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THE BEHAVIOR OF THE ANALYTICALLY CONTINUED RESOLVENT OPERATOR NEAR $\kappa = 0$ AND AN APPLICATION TO ENERGY DECAY

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ABSTRACT. We shall study the behavior of the analytically continued resolvent operator $R^+(\kappa)$ for perturbations of $-\Delta$ in a neighborhood of $\kappa = 0$. As an application, making use of Vainberg's argument, we shall show the local energy decay of solutions to generalized wave equations whose stationary problems are not positive definite.

1. INTRODUCTION AND RESULTS

In this paper we shall study the behavior of the analytically continued resolvent operator $R^+(\kappa)$ near $\kappa = 0$ of the following problem:

$$(1.1)\qquad \begin{cases} (L-\kappa^2)u = -\alpha(x)[\partial_j a_{jk}(x)\partial_k - q(x)]u - \kappa^2 u = f & \text{in } \Omega, \\ Bu = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is R^n or an exterior domain of R^n $(n \geq 3)$ with the C^2-class boundary $\partial\Omega$ and B is either the Dirichlet boundary condition or the third boundary condition of the form $Bu = \nu_j(x)a_{jk}\partial_k u + \sigma(x)u$ with the unit outer normal vector $\nu(x) = (\nu_1, \dots, \nu_n)$ at $x \in \partial\Omega$. Assumptions on the coefficients of L and B are as follows:

(1.2) The function $\alpha(x)$ is bounded, measurable, and real valued, and is uniformly positive in R^n. The real symmetric matrix $(a_{jk}(x))$ is uniformly positive in R^n and its components are in $C^1(R^n)$. Moreover there exist positive constants C and b such that

$$|\alpha(x) - 1| + |a_{jk}(x) - \delta_{jk}| + |\nabla a_{jk}(x)| \leq Ce^{-2b|x|} \quad \text{in } R^n.$$

(1.3) The function $q(x)$ is real and in $L^p_{\text{loc}}(R^n)$, where $p = n/2$ for $n \geq 5$, $p > 2$ for $n = 4$ and $p = 2$ for $n = 3$. The real valued function $\sigma(x)$ is in $C^2(\partial\Omega)$. Moreover there exist positive numbers C and b such that for sufficiently large $|x|$ $|q(x)| \leq Ce^{-2b|x|}$.

Let $H^m_{\rho_1}(\Omega)$ be a set $\{f \in H^m_{\text{loc}}\{\overline{\Omega}); \ \rho_1\partial_x^\alpha f \in L^2(\Omega) \text{ for } |\alpha| \leq m\}$ and $L^2_{\rho_1}(\Omega) = H^0_{\rho_1}(\Omega)$. We denote by Γ_a a set $\{\kappa \in C; \operatorname{Im}\kappa > -a\}$ if n is odd,

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and a set $\{\kappa \in C \setminus 0;\ |\operatorname{Im} a| < a,\ -\infty < \arg\kappa < \infty\} \cup \{\kappa \in C;\ 0 < \arg\kappa < \pi\}$ if n is even. Then in Theorem 4.6 of our previous paper [9] under the above assumptions we show that for any positive $a < b$ there exists $R^+(\kappa)$ which is a finitely meromorphic function in Γ_a with values in $\mathscr{B}(L^2_\rho(\Omega),\ H^2_{\rho^{-1}}(\Omega))$ such that $u(x) = R^+(\kappa)f$ is the solution of (1.1) if κ is an analytic point of $R^+(\kappa)$, where $\rho(x) = e^{-a|x|}$.

The purpose of this paper is to study the behavior of $R^+(\kappa)$ in a neighborhood of $\kappa = 0$. It is well known that L is a selfadjoint operator on $\mathscr{H}$ with the domain $D(L) = \{f \in H^2(\Omega);\ Bu = 0 \text{ on } \partial\Omega\}$, where $\mathscr{H} = \{f(x);\ \|f\|^2_{\mathscr{H}} = (f,\ f\alpha^{-1})_{L^2(\Omega)} < \infty\}$ (see Lemma 2.1 of [8]). We denote by $H_D(\Omega)$ the completion of $C_0^\infty(\overline{\Omega})$ $(C_0^\infty(\Omega))$ by the Dirichlet norm $\|\nabla f\|_{L^2(\Omega)}$ if B is the third (Dirichlet) boundary condition. Let H be $H_D(\Omega) \times \mathscr{H}$ and $A = \left(\begin{smallmatrix} 0 & 1 \\ -L & 0 \end{smallmatrix}\right)$ be an operator with the domain $\{f = {}^t(f_1,\ f_2) \in H;\ f_2 \in L^2(\Omega) \cap H_D(\Omega),\ \partial_x^2 f_1 \in L^2(\Omega) \text{ for } |\alpha| = 2,\ Bf_1 = 0 \text{ (on } \partial\Omega\}.)$ Put $E_n = \{\kappa;\ \kappa^2$ is a nonpositive eigenvalue of $L\}$ if $n \geq 5$ and put $E_n = \{\kappa;\ \kappa = i\mu,\ \mu \geq 0,\ \pm\mu$ is an eigenvalue of $A\}$ if $n = 3, 4$. We note that $E_n \setminus \{0\} = \{\kappa;\ \kappa^2$ is a negative eigenvalue of $L\}$ $(n = 3, 4)$ and that from Lemma 2.1 of [8] the number of elements of E_n is finite and the dimensions of each corresponding eigenspace are finite. We shall prove the following:

