DENSITY OF THE POLYNOMIALS
IN THE HARDY SPACE OF CERTAIN SLIT DOMAINS

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(Communicated by Paul S. Muhly)

Abstract. In this article we construct a Jordan arc \( \Gamma \) in the complex plane, with endpoints 0 and 1, such that the polynomials are dense in the Hardy space \( H^2(D \setminus \Gamma) ; \quad D := \{ z \in \mathbb{C} : |z| < 1 \} \).

It is well known that if \( G = \{ z \in \mathbb{C} : |z| < 1 \} \setminus [0, 1] \) (\( \mathbb{C} \) denotes the complex plane), then the polynomials are not dense in the Hardy space \( H^2(G) \). One of the assertions of this paper, however, is that there are regions \( D \) of the same sort as \( G \) such that the polynomials are dense in \( H^2(D) \). In fact, we construct a homeomorphic image \( \Gamma \) of the interval \([0, 1]\), where \( \Gamma \) has endpoints 0 and 1, and \( \Gamma \setminus \{1\} \subseteq \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \), such that the polynomials are dense in \( H^2(\mathbb{D} \setminus \Gamma) \).

Recall that if \( D \) is a bounded Dirichlet region, then the Hardy space \( H^2(D) \) is the collection of functions \( f \) that are analytic in \( D \) such that \( |f|^2 \) has a harmonic majorant on \( D \). Furthermore, for any point \( z_0 \) in \( D \) (norming point), the mapping \( \| \cdot \|_{z_0} : H^2(D) \to \mathbb{R} \) defined by \( \|f\|_{z_0} = (u_f(z_0))^{1/2} \), where \( u_f \) is the least harmonic majorant of \( |f|^2 \) on \( D \), is a norm on \( H^2(D) \), and, under this norm, \( H^2(D) \) forms a Banach space (cf. [6]). By Harnack's inequality, different norming points yield equivalent norms. We let \( \omega(\cdot, D, z_0) \) denote harmonic measure on \( \partial D \) evaluated at \( z_0 \). Notice that if \( f \) is analytic on \( D \) and continuous on \( \overline{D} \), then \( f \in H^2(D) \) and

\[
\|f\|_{z_0} = \left\{ \int |f(\zeta)|^2 d\omega(\zeta, D, z_0) \right\}^{1/2}.
\]

1. Definition. A function \( \gamma : [0, 1] \to \mathbb{C} \) is said to be a Jordan arc if and only if it is both continuous and one-to-one. Throughout this paper we shall identify a Jordan arc \( \gamma \) with its trace \( \Gamma := \gamma([0, 1]) \).

In order to minimize technical details we do much of our work on a particular "annular" region which has rectilinear boundary. For the rest of the paper let \( E = \{ z = x + iy : 1 < \max\{|x|, |y|\} < 2 \} \), \( S = \{ z = x + iy : \max\{|x|, |y|\} = 1 \} \), and \( T = \{ z = x + iy : \max\{|x|, |y|\} = 2 \} \). Let us say that a Jordan arc \( \Gamma := \ldots \)

Received by the editors August 14, 1989 and, in revised form, January 15, 1991.
1980 Mathematics Subject Classification (1985 Revision). Primary 30E10, 30D55; Secondary 30C85, 47B20.

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0002-9939/92 $1.00 + $.25 per page

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\(\gamma([0, 1])\) connects \(S\) to \(T\) if \(\gamma(0) \in S\), \(\gamma(t) \in E\) for \(0 < t < 1\), and \(\gamma(1) \in T\). If \(z\) and \(\zeta\) are complex numbers, then let \([z, \zeta] = \{(1 - t)z + t\zeta : 0 \leq t \leq 1\}\) (observe that \([z, \zeta] = [\zeta, z]\)) be the segment that connects \(z\) to \(\zeta\).

2. Definition. Let \(G\) be a bounded, simply connected region in \(\mathbb{C}\). A Jordan arc \(\gamma\) is called a cross-cut of \(G\) if both \(\gamma(0)\) and \(\gamma(1)\) are in \(\partial G\) and \(\gamma((0, 1)) \subset \mathbb{C}\).

