcal{M}|_{\mathcal{H}_0} \) is a maximal abelian selfadjoint subalgebra of \( \mathcal{B}(_{\mathcal{H}_0}) \). We can regard \( \mathcal{H}_0 \) as \( L^2(\mu) \) and identify \( \mathcal{M} \) with \( L^\infty(\mu) \) for a suitable finite measure \( \mu \). The lattice \( \mathcal{L}|_{\mathcal{H}_0} \) of idempotents in \( \mathcal{M}_0 \) has the form \( \{PL : L \in \mathcal{L}_0\} \) for some sublattice \( \mathcal{L}_0 \) of \( \mathcal{F} \). The restriction to \( \mathcal{H}_0 \) of \( Q \) is \( P_F \) for some \( F \in \mathcal{F} \). The existence in \( \mathcal{H}_0 = L^2(\mu) \) of the elements \( f_1, \ldots, f_m \), implies that there is some \( \delta > 0 \) such that \( \mu(F \Delta L) \geq \delta \) for all \( L \in \mathcal{L}_0 \). The theorem gives some \( T_0 \in \mathcal{B}(_{\mathcal{H}_0}) \) such that \( T_0PL = PLT_0P \) for all \( L \in \mathcal{L}_0 \) but \( T_0P_F \neq P_F T_0P_F \). If we define \( T \in \mathcal{B}(_{\mathcal{H}}) \) by \( T = T_0P_0 \), then \( T \in \text{Alg} \mathcal{L} \) but \( TQ \neq QTQ \).
References


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