Theorem 1.1. *We have the following two statements*:

(i) *If n is odd and $0 \notin E_n$, then $R^+(\kappa)$ is analytic at $\kappa = 0$.*

(ii) *If n is even and $0 \notin E_n$, then there exists δ such that*

$$R^+(\kappa) = \sum_{p,q=0}^{\infty} R_{pq}(\kappa^{n-2}\ln\kappa)^p \kappa^q, \tag{1.4}$$

where $\kappa \in \{\kappa \in \Gamma_a;\ |\kappa| < \delta,\ -\pi/2 < \arg\kappa < 3\pi/2\}$, $R_{pq} \in \mathscr{B}(L^2_\rho(\Omega),\ H^2_{\rho^{-1}}(\Omega))$ and the double series is absolutely convergent in the uniform operator norm.

Under the assumptions $\alpha(x) = 1$, $q(x) = 0$, and $\sigma = 0$ the above theorem is proved in Lemma 1 of [4] and Theorem 1.2 of [2]. In this case $0 \notin E_n$ is automatically satisfied.

As an application of Theorem 1.1 we shall consider the local energy decay of the solution of the following wave equation:

$$\begin{cases} \partial_t^2 u + Lu = 0 & \text{in } R \times \Omega, \\ Bu = 0 & \text{on } R \times \partial\Omega,\ \partial_t^{j-1} u = f_j \text{ on } t = 0\ (j = 1, 2), \end{cases} \tag{1.5}$$

where we assume that all coefficients of L and B and $\partial\Omega$ are in the C^∞ class. It is known that there exists a group of linear operators $U(t)$ on H such that A is the infinitesimal generator of $U(t)$. (See Theorem 2.7 in [8].) We remark that for any $f = {}^f(f_1,\ f_2) \in H$ the first component of $U(t)f$ satisfies (1.5) in the distribution sense. (See the proof of Lemma 3.2.) In order to show a decay of the local energy $E(f;\ \Omega_a) = \frac{1}{2}\int_{\Omega_a}\{a_{jk}\partial_k f_1 \partial_j \overline{f}_1 + q|f_1|^2 + \alpha^{-1}|f_2|^2\} + \frac{1}{2}\int_{\partial\Omega}\sigma|f_1|^2\,dS$, where $\Omega_a = \Omega \cap \{|x| < a\}$, $f = {}^t(f_1,\ f_2) \in H$ and $\sigma = 0$ if B is the Dirichlet condition, we suppose a nontrapping condition of (1.5), which is stated in Definition 3.1. We note that if $\alpha(x) = 1$, $a_{jk}(x) = \delta_{jk}$, and the complement set of Ω is convex or star-shaped, a nontrapping condition of (1.5)

holds. Making use of Vainberg's argument in [6], we can show the following local energy decay:

Theorem 1.2. *We assume that* $\alpha(x)-1$, $a_{jk}(x)-\delta_{jk}$, $q(x)$ *belongs to* $C_0^\infty(\overline{\Omega})$. *If the nontrapping condition of Definition* 3.1 *holds and* $0 \notin E_n$, *then for any* $f \in H \cap \mathscr{E}'(\Omega_a)$,

$$E(U(t)(1-P)f;\Omega_a) \le a(t)\|f\|_H^2 \quad \text{for all } t>0, \tag{1.6}$$

where P *is the projection to the eigenspace associated to negative eigenvalues of* L, *and if* n *is odd, then* $a(t) = C_1\exp(-C_2 t)$ *with positive constants* C_j $(j=1,2)$ *and if* n *is even, then* $a(t) = C_3(1+t)^{2(1-n)}$ $(C_3>0)$. *Moreover if* $E_n = \varnothing$ *and* $\operatorname{Ker} A = \{0\}$, *then by putting* $E(f) = E(f;\Omega_\infty)$ *we see that for* $f \in H \cap \mathscr{E}'(\Omega_a)$

$$E(U(t)f;\Omega_a) \le a(t)E(f). \tag{1.7}$$

Under the assumption $\alpha(x)=1$, $q(x)=0$, *and* $\sigma=0$, (1.7) *is proved in* [4] *and Theorem* 4.3 *of* [2].

2. The proof of Theorem 1.1

In the odd case Theorem 1.1 is proved in Theorem 4.6 of [9]. So in this section we always assume that n is even and $n \ge 4$. First we shall consider the fundamental solution $R_0^+(\kappa)$ of $-\Delta+\kappa^2$ defined by $[R_0^+(\kappa)f](x) = \int F_\kappa^+(x-y)f(y)\,dy$. Here by making use of the Hankel function $H_p^{(1)}(z)$ of the first kind with $p=(n-2)/2$, $F_\kappa^+(x)$ is defined by $i(\kappa/2\pi|x|)^p H_p^{(1)}(\kappa|x|)/4$. The behavior of $R_0^+(\kappa)$ near $\kappa=0$ is as follows:

Lemma 2.1. *There exists* $A(\kappa)$ *and* $B(\kappa)$ *which are analytic in* $\{\kappa \in C;\ |\kappa| < a\}$ *with values in* $\mathscr{B}(L_\rho^2(R^n),\ H_{\rho^{-1}}^2(R^n))$, *where* a *is an arbitrary positive number and* $\rho(x) = e^{-a|x|}$, *such that*

$$R_0^+(\kappa) = A(\kappa)\kappa^{n-2}\ln\kappa + B(\kappa), \tag{2.1}$$

where $[B(0)f](x) = c_0\int |x-y|^{2-n}f(y)\,dy$ *with some constant* c_0.