3. Lemma. (a) Let \(D\) and \(G\) be bounded, simply connected regions in \(\mathbb{C}\) such that \(z_0 \in D \subset G\). If \(B\) is a Borel subset of \((\partial D) \cap (\partial G)\), then \(\omega(B, D, z_0) \leq \omega(B, G, z_0)\).

(b) Let \(G\) be a bounded, simply connected region in \(\mathbb{C}\). If \(\gamma\) is a cross-cut of \(G\) (\(\Gamma = \gamma([0, 1])\)), the components of \(G \setminus \Gamma\) are \(G_1\) and \(G_2\), \(z_0 \in G_1\) and \(F = (\partial G_2) \setminus \Gamma\), then \(\omega(F, G, z_0) \leq \omega(\Gamma, G_1, z_0)\).

Proof (sketch). Part (a) follows from the maximum principle for harmonic functions. Part (b) is a consequence of (a) and the fact that harmonic measure is a probability measure.

4. Lemma. Suppose \(0 < \varepsilon < 1/4\) and \([\zeta, \eta]\) is a segment of length \(2\varepsilon\) in \(\{z \in E : \Re(z) > 0\}\). Then \(\omega([\zeta, \eta], E \setminus [\zeta, \eta], -3/2) \leq -1/\log(\varepsilon)\).

Proof. Let \(D = \{z \in \mathbb{C} : |z| < 4\}\) and \(\Delta = \{z \in \mathbb{C} : |z - ((\zeta + \eta)/2)| < \varepsilon\}\); notice that \(\overline{\Delta} \subseteq \{z \in \mathbb{C} : |z| < 3\}\). Since \(E \setminus [\zeta, \eta] \subseteq D \setminus [\zeta, \eta]\), it follows from Lemma 3(a) that \(\omega([\zeta, \eta], E \setminus [\zeta, \eta], -3/2) \leq \omega([\zeta, \eta], D \setminus [\zeta, \eta], -3/2)\). Likewise, since \(D \setminus \Delta \subseteq \Delta \setminus [\zeta, \eta]\) we have \(\omega(\partial D, D \setminus \Delta, -3/2) \leq \omega(\partial D, D \setminus [\zeta, \eta], -3/2)\), and therefore \(\omega([\zeta, \eta], D \setminus [\zeta, \eta], -3/2) \leq \omega(\partial \Delta, D \setminus \Delta, -3/2)\). Consequently, \(\omega([\zeta, \eta], E \setminus [\zeta, \eta], -3/2) \leq \omega(\partial \Delta, D \setminus \Delta, -3/2)\).

Next we let \(\varphi\) be a Möbius transformation that maps \(D\) onto the unit disk \(\mathbb{D}\) and \(\Delta\) onto a disk \(\Delta_\varphi\) with center \(z = 0\). Elementary calculations give us that \(|\varphi(-3/2)| \geq 3/8\) and that the radius of \(\Delta_\varphi\) is at most \(\varepsilon\). So

\[
\log(3/8) \leq \log(|\varphi(-3/2)|) = \int \log|z| d\omega(z, \mathbb{D} \setminus \Delta_\varphi, \varphi(-3/2))
\]

\[
= [\log(\text{radius } (\Delta_\varphi))] \cdot \omega(\partial \Delta_\varphi, \mathbb{D} \setminus \Delta_\varphi, \varphi(-3/2))
\]

\[
\leq [\log(\varepsilon)] \cdot \omega(\partial \Delta_\varphi, \mathbb{D} \setminus \Delta_\varphi, \varphi(-3/2)).
\]

Therefore,

\[
\omega([\zeta, \eta], E \setminus [\zeta, \eta], -3/2) \leq \omega(\partial \Delta, D \setminus \Delta, -3/2)
\]

\[
= \omega(\partial \Delta_\varphi, \mathbb{D} \setminus \Delta_\varphi, \varphi(-3/2)) \leq \frac{\log(3/8)}{\log(\varepsilon)} < -\frac{1}{\log(\varepsilon)}. \quad \Box
\]

5. Lemma. If \(\Gamma\) is a Jordan arc that connects \(S\) to \(T\), \(\omega := \omega(\cdot, E \setminus \Gamma, z_0)\), and \(1/z\) can be approximated by polynomials in the \(L^2(\omega)\) norm, then the polynomials are dense in the Hardy space \(H^2(E \setminus \Gamma)\).