Proof. From [7] (see (5), p. 74; (1), p. 61; (2), p. 62) it follows that $F_\kappa^+(x) = A_1(\kappa|x|)\kappa^{n-2}(\ln\kappa|x|/2+1) + |x|^{-p}\sum_{m=0}^{p-1} c_m(\kappa|x|)^{2m} + \kappa^{n-2}B_1(\kappa|x|)$, where $A_1(z)$ and $B_1(z)$ are entire functions such that for any j $|\partial_z^j A_1(z)| + |\partial_z^j B_1(z)| \le C_j e^{|z|}$. Let $A(\kappa)f = \int A_1(\kappa|x-y|)f(y)\,dy$. Then by the argument of proving Propositions 2.4 and 2.5 in [9] we see that $A(\kappa)$ is analytic in $\{\kappa \in C;\ |\kappa| < a\}$ with values in $\mathscr{B}(L_\rho^2(R^n),\ H_{\rho^{-1}}^2(R^n))$. Similarly $B_2(\kappa)f = \int B_2(\kappa|x-y|)f(y)\,dy$, where

$$B_2(\kappa|x|) = A_1(\kappa|x|)\kappa^{n-2}(\ln|x|/2+1) + |x|^{-p}\sum_{m=1}^{p-1} c_m(\kappa|x|)^{2m} + \kappa^{n-2}B_1(\kappa|x|),$$

has the same property. The remainder term $(B_3 f)(x) = c_0\int |x-y|^{2-n}f(y)\,dy$ belongs to $H_{\rho^{-1}}^1(R^n)$ for $f \in L_\rho^2(R^n)$ and satisfies the relation $-\Delta(B_3 f) = c_0' B_3 f$ with some constant c_0'. Thus by the argument of proving Proposition 2.4 of [9] it follows that $B_3 \in \mathscr{B}(L_\rho^2(R^n),\ H_{\rho^{-1}}^2(R^n))$. The proof is completed.

Next according to the argument in §3 of [9] we shall check the behavior of the resolvent operator of $-\partial_j a_{jk}(x)\partial_k$ in R^n. Let $\varphi_m(x)$ be a $C_0^\infty(R^n)$ function such that $\varphi_m(x) = 1$ for $|x| < m$, $\varphi_m(x) = 0$ for $|x| > m+1$, $0 \le \varphi_m(x) \le 1$, and for any multi-index β $|\partial_x^\beta \varphi_m(x)| \le C_\beta$, where C_β does not depend on m. Put $A = \partial_j(\delta_{jk} - a_{jk}(x))(1-\varphi_m(x))\partial_k$. Then A belongs to $\mathscr{B}(H^2_{\rho^{-1}}(R^n), L^2_\rho(R^n))$ and if m is sufficiently large, then $\|A\|$ is sufficiently small. Put $T_1(\kappa) = AR_0^+(\kappa)$. Then for any $g \in L^2_\rho(R^n)$, $-(\partial_j \tilde{a}_{jk}(x)\partial_k + \kappa^2) \times R_0^+(\kappa)g = (1+T_1(\kappa))g$, where $\tilde{a}_{jk}(x) = \delta_{jk}(x) + (a_{jk}(x) - \delta_{jk})(I - \varphi_m(x))$. From Lemma 2.1 it follows that $R_1^+(\kappa) = R_0^+(\kappa)(I+T_1(\kappa))^{-1}$ is denoted by

$$F(\kappa^{n-2}\ln\kappa, \kappa) = \sum_{p,q=0}^{\infty} A_{pq}(\kappa^{n-2}\ln\kappa)^p \kappa^q,$$

where $A_{pq} \in \mathscr{B}(L^2_\rho(R^n), H^2_{\rho^{-1}}(R^n))$ and $\sum_{p,q=0}^\infty \|A_{pq}\||\lambda|^p|\kappa|^q < \infty$ if $|\lambda|$ and $|\kappa|$ are sufficiently small.

We shall consider the following problem:

$$\text{(2.2)} \qquad \begin{cases} (\partial_j a_{jk}(x)\partial_k + \lambda_0)v_1 = \partial_j(\tilde{a}_{jk} - a_{jk})(x)\partial_\kappa A_{pq} g & \text{in } |x| < N, \\ v_1(x) = 0 \quad \text{on } |x| = N. \end{cases}$$

where $\operatorname{Im}\lambda_0 \ne 0$, $g \in L^2_\rho(R^n)$, and N is sufficiently large. Put $V_1(\kappa) = \sum_{p,q=0}^\infty V_{pq}(\kappa^{n-2}\ln\kappa)^p\kappa^q$, where $V_{pq}g = v_1(x)$. Then from $V_{pq} \in \mathscr{B}(L^2_\rho(R^n), H^2(\{|x| < N\}))$ and $\|V_{pq}\| \le C\|A_{pq}\|$, where C does not depend on p and q, we see that $\sum_{p,q=0}^\infty \|V_{pq}\||\lambda|^p|\kappa|^q < \infty$ if $|\lambda|$ and $|\kappa|$ are sufficiently small, and that $V_1 \in \mathscr{B}(L^2_\rho(R^n), H^2(\{|x| > N\}))$. Now from (2.2) it follows that

$$-(\partial_j a_{jk}(x)\partial_k - \kappa^2)(R_1^+(\kappa) + \varphi_1^2 V_1(\kappa))g = (I + T_2(\kappa))g,$$

where $\varphi_1(x) \in C_0^\infty(\{|x| < N\})$ such that $\varphi_1(x) = 1$ on $\operatorname{supp}(a_{jk} - \tilde{a}_{jk})$ and $T_2(\kappa) = \varphi_1^2(\lambda_0 - \kappa^2)V_1(\kappa) - [\partial_j a_{jk}\partial_k, \varphi_1^2]V_1(\kappa)$, where $[A, B] = AB - BA$. We have the following:

Lemma 2. *$I + T_2(0)$ is an invertible operator in $\mathscr{B}(L^2_\rho(R^n), L^2_\rho(R^n))$.*

Proof. Since $T_2(0)$ is a compact operator, we may show that $\operatorname{Ker}(I + T_2(0)) = \{0\}$. We suppose $(I+T_2(0))g = 0$. Then $-\partial_j a_{jk}(x)\partial_k(R_1^+(\kappa) + \varphi_1^2 V_1(0))g = 0$. If we put $h(x) = (I + T_1(0))^{-1}g$, then $(1+|x|)^N h(x) \in L^2(R^n)$ for any N and $R_1^+(0)g = R_0^+(0)h$. From Lemma 2.1, $R_0^+(0)h(x) = c_0 \int |x-y|^{2-n}h(y)\,dy$. By $|x|^{2-n} \in L^1(\{|x| < 1\})$ it follows that $|\int_{D_1} |x-y|^{2-n}h(y)\,dy| \le (1+|x|)^{-N}h_1(x)$, where $D_1 = \{y; |x-y| < 1\}$ and $h_1(x) \in L^2(R^n)$. On the other hand $|\int_{D_2} |x-y|^{2-n}h(y)\,dy|^2 \le C\int_{D_2} |x-y|^{2\varepsilon-n}(1+|y|^{-2N}\,dy$, where ε is sufficiently small and $D_2 = \{y; |x-y| > 1\}$. From the well-known inequality (see (2.2) in [9]) we see that $|R_0^+(0)h(x)| \le C(1+|x|)^{\varepsilon-n/2} + (1+|x|)^{-N}h_1(x)$. Similarly $\nabla R_0^+(0)h(x)$ is dominated by a function of the same type. So we can use the argument in the proof of Proposition 3.1 in [9] and can show that $g = 0$. The proof is completed.

We put $R_2^+(\kappa) = (R_1^+(\kappa) + \varphi_1^2 V_1(\kappa))(I + T_2(\kappa))^{-1}$. Then $R_2^+(\kappa)$ is the resolvent operator of $-\partial_j a_{jk}(x)\partial_\kappa - \kappa^2$ and is denoted by $F_1(\kappa^{n-2}\ln\kappa, \kappa) = \sum_{p,q=0}^\infty B_{pq}(\kappa^{n-2}\ln\kappa)^p\kappa^q$, where $B_{pq} \in \mathscr{B}(L^2_\rho(R^n), H^2_{\rho^{-1}}(R^n))$ and

$\sum_{p,q=0}^{\infty} \|B_{pq}\| |\lambda|^p |\kappa|^q < \infty$ if $|\lambda|$ and $|\kappa|$ are sufficiently small. We shall consider the following problem:

$$(2.3) \qquad \begin{cases} (\partial_j a_{jk} \partial_k - \lambda_1) v_2 = (\partial_j a_{jk} \partial_k - \lambda_1)(1 - \psi) B_{pq} E g & \text{in } \Omega_1, \\ B v_2 = 0 \quad \text{on } \partial\Omega, \ v_2 = 0 \text{ on } |x| = N_1, \end{cases}$$

where $\operatorname{Im} \lambda_1 \neq 0$, $g \in L^2_\rho(\Omega)$, $Eg = g$ in Ω and $Eg = 0$ in Ω^c, $\Omega_1 = \Omega \cap \{|x| < N_1\}$ and $\psi(x) \in C^\infty(R^n)$ such that $\psi(x) = 1$ if $|x| > N_1 - 1$ and $\psi(x) = 0$ near $\partial\Omega$. Put $V_2(\kappa) = \sum_{p,q=0}^{\infty} W_{pq}(\kappa^{n-2} \ln \kappa)^p \kappa^q + \psi R_2^+(\kappa) E$, where $W_{pq} \in \mathscr{B}(L^2_\rho(\Omega), H^2(\Omega_1))$ is defined by $W_{pq} g = v_2$. We note that $\sum_{p,q=0}^{\infty} \|W_{pq}\| |\lambda|^p |\kappa|^q < \infty$ if $|\lambda|$ and $|\kappa|$ are sufficiently small. From (2.3) we see that $-(\partial_j a_{jk} \partial_k - q + \alpha^{-1}\kappa^2)\{R_2^+(\kappa) E g - \varphi_2(R_2^+(\kappa) E - V_2(\kappa)) g\} = (I + T_3(\kappa)) g$, where $\varphi_2 \in C_0^\infty(\{x; |x| < N_1\})$ such that $\varphi_2(x) = 1$ near Ω^c and $T_3(\kappa) \in \mathscr{B}(L^2_\rho(\Omega), L^2_\rho(\Omega))$ is defined by

$$\{(\alpha^{-1} - 1)\kappa^2 - q\} R_2^+ E g + \{[\partial_j a_{jk} \partial_k, \varphi_2] + \varphi_2(\alpha^{-1}\kappa^2 - \lambda_1 - q)\}(R_2^+(\kappa) E - V_2(\kappa)) g.$$