Proof. Let \(\varphi\) be a conformal map from \(\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}\) one-to-one and onto \(E \setminus \Gamma\) such that \(\varphi(0) = z_0\), and define \(\| \cdot \| : H^2(E \setminus \Gamma) \to \mathbb{R}\) by

\[
\|f\|^2 = |f(z_0)|^2 + \int_{E \setminus \Gamma} |f'|^2(1 - |\varphi^{-1}|^2) dA.
\]
By Green's Theorem and a change of variables, \( \| \cdot \| \) defines a norm on \( H^2(E \setminus \Gamma) \) that is equivalent to the Hardy space \( H^2(E \setminus \Gamma) \) norm.

Now from our hypothesis it follows that no point in \( \partial (E \setminus \Gamma) \) can be an analytic bounded point evaluation for the polynomials with respect to the \( L^2(\omega) \) norm. Since the \( L^2(\omega) \) and \( H^2(E \setminus \Gamma) \) norms are equivalent for the polynomials, we can conclude that no point in \( \partial (E \setminus \Gamma) \) is an analytic bounded point evaluation for the polynomials with respect to the \( L^2((1-|\phi^{-1}|^2) \, dA) \) norm. Therefore, by [5, Theorem 4], the polynomials are dense in \( L^2(E \setminus \Gamma, (1-|\phi^{-1}|^2) \, dA) \), and thus are dense in \( H^2(E \setminus \Gamma) \) by [4, Corollary 3.4].

Let \( \Gamma \) be a Jordan arc that connects \( S \) to \( T \). How pathological must \( \Gamma \) be so that the polynomials have a chance of being dense in \( H^2(E \setminus \Gamma) \)? If \( \phi \) is a conformal map from the unit disk \( \mathbb{D} \) one-to-one and onto \( E \setminus \Gamma \) and the polynomials are dense in \( H^2(E \setminus \Gamma) \), then by [4, Corollary 3.5] \( \phi \) must be univalent almost everywhere on \( \partial \mathbb{D} \). This can be rephrased in terms of \( \omega(\cdot, E \setminus \Gamma, z_0) \) to give us that the set of tangent points of \( \Gamma \) (see [3]) has one-dimensional Hausdorff measure equal to zero; but much more can be said. Indeed, if there exists one point \( z \) in \( \Gamma \) and a crescent \( \Omega \) in \( E \setminus \Gamma \), with multiple boundary point \( z \), such that the bounded component of \( C \setminus \Omega \) contains \( \{z = x + iy : \min\{|x|, |y|\} < 1\} \) and the polynomials are not dense in \( H^2(\Omega) \), then the polynomials are not dense in \( H^2(E \setminus \Gamma) \). A consequence of this (cf. [1]) is that if there exists one point \( z \) in \( \Gamma \) such that from each side of \( \Gamma \) we can approach \( z \) through a cone in \( E \setminus \Gamma \), then the polynomials are not dense in \( H^2(E \setminus \Gamma) \).

6. Theorem. There exists a Jordan arc \( \Gamma \) that connects \( S \) to \( T \) such that the polynomials are dense in \( H^2(E \setminus \Gamma) \).

Proof. By Lemma 5, it is sufficient to produce a Jordan arc \( \Gamma \) that connects \( S \) to \( T \) such that \( -3/2 \notin \Gamma \) and \( 1/z \) can be approximated by polynomials in the \( L^2(\omega) \) norm; \( \omega := \omega(\cdot, E \setminus \Gamma, -3/2) \). A reasonable strategy for producing this \( \Gamma \) is to find a sequence of polynomials \( \{p_n\} \) and a sequence of polygonal Jordan arcs \( \{\Gamma_n\} \) such that

\[
\begin{align*}
(a) & \quad \text{for all } n, \Gamma_n \text{ connects } S \text{ to } T, \Re(z) > 0 \text{ for all } z \text{ in } \Gamma_n, \text{ and } \{\Gamma_n\} \text{ converges uniformly to a Jordan arc } \Gamma \text{ that connects } S \text{ to } T; \\
(b) & \quad \int |1/z - p_k|^2 \, d\omega_n < 1/k \text{ whenever } 1 \leq k \leq n, \text{ where } \\
& \quad \omega_n := \omega_n(\cdot, E \setminus \Gamma_n, -3/2).
\end{align*}
\]