Lemma 2.3. *If $0 \notin E_n$, then $I + T_3(0)$ is invertible.*

Proof. From Proposition 4.2 in [9], $T_3(0)$ is a compact operator. So we may prove $\operatorname{Ker}(I + T_3(0)) = \{0\}$. We shall show that if $(I + T_3(0)) g = 0$, then $R_2^+(0) E g - \varphi_2(R_2^+(0) E - V_2(0)) g$ belongs to $L^2(\Omega)$, if $n \geq 6$, and it belongs to $H_D(\Omega)$, if $n = 4$. From the definition $R_2^+(0) E g - R_0^+(0) h$ has a compact support, where $h = (I + T_1(0))^{-1}(I + T_2(0))^{-1} E g$ and $(1 + |x|)^N h(x) \in L^2(R^n)$ for all N. Since $|x|^{2-n} \in L^2(\{x; |x| \geq 1\}) \cap L^1(\{x; |x| \leq 1\})$ if $n \geq 6$, we see that $R_2^+(0) E g$ belongs to $L^2(R^n)$, if $n \geq 6$. By similar argument and one of Lemma 2.2, $\nabla R_2^+(0) E g \in L^2(R^n)$ and $|R_2^+(0) E g(x)| \leq C(1 + |x|)^{\varepsilon - n/2} + (1 + |x|)^{-N} h_2(x)$, where $h_2(x) \in L^2(R^n)$. So by the argument in the proof of Proposition 4.4 in [9] when $\kappa = 0$ and $n = 3$ $R_2^+(0) E g - \varphi_2(R^+(0) E - V_2(0)) g \in H_D$ if $n = 4$. It follows that if $0 \notin E_n$, then $R_2^+(0) E g - \varphi_2(R_2^+(0) E - V_2(0)) g = 0$. The argument of deriving $g = 0$ from this condition is similar to one of the proof of Proposition 4.4 in [9]. The proof is completed.

Finally we put

$$R^+(\kappa) = \{R_2^+(\kappa) E - \varphi_2(R_2^+(\kappa) - V_2(\kappa))\}(I + T_3(\kappa))^{-1} \alpha^{-1},$$

where $(\alpha^{-1} f)(x) = \alpha^{-1}(x) f(x)$. Then $R^+(\kappa)$ has all the properties stated in Theorem 1.1.

3. The local energy decay

In this section we shall show the local energy decay of solutions to (1.5). Here we assume that all coefficients of L and B and $\partial\Omega$ are in the C^∞ class, and the supports of $\alpha(x) - 1$, $a_{jk}(x) - \delta_{jk}$, $q(x)$ are compact. First, we state the definition of a nontrapping condition. The generalized bicharacteristics of problem (1.5) are defined in Definition 3.1 in [3]. The projection into $\overline{\Omega}$ of generalized bicharacteristics is called generalized geodesics. We note that these geodesics are parameterized by time t.

Definition 3.1. Problem (1.5) is said to be nontrapping if for any sufficiently large a there exists T_a such that there are no generalized geodesics $\gamma(t)$ which satisfy the condition $\{\gamma(t) \in \overline{\Omega};\ t \in [0, T_a]\} \subset \{x;\ |x| < a\}$.

It is known that if $\alpha(x) = 1$, $a_{jk}(x) = \delta_{jk}$, and Ω^c is convex or star-shaped, then problem (1.5) is nontrapping. In this section we always assume this nontrapping condition.

Lemma 3.2. *Assume that the support of $f \in H_D$ is contained in $\Omega_a = \Omega \cap \{x;\ |x| < a\}$. If a is sufficiently large, then $[U(t)f](x) \in C^\infty(D_a)$, where $D_a = \{(t, x) \in R \times \overline{\Omega};\ t > T_a,\ |x| < t - T_a + a\}$.*

Proof. Let $u(t, x)$ be the first component of $[U(t)f](x)$. Since $D(A)$ is a dense set in H, we see that $\partial_t^2 u + Lu = 0$ as $\mathscr{D}'(R \times \Omega)$. From this fact, Theorem 4.3.1, and Theorem 2.5.6 of [1] it follows that the trace of Bu on $R \times \partial\Omega$ exists as an element of $\mathscr{D}'(R \times \partial\Omega)$. From $Bu = 0$ on $R \times \partial\Omega$ for $f \in D(A)$ by approximating elements of $D(A)$ to u, we see that $Bu = 0$ on $R \times \partial\Omega$. Thus we can use theorems on a propagation of singularities for $u(t, x)$. From the finite propagation property and the fact that $D(A)$ is a dense subset of H, $u(t, x) = 0$ in $\{(t, x);\ |x| > |t| + a\}$. Let $\gamma(t)$ be a generalized geodesic such that $|\gamma(t_0)| < t_0 - T_a + a$ and $t_0 > T_a$. Then from $L = -\Delta$ in $\{x;\ |x| > a\}$ and the nontrapping condition it follows that $|\gamma(0)| > a$. Thus $u(t, x)$ is C^∞ near $\{(t, \gamma(t));\ |t| < \delta\}$, where δ is sufficiently small. By theorems on a propagation of singularities (see for example Theorem 5.10 in [3]) it follows that $u(t, x)$ is C^∞ near $(t_0, \gamma(t_0))$. We note that the condition $(2.2)_\pm$ in [3] is valid for the third boundary condition. The proof is completed.