In fact, for convenience of proof, we shall choose \( \Gamma_n \) so that its angle of incidence with both \( S \) and \( T \) is \( \pi/2 \) and that the angle formed by \( \Gamma_n \) at any of its vertices is at least \( \pi/3 \). The limiting arc \( \Gamma \) is the one we are after.

Let \( W_1 = \{z \in E : \text{dist}(z, [1, 2]) < 1/8\} \). By Runge's Theorem, there is a polynomial \( p_1 \) such that

\[
\|(1/z - p_1)^2\|_{E \setminus W_1} := \sup\{|1/z - p_1(z)|^2 : z \in E \setminus W_1\} < 1/2.
\]

Now we construct \( \Gamma_1 \). Let \( \Omega_1 = \{1 - i/4, 5/4 + i/4, 3/2 - i/4, 7/4 + i/4, 2 - i/4\} \); obviously \( 5 = \text{cardinality of } \Omega_1 : = |\Omega_1| \). Choose \( 0 < \epsilon_1 < 1/16 \) small.
Let $K_1 = ([1 + i/4, 2 + i/4] \cup [1 - i/4, 2 - i/4]) \setminus \bigcup_{z \in \Omega_1} B(z; \epsilon_1)$, where $B(z; \epsilon_1) := \{\zeta \in \mathbb{C} : |z - \zeta| < \epsilon_1\}$. Now $K_1$ is a closed set with five components: $I_1(1), I_1(3), I_1(5), I_1(7)$, and $I_1(9)$, each of which is a segment. The numbering scheme is as follows: $I_1(j) \subseteq [1 + i/4, 2 + i/4]$ for $j = 1, 5, 9$; $I_1(j) \subseteq [1 - i/4, 2 - i/4]$ for $j = 3, 7$; $\text{Re}(z) < \text{Re}(\zeta) < \text{Re}(\eta)$ whenever $z \in I_1(1), \zeta \in I_1(5)$, and $\eta \in I_1(9)$, and $\text{Re}(z) < \text{Re}(\zeta)$ whenever $z \in I_1(3)$ and $\zeta \in I_1(7)$. Connect the right endpoint of $I_1(1)$ to the left endpoint of $I_1(3)$, the right endpoint of $I_1(3)$ to the left endpoint of $I_1(5)$, the right endpoint of $I_1(5)$ to the left endpoint of $I_1(7)$, and the right endpoint of $I_1(7)$ to the left endpoint of $I_1(9)$, with segments $I_1(2), I_1(4), I_1(6)$, and $I_1(8)$, respectively. Let $\Gamma_1 = \bigcup_{j=1}^9 I_1(j)$ (see Figure 1). Notice that $\Gamma_1$ is a polygonal Jordan arc that connects $S$ to $T$, the angle of incidence of $\Gamma_1$ with both $S$ and $T$ is $\pi/2$, and the angle formed by $\Gamma_1$ at any of its vertices is at least $\pi/3$.

Now $W_1$ is only accessible in $E \setminus \Gamma_1$ from $z = -3/2$ through five “gaps” in $\Gamma_1$, each of size at most $2\epsilon_1$. Consequently, by (6.2), Lemma 4, and Lemma 3(b), $\omega_1(W_1) < 1/(2\|1/z - p_1\|^2\|E\setminus \Gamma_1\|_{W_1})$; $\omega_1 := \omega_1(\cdot, E \setminus \Gamma_1, -3/2)$. Therefore, since $\|1/z - p_1\|^2\|E\setminus \Gamma_1\|_{W_1} < 1/2$, we have that