Next we shall state the behavior of $R^+(\kappa)$ near $|\operatorname{Re}\kappa| = \infty$.

Theorem 3.3. *Assume that problem* (1.5) *is nontrapping. Then there exist positive constants α and β such that $R^+(\kappa)$ is analytic in $U_{\alpha,\beta} = \{\kappa;\ -\pi/2 < \arg\kappa < 3\pi/2,\ |\operatorname{Im}\kappa| < \alpha \ln|\operatorname{Re}\kappa| - \beta\}$ with values in $\mathscr{B}(L_a^2(\Omega),\ H^2(\Omega_a))$, where $L_a^2(\Omega) = L^2(\Omega) \cap \mathscr{E}'(\Omega_a)$ with $\Omega_a = \Omega \cap \{|x| < a\}$. Moreover there exist positive constants C and T such that for $j = 0, 1, 2$, $\kappa \in U_{\alpha,\beta}$,*

$$\|R^+(\kappa)f\|_{H^{2-j}(\Omega_a)} \le C|\kappa|^{1-j} e^{T|\operatorname{Im}\kappa|} \|f\|_{L^2(\Omega_a)}. \tag{3.1}$$

The statement of the above theorem is the same as Theorem 7 in [6]. However, in order to prove it the author of [6] assumes an existence of the Green function of problem (1.5) which is defined in condition D' in [6, p. 11]. Since there is no guarantee on the existence of it, we shall give a proof without assuming the existence of the Green function. But the outline is almost the same as the proof of Theorem 7 in [6].

The following is a sketch of the proof of Theorem 3.3 without assuming the existence of the Green function. Let $h(t, x)$ be a $C^\infty(R^{n+1})$ function such that $h(t, x)$ depends only on t in a neighborhood of $\partial\Omega$, $h(t, x) = 1$ in the complement set of $\{(t, x);\ t > T_a + 1,\ |x| < t - (T_a + 1) + a\}$ and $h(t, x) = 0$ in $\{(t, x);\ t > T_a + 2,\ |x| < t - (T_a + 2) + a\}$. Put $(E\varphi)(t, x) = h(t, x)u(t, x)$, where $u(t, x)$ is the first component of $U(t)^t(0, \varphi)$ and define $(\widetilde{E}(\kappa)\varphi) = \int_0^\infty e^{i\kappa t}(E\varphi)(x, t)\,dt$. Then we have the following:

Lemma 3.4. *The operator $\widetilde{E}(\kappa)$ is an entire function of κ with values in $\mathscr{B}(H_a^s(\Omega),\ H^{s+2}(\Omega_a))$ $(s = 0, 1)$ and there exist positive constants C and T such that for $s = 0, 1$ and $j = 0, 1, 2$,*

$$\|\widetilde{E}(\kappa)\varphi\|_{H^{s+2-j}(\Omega_a)} \le C|\kappa|^{1-j} e^{T|\operatorname{Im}\kappa|} \|\varphi\|_{H^s(\Omega_a)}. \tag{3.2}$$

Proof. By Lemma 3.2 and the closed graph theorem the mapping from $L_a^2(\Omega)$ to $u(t, x) \in C^\infty(D)$, where $D = \{(t, x) \in R \times \overline{\Omega}: t > T_a, |x| < t - T_a + a\}$, is continuous. So for any j, β, and $D_1 \Subset D$ there exists a constant $C = C(j, \beta, D_1)$ so that

$$\sup_{D_1} |(\partial_x^\beta \partial_t^j u)(t, x)| \le C\|\varphi\|_{L^2(\Omega_a)}. \tag{3.3}$$

By $\partial_t(E\varphi)(0, x) = \varphi$ and integration by parts it follows that

$$(L - \kappa^2)\widetilde{E}\varphi = -\varphi + \int_0^\infty f(t, x)e^{i\kappa t}\, dt, \tag{3.4}$$

where $f = [\partial_t^2 + L, h]u$ belongs to $C^\infty(R \times \overline{\Omega})$ from (3.3). Making use of the properties of $U(t)$, the continuous inclusion $H_a^1(\Omega) \subset H^1(\Omega_a)$, and the elliptic estimate for $L(u(t, \cdot)) = -(\partial_t^2 u)(t, x)$, we can prove that $\partial^j E/\partial t^j \in C(R; \mathscr{B}(H_a^s(\Omega), H^{s+1-j}(\Omega_a)))$, where $s = 0, 1$ and $0 \le j \le s + 1$. Now the estimate (3.2) is derived from the above property of E, (3.3), the elliptic estimate for (3.4), and $\widetilde{E}\varphi = i\kappa^{-1}\int_0^\infty e^{i\kappa t}\partial_t(E\varphi)\, dt$. The proof of Lemma 3.4 is completed.