$$\int |1/z - p_1|^2 d\omega_1 < 1.$$ 

For $n \geq 2$, $\Gamma_n$ is constructed inductively so that “over” each segment of $\Gamma_{n-1}$, $\Gamma_n$ looks like $\Gamma_1$. In order to construct $\Gamma_n$, certain other items need to be defined inductively. For $n \geq 2$, let

$$W_n = \{z \in E : \text{dist}(z, \Gamma_{n-1}) < \epsilon_{n-1}/16\}$$

and let

$$W_n' = \{z \in E : \text{dist}(z, \Gamma_{n-1}) < \epsilon_{n-1}/8\}.$$
By Runge's Theorem there is a polynomial \( p_n \) such that \( \| (1/z - p_n)^2 \|_{E \setminus W_n} < 1/(2n) \). The substance of the inductive step is found in the construction of \( \Gamma_2 \) and so, for the most part, we focus our attention there.

Recall that \( \Gamma_1 = \bigcup_{j=1}^{9} I_1(j) \). For \( 1 \leq j \leq 9 \), let \( I_1^*(j) \) be the straight line that contains \( I_1(j) \), and let \( D_2(j) = \{ z \in \mathbb{C} : \text{dist}(z, I_1^*(j)) < \epsilon_1/8 \} \). For \( j = 2, 4, 6, 8 \), let \( V_2(j, j + 1) = D_2(j) \cap D_2(j + 1) \) and \( V_2(j, j - 1) = D_2(j) \cap D_2(j - 1) \). Let

\[
W_2'' = W_2' \cup \left\{ \bigcup_{j=1}^{4} (V_2(2j, 2j + 1) \cup V_2(2j, 2j - 1)) \right\}.
\]

Notice that, unlike \( \partial W_2' \), \( \partial W_2'' \) is a polygon. Moreover, since the angle formed by \( \Gamma_1 \) at any of its vertices is at least \( \pi/3 \), it follows that \( \text{dist}(z, \Gamma_1) < \epsilon_1/4 \) whenever \( z \in W_2'' \). We shall construct \( \Gamma_2 \) using \( \text{cl}(E \cap (\partial W_2'')) \).

Now \( \partial V_2(j, j+1) \) [resp., \( \partial V_2(j, j-1) \)] is a parallelogram \( (j = 2, 4, 6, 8) \). Let \( a_2(j, j+1) \) [resp., \( a_2(j, j-1) \)] be the unique vertex of \( \partial V_2(j, j+1) \) [resp., \( \partial V_2(j, j-1) \)] which is in \( \text{co}(I_1(j) \cup I_1(j+1)) \) := closed convex hull of \( (I_1(j) \cup I_1(j+1)) \). Let \( b_2(j, j+1) \) [resp., \( b_2(j, j-1) \)] be the unique point in \( \partial D_2(j+1) \) [resp., \( \partial D_2(j-1) \)] such that the segment \( [a_2(j, j+1), b_2(j, j+1)] \) [resp., \( [a_2(j, j-1), b_2(j, j-1)] \)] is perpendicular to \( I_1^*(j+1) \) [resp., \( I_1^*(j-1) \)], and let \( c_2(j, j+1) \) [resp., \( c_2(j, j-1) \)] be the unique point in \( \partial D_2(j) \) such that the segment \( [a_2(j, j+1), c_2(j, j+1)] \) [resp., \( [a_2(j, j-1), c_2(j, j-1)] \)] is perpendicular to \( I_1^*(j) \). Let \( a_2(0, 1) \) [resp., \( a_2(10, 9) \)] be the intersection of the component of \( \text{cl}(E \cap (\partial W_2'')) \) that contains \( a_2(2, 1) \) with \( S \) [resp., \( T \)], and let \( b_2(0, 1) \) [resp., \( b_2(10, 9) \)] be the intersection of the component of \( \text{cl}(E \cap (\partial W_2'')) \) that contains \( b_2(2, 1) \) with \( S \) [resp., \( T \)]. For \( 1 \leq j \leq 9 \), if \( j \) is odd, then let \( R_2(j) \) be the rectangle with vertices \( a_2(j+1, j), a_2(j-1, j), b_2(j+1, j), b_2(j-1, j) \), and if \( j \) is even, then let \( R_2(j) \) be the rectangle with vertices \( a_2(j, j+1), a_2(j, j-1), c_2(j, j+1), c_2(j, j-1) \). Call a rectangle \( R_2(j) \) even if its \( a_2 \)-vertices are diagonal, and odd otherwise. Notice that \( R_2(j) \) is even if \( j \) is odd, and odd if \( j \) is even.

With straight lines that are perpendicular to \( I_1^*(j) \), partition \( R_2(j) \) into congruent subrectangles so that the number of subrectangles is even [resp., odd] if \( R_2(j) \) is even [resp., odd], the greatest dimension of any subrectangle is \( \epsilon_1/4 \) (the width of \( R_2(j) \)) and the least dimension is no less than \( \epsilon_1/8 \); it is possible to partition in this way because the length of any \( R_2(j) \) is at least twice its width—see Figure 2 on next page (labeled in part). Let \( \Omega_2 \) be the collection of points defined by:

(i) \( a_2(0, 1) \in \Omega_2 \)
(ii) \( z \in \Omega_2 \) if and only if \( z \) is a vertex of some subrectangle of some \( R_2(j) \) and the vertex diagonal to \( z \) in this subrectangle is in \( \Omega_2 \).

Now choose \( 0 < \epsilon_2 < \epsilon_1/16 \) (for \( n \geq 2, 0 < \epsilon_n < \epsilon_{n-1}/16 \)) small enough so that

\[
\| \Omega_2 \| (-1/\log(\epsilon_2)) < 1/(4||1/z - p_2)^2\|_{W_2},
\]

and let \( K_2 = \text{cl}(E \cap (\partial W_2'')) \setminus \bigcup_{z \in \Omega_2} B(z; \epsilon_2) \). Notice that \( K_2 \) is made up of finitely many components, each of which is either a segment or a polygonal Jordan arc that is the union of two segments. In the same way that \( \Gamma_1 \) was pieced
together, construct $\Gamma_2$ by connecting, with segments, the right endpoint of the component of $K_2$ which contains $b_2(0, 1)$ to the left endpoint of the component which contains the vertex that is diagonal to $b_2(0, 1)$ in the subrectangle of $R_2(1)$ which has $b_2(0, 1)$ as a vertex, etc. (see Figure 3). The resulting arc $\Gamma_2$ is a polygonal Jordan arc whose angle of incidence with both $S$ and $T$ is
\(\pi/2\) and whose angle at any vertex is at least \(\pi/3\). Moreover, any maximal segment of \(\Gamma_2\) (i.e., a segment of \(\Gamma_2\) that is properly contained in no other segment of \(\Gamma_2\)) has length at least \(2e_2\). Now since \(\| (1/z - p_2)^2 \|_{E\setminus W_2} < 1/4\) and \(W_2\) is only accessible in \(E\setminus \Gamma_2\) from \(z = -3/2\) through \(|\Omega_2|\) "gaps" in \(\Gamma_2\) each of size at most \(2e_2\), it follows from (6.3), Lemma 4, and Lemma 3(b) that

\[
\int |1/z - p_2|^2 \, d\omega_2 < 1/2,
\]

where \(\omega_2 := \omega_2(\cdot, E\setminus \Gamma_2, -3/2)\). Also notice that by our choice of \(\Gamma_2\), in order to access \(W_1\) in \(E\setminus \Gamma_2\) from \(z = -3/2\), one must pass through one of five gaps in \(\Gamma_2\), each of which represents a narrowing of one of the gaps in \(\Gamma_1\). Consequently,

\[
\int |1/z - p_1|^2 \, d\omega_2 < 1.
\]

For \(n \geq 3\), \(p_n\) is chosen and \(\Gamma_n\) is constructed in basically the same way we chose \(p_2\) and constructed \(\Gamma_2\).