From Lemma 3.2 we can extend $u(t, x)$ near $D_2 = [T_a + 1, T_a + 2] \times \{x \in R^n: |x| < a\}$ to be an element of $C^\infty(D_2)$. Here we may assume that the extended function $\tilde{u}(t, x)$ also satisfies (3.3) for any compact subset D_1 of $\{(t, x) \in R \times \overline{\Omega}): t > T_a, |x| < t - T_a + a\} \cup D_2$. Let us consider the solution of the wave equation $(\partial_t^2 - \Delta)w = -\tilde{f}$ in R^n, $\partial_t^j w = 0$ on $t = 0$ $(j = 0, 1)$, where $\tilde{f} = [\partial_t^2 + L, h]\tilde{u} \in C^\infty(R^{n+1})$. Put $\widetilde{V}(\kappa)\varphi = \int_0^\infty e^{i\kappa t}w(t, x)\, dt$. Then we have the following:

Lemma 3.5. *$\widetilde{V}(\kappa)$ is an analytic function of K in $D = \{\kappa \in C \setminus \{0\}: = \pi/2 < \arg\kappa < 3\pi/2\}$ with values in $\mathscr{B}(L_a^2(R^n), H^s(\{x: |x|, a\}))$, where $L_a^2(R^n) = L^2(R^n) \cap \mathscr{E}'(\{|x| < a\})$ and s is arbitrary integer. Moreover there exist constants C_s and T so that for any $\kappa \in D$ there exist constants C_s and T so that for any $\kappa \in D$*

$$\|\widetilde{V}(\kappa)\varphi\|_{H^s(\{|x|<a\})} \le C_s|\kappa|^{-2}e^{T|\operatorname{Im}\kappa|}\|\varphi\|_{L^2(R^n)}. \tag{3.5}$$

Proof. Let $\psi(x) \in C^\infty(R^n)$ such that $\psi(x) = 0$ if $|x| < a_1$ $(a_1 > a)$, $\psi(x) = 1$ if $|x| > A_1$, and $L = -\Delta$ in supp ψ. Put $v(t, x) = \psi E\varphi + w$. Then v satisfies the Cauchy problem $(\partial_t^2 - \Delta)v = \psi f - \tilde{f} - [\psi, \Delta]E\varphi$ in $[0, \infty) \times R^n$, $\partial_t^j v = 0$ on $t = 0$, $(j = 0, 1)$. Here the support of $\psi f - \tilde{f} - [\psi, \Delta]E\varphi$ is contained in $\{(t, x): |x| < M_0, 0 < t < T_0\}$ for some M_0 and T_0. From the finite progagation property of v it follows that $(\partial_t^2 - \Delta)v = 0$ in $[T_0, \infty) \times R^n$, $\partial_t^j v = \varphi_j$ on $t = T_0$ $(j = 0, 1)$, where the supports of φ_j are contained in $\{|x| < M\}$. Making use of the inequality $\sup_{t\in[0,T]} \|\partial_t^j w(t, \cdot)\|_{H^s(R^n)} \le C_{T,j,s}\|\varphi\|_{L^2(\Omega)}$ which is derived from (3.3) and a strictly hyperbolic estimate for w, and the continuous inclusion $H_a^1(\Omega) \subset H^1(\Omega_a)$, we see that $\|\varphi_j\|_{L^2(R^n)} \le C\|\varphi\|_{L^2(\Omega)}$ $(j = 0, 1)$. If n is odd, by the existence of a lacuna for v, $w(t, x) = 0$ for $t > T_1$ and $|x| < a$. If n is even, $v = d(E_x(t) * \varphi_0)/dt + E_x(t) * \varphi_1$, where,

$$E_x(t) * \varphi_1(2\pi)^{-p}\left(\frac{1}{t}\frac{d}{dt}\right)^{p-1}\int_{|y|\le t} \varphi_1(x - y)/(t^2 - |y|^2)^{1/2}\, dy$$

with $p = n/2$. If $t > T_1 = a + M$ and $|x| < a$, then

$$\int_{|y|\le t} \varphi_1(x-y)/(t^2-|y|^2)^{1/2}\,dy = \int_{|y|\le M} \varphi_1(y)/(t^2-|x-y|^2)^{1/2}\,dy.$$

It follows that for $\operatorname{Re} t > T_1$, $|x| < a$, $w(t, x)$ is infinitely differentiable jointly with respect to t, x is analytic in t and satisfies $\sup|\partial_t^j \partial_x^\alpha w| \le C_{j,\alpha} t^{-3}\|\varphi\|_{L^2(\Omega)}$, where the supremum is taken in $|x| < a$. Thus by the argument in the proof of Lemma 6 of [6] we have the desired properties on $\widetilde{V}(\kappa)$. The proof is completed.

Let $\psi_1(x) \in C^\infty(R^n)$ such that $\psi_1(x) = 1$ for $|x| > a$ and $\psi_1(x) = 0$ near $\partial\Omega$, and put $\widetilde{W}(\kappa)\varphi = \widetilde{E}(\kappa)\varphi + \psi_1\widetilde{V}(\kappa)\varphi$. Then $(L-\kappa^2)\widetilde{W}\varphi = -(I+T(\kappa))\varphi$ and $B\widetilde{W}(\kappa)\varphi = 0$, where $T(\kappa)\varphi = -(L+\Delta)\widetilde{V}\varphi - (L-\kappa^2)(\psi_1 - 1)\widetilde{V}\varphi \in L_a^2(\Omega)$. From (3.2) and (3.5) there exist positive constants α, β such that $\|T(\kappa)\|_{\mathscr{B}(L_a^2(\Omega), L_a^2(\Omega))} \le 1/2$ for $\kappa \in U_{\alpha,\beta}$. Put $\widetilde{R}(\kappa) = -\widetilde{W}(\kappa)(I+T(\kappa))^{-1}$, which satisfies (3.1). The final problem to complete the proof of Theorem 3.3 is to show the equality $R^+(\kappa)\varphi = \widetilde{R}(\kappa)\varphi$ for $\varphi \in L_a^2(\Omega)$. In order to prove this we may show the following:

Lemma 3.6. *There exists a positive constant A such that for any $\varphi \in L_a^2(\Omega)$ and κ with $\operatorname{Im}\kappa > A$, $\widetilde{R}(\kappa)\varphi \in H^2(\Omega)$.*

Proof. Let $\psi_1(x)$ be the function that appeared in the definition of $\widetilde{W}(\kappa)$ and put $v_1(t, x) = \psi_1(E\varphi + w)$, which satisfies $(\partial_t^2 + L)v_1 = F(t, x)$, where $F = \psi_1(\partial_t^2 + L)(E\varphi + w) + \psi_1(\Delta - L)w$. From the properties of $U(t)$ and a strictly hyperbolic estimate for w that appeared in the proof of Lemma 3.5 it follows that for all $t \ge 0$ $\|F(t, \cdot)\|_{L^2(R^n)} \le C_\varphi$. By the proof of Theorem 3.8 in [8], $u(t, x) = [U(t)\varphi]_1(x) = \sum_{j=1}^m \lambda_j^{-1}(\varphi, p_j)_{\mathscr{H}}(e^{\lambda_j t} - e^{-\lambda_j t})p_j/2 + L_+^{-1/2}\sin tL_+^{1/2}\varphi_1$, where $\{-\lambda_j^2\}$ are negative eigenvalues of L, $\{p_j\}$ are linearly independent eigenvectors of $\{-\lambda_j^2\}$, $L_+ = \int_0^\infty \lambda\, dE(\lambda)$ with the spectral resolution $\{E(\lambda)\}$ of L, and $\varphi_1 = \varphi - \sum_{j=1}^m (\varphi, p_j)_{\mathscr{H}} p_j$. Thus we have $v_1 \in C(R; H^1(R^n))$ and $\partial_t v_1 \in C(R; L^2(R^n))$. Making use of an estimate for hyperbolic operator on v_1, we see that there exist $A > 0$ and $\gamma > 0$ so that $e^{-tA}\|v_1(t, \cdot)\|_{H^1(R^n)} \le C\{\|\psi_1\varphi\|_{L^2(R^n)} + \int_0^t e^{-s\gamma}\|F(s, \cdot)\|_{L^2(R^n)}\,ds\}$. Thus $\widetilde{R}(\kappa)\varphi \in H^1(\Omega)$ if $\operatorname{Im}\kappa > A$. Since $\widetilde{R}(\kappa) \in H^1_{\mathrm{loc}}(\overline{\Omega})$, from the Fourier transform and $(\Delta - \kappa^2)\widetilde{R}(\kappa)\varphi \in L^2(\Omega)$ it follows that $\widetilde{R}(\kappa)\varphi \in H^2(\Omega)$. The proof is completed.

Proof of Theorem 1.2. We use the argument in the proof of Theorem 8 in [6]. The solution $u(t, x)$ of (1.5) is equal to $\int_{i\gamma-\infty}^{i\gamma+\infty} R^+(\kappa)\varphi e^{-i\kappa t}\,d\kappa/2\pi$, where $(f_1, f_2) = (0, \varphi)$ with $\varphi \in D(L) \cap L_a^2(\Omega)$ and γ is sufficiently large. We note that the integral converges from the equality $R^+(\kappa)\varphi = \kappa^{-2}(R^+(\kappa)L\varphi - \varphi)$. By the unique continuation property of solutions to $(L - \kappa^2)u = 0$, where $\kappa \in R$, and Theorem 4.7(a) in [9] it follows that any point in $R \setminus \{0\}$ is an analytic point of $R^+(\kappa)$. By deforming the contour of the integration, we see that $u(t, x) = i\sum_{j=1}^m \operatorname{Res}[R^+(\kappa)\varphi e^{-\kappa t}; i\lambda_j] - \int_\Gamma R^+(\kappa)\varphi e^{-i\kappa t}\,d\kappa$, where t is sufficiently large and Γ is the curve shown in Figures 1 or 2 in [6]. From $R^+(\kappa) = (L - \kappa^2)^{-1}$ for $0 < \arg\kappa < \pi$ it is not difficult to show that $i\operatorname{Res}[R^+(\kappa)\varphi e^{-i\kappa t}; i\lambda_j] = e^{\lambda_j t}\sum_l (\varphi, p_l)_{\mathscr{H}} p_l/2\lambda_j$, where $\{p_l\}$ is a base of the

eigenvector space of the negative eigenvalues $-\lambda_j^2$ of L. From Theorems 1.1, 3.3, and Theorem 8 and Lemma 9 of [6] we have (1.6) for $f = (0, \varphi)$ with $\varphi \in D(L) \cap L_a^2(\Omega)$. We note that the solution $u(t, x)$ of (1.5) is equal to the first component of $\partial_t U(t)^t(0, f_1) + U(t)^t(0, f_2)$. Moreover if $E_n = \varnothing$ and $\operatorname{Ker} A = \{0\}$, then we can show that $\|f\|_{H_D}^2$ is equivalent to $E(f)$. We complete the proof of Theorem 1.2.

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