Let us parametrize \(\Gamma_n\). Define \(\gamma_1: [0, 1] \rightarrow \Gamma_1\) by \(\gamma_1(x)\) is the point on \(\Gamma_1\) whose distance along \(\Gamma_1\) from \(S\) is \(x \cdot [\text{length}(\Gamma_1)]\). Now we turn to \(\Gamma_2\). For \(j = 2, 4, 6, 8\) let \(a'_2(j, j + 1)\) [resp., \(a'_2(j, j - 1)\)] be the vertex of \(\partial V(j, j + 1)\) [resp., \(\partial V(j, j - 1)\)] that is diagonal to \(a_2(j, j + 1)\) [resp., \(a_2(j, j - 1)\)]. Notice that \(a'_2(j, j + 1)\) and \(a'_2(j, j - 1)\) are in \(\Gamma_2\). Now \(\Gamma_2\) \(\setminus \{ \cup_{i=1}^4 \{a'_2(2i, 2i + 1), a'_2(2i, 2i - 1)\} \}\) has nine components; number them as to the order in which each is encountered when traversing \(\Gamma_2\) from \(S\) to \(T\). Define a continuous one-to-one function \(\beta_2: \Gamma_1 \rightarrow \Gamma_2\) by mapping \(I_1(j)\) (recall that \(\Gamma_1 = \bigcup_{j=1}^9 I_1(j)\)) onto the closure of the \(j\)th component of \(\Gamma_2\) \(\setminus \{ \cup_{i=1}^4 \{a'_2(2i, 2i + 1), a'_2(2i, 2i - 1)\} \}\) in the same way that \(\gamma_1\) maps \([0, 1]\) onto \(\Gamma_1\). Let \(\gamma_2 = \beta_2 \circ \gamma_1\). Similarly, for any \(n\), define a continuous one-to-one function \(\beta_n: \Gamma_{n-1} \rightarrow \Gamma_n\) by mapping any maximal segment of \(\Gamma_{n-1}\) (i.e., a segment of \(\Gamma_{n-1}\) that is properly contained in no other segment of \(\Gamma_{n-1}\)) to the part of \(\Gamma_n\) that "covers" the segment. Then let \(\gamma_n = \beta_n \circ \gamma_{n-1}\).

Choose \(\delta > 0\). Now there exists \(N \geq 3\) such that \(e_{n-2} < \delta\). No maximal segment of \(\Gamma_n\) has length greater than \((3/4) \cdot e_{N-2}\). So, by the construction of \(\Gamma_k\) and the definition of \(\gamma_k\), if \(m, n \geq N\) and \(t \in [0, 1]\), then \(|\gamma_m(t) - \gamma_n(t)| < \delta\). Therefore, \(\{\gamma_n\}\) is uniformly Cauchy and hence converges uniformly to a continuous function \(\gamma: [0, 1] \rightarrow \Gamma := \gamma([0, 1])\).

To show that \(\gamma\) is one-to-one, choose \(s\) and \(t\) in \([0, 1]\) such that \(s \neq t\). By the definition of \(\gamma_n\) there exists \(N\) such that \(\gamma_n(s)\) and \(\gamma_n(t)\) are in nonadjacent maximal segments of \(\Gamma_N\). Reviewing the construction of \(\Gamma_N\), we find that \(|\gamma_N(s) - \gamma_N(t)| \geq 2e_N\). In fact, if \(n \geq N + 1\), then

\[
|\gamma_n(s) - \gamma_n(t)| \geq 2e_N - 2 \cdot \sum_{k=N+1}^n \frac{e_n}{4^k - N} > e_N.
\]

Hence, \(|\gamma_n(s) - \gamma_n(t)| \rightarrow 0\) as \(n \rightarrow \infty\), and so \(\gamma(s) \neq \gamma(t)\). Therefore \(\gamma\) is one-to-one, and \(\Gamma\) is a Jordan arc.

We now have a sequence of polynomials \(\{p_n\}\) and a sequence of polygonal Jordan arcs \(\{\Gamma_n\}\) which satisfy (6.1)(a) and (b). Let \(\omega := \omega(\cdot, E\setminus \Gamma, -3/2)\).
Since $\Gamma_n$ converges uniformly to $\Gamma$, it follows that, for fixed $k$,

$$\int \frac{1}{|z - p_k|^2} \, d\omega_n \to \int \frac{1}{|z - p_k|^2} \, d\omega,$$

as $n \to \infty$; $\omega_n := \omega_n(\cdot, E\setminus\Gamma_n, -3/2)$. Therefore, because $\int |1/z - p_k|^2 \, d\omega_n < 1/k$ whenever $1 \leq k \leq n$, we have that

$$\int |1/z - p_k|^2 \, d\omega \leq 1/k \to 0,$$

as $k \to \infty$. By Lemma 5, the proof is now complete. □

7. Theorem. There exists a Jordan $\Gamma := \gamma([0, 1])$, where $\gamma(0) = 0$, $\gamma(t) \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ for $0 < t < 1$, and $\gamma(1) = 1$, such that the polynomials are dense in $H^2(\mathbb{D}\setminus\Gamma)$.

Proof (sketch). In a way similar to the proof of Theorem 4, we produce a sequence of Jordan arcs $\{\Gamma_n := \gamma_n([0, 1])\}$ ($\Gamma_n$ having the same geometry as in the proof of Theorem 4), where $\gamma_n([0, 1]) \subseteq \{z \in \mathbb{D} : \text{Re}(z) > 0\}$ and $|\gamma_n(1)| = 1$ for all $n$, a sequence of points $\{t_n\}$, where $0 < t_n < 1$ and $|\gamma_n(t_n) - \gamma_n(0)| \to 0$ as $n \to \infty$, and a sequence of polynomials $\{p_n\}$ such that:

(a) $\Gamma_n$ converges uniformly to a Jordan arc $\Gamma := \gamma([0, 1])$, where $\gamma(0) = 0$, $\gamma(t) \in \mathbb{D}$ for $0 < t < 1$, and $\gamma(1) = 1$;

(b) $|p_n(\gamma_n(t_n))| \geq 1$ for all $n$, and $\int |p_k|^2 \, d\omega_n < 1/k$ whenever $1 \leq k \leq n$, where $\omega_n := \omega_n(\cdot, \mathbb{D}\setminus\Gamma_n, -1/2)$.

The limiting arc $\Gamma := \gamma([0, 1])$ will then have the property that, for all $n$, $\int |p_n|^2 \, d\omega \leq 1/n$ ($\omega := \omega(\cdot, \mathbb{D}\setminus\Gamma, -1/2)$) and yet $|p_n(\gamma_n(t_n))| \geq 1$, where $\gamma_n(t_n) \to \gamma(0)$ as $n \to \infty$. So, $\gamma(0)$ is not an analytic bounded point evaluation for the polynomials with respect to the $H^2(\mathbb{D}\setminus\Gamma)$ norm, and hence nor is any point in $\partial(\mathbb{D}\setminus\Gamma)$. Following an argument similar to the proof of Lemma 3, we get that the polynomials are dense in $H^2(\mathbb{D}\setminus\Gamma)$. □

8. Remark. Theorems 4 and 5 provide us with new examples of analytic Toeplitz operators $T_\varphi$, where $\varphi$ is a Riemann map from the unit disk $\mathbb{D}$ onto $E\setminus\Gamma$ (of Theorem 4) or onto $\mathbb{D}\setminus\Gamma$ (of Theorem 5), such that $T_\varphi$ is cyclic (with cyclic vector 1) and yet $\varphi$ is not a weak-star generator of $H^\infty$ (cf. [8]).

There is unfinished business here, and yet very little of it is easily approachable.

9. Problem. Find a condition on $\Gamma$ which is both necessary and sufficient for density of the polynomials in $H^2(\mathbb{D}\setminus\Gamma)$, where $\Gamma := \gamma([0, 1])$ is a Jordan arc such that $\gamma([0, 1]) \subseteq \mathbb{D}$ and $\gamma(1) = 1$.

10. Question. Does there exist a Jordan arc $\Gamma$, with endpoints 0 and 1, such that the polynomials are dense in $L^2_{\omega}(\mathbb{D}\setminus\Gamma, dA)$?

Acknowledgment

The author is grateful to Daniel Luecking for pointing out some useful references and to the referee for helpful suggestions.
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