LIPSCOMB'S $L(A)$ SPACE FRACTALIZED IN HILBERT'S $l^2(A)$ SPACE

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Abstract. By extending adjacent-endpoint identification in Cantor's space $N((0, 1))$ to Baire's space $N(A)$, we move from the unit interval $I = L\{0, 1\}$ to $L(A)$. The metric spaces $L(A)^{n+1}$ and $L(A)^\infty$ have provided nonseparable analogues of Nöbeling's and Urysohn's imbedding theorems. To date, however, $L(A)$ has no metric description. Here, we imbed $L(A)$ in $l^2(A)$ and the induced metric yields a geometrical interpretation of $L(A)$. Except for the small last section, we are concerned with the imbedding. Once inside $l^2(A)$, we see $L(A)$ as a subspace of a "closed simplex" $\Delta^A$ having the standard basis vectors together with the origin as vertices. The part of $L(A)$ in each $n$-dimensional face $\omega^n$ of $\Delta^A$ is a "generalized Sierpiński Triangle" called an $n$-web $\omega^n$. Topologically, $\omega^n$ is $L\{0, 1, \ldots, n\}$. For $n = 2$, $\omega^2$ is just the usual Sierpiński Triangle in $E^2$; for $n = 3$, $\omega^3$ is Mandelbrot's fractal skewed web. Thus, $L(A) \rightarrow l^2(A)$ invites an extension of fractals. That is, when $|A|$ infinite, Baire's Space $N(A)$ is a "generalized code space" on $|A|$ symbols that addresses the points of the "generalized fractal" $L(A)$.

1. Notation and definitions

For any set $A$, let $N(A)$ be the set of all sequences of elements of $A$. For $a = a_1a_2\cdots$ and $b = b_1b_2\cdots$ in $N(A)$, define

$$\rho(a, b) = \frac{1}{\min\{k|a_k \neq b_k\}}.$$ 

Then $N(A)$ with this metric $\rho$ is a generalized Baire's 0-dimensional space. Topologically, $N(A)$ is the countable product of discrete space $A$ [11, p. 51]. In particular, when $A = \{0, 1\}$ we have the mappings

$$a_1a_2\cdots \rightarrow \sum_{i=1}^{\infty} \frac{2a_i}{3^i} \rightarrow \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$ 

That is, there is the topological correspondence ("\leftrightarrow" in (1)) between $N(A)$ and Cantor's middle-thirds space $\mathcal{C}(0, 1)$; also, there is an identification map

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("\rightarrow\) in (1)) onto the unit interval \(I\)—the so-called identification of adjacent endpoints.

The equivalence, "\(\rightarrow\)" in (1), reveals the endpoints of Cantor's space as eventually constant sequences of 0's and 1's. Generalizing this observation to arbitrary \(A\), endpoints of \(N(A)\) are defined [4] as the eventually constant sequences in \(A\). If an endpoint is a constant sequence, then its tail-index is defined as zero. Otherwise, the tail-index of an endpoint \(a_1a_2\cdots\) is the unique index \(j\) such that \(a_j \neq a_{j+1} = a_{j+2} = a_{j+3} = \cdots\). Two distinct endpoints \(a\) and \(b\) of \(N(A)\) are adjacent when \(a\) and \(b\) have the same tail-index \(j \geq 1\); \(a_k = b_k\) for each \(k < j\); \(a_j = b_{j+1} = b_{j+2} = \cdots\); and \(b_j = a_{j+1} = a_{j+2} = \cdots\).

A string of repeated \(a\)'s is denoted \(\overline{a}\), e.g., \(a_1a_2aa\cdots = \overline{a_1a_2}\).

For \(a, b \in N(A)\) define \(a \sim b\) when either \(a = b\) or \(a \neq b\) are adjacent endpoints. An induced equivalence class \(\alpha\) is either a singleton or a doubleton subset of \(N(A)\). Any member \(a = a_1a_2\cdots\) of \(\alpha\) is an expansion of \(\alpha\). Thus, each \(\alpha\) has at most two expansions. The space \(L(A)\) is the quotient space \(N(A)/\sim\) and the natural map \(p: N(A) \rightarrow L(A)\) is perfect [4], i.e., \(p\) is a continuous closed surjection with compact point-inverse sets. It follows that \(L(A)\) is metric [2, p. 235; 9; 15] and that \(L(A)\) is one-dimensional [4] (the dimension is the covering dimension [11]). For applications of \(L(A)\), [5–7, 12] contains nonseparable versions of Nöbeling's and Urysohn's imbedding theorems [14, 16].

A doubleton set \(\alpha \in L(A)\) is a rational point in \(L(A)\). All other points of \(L(A)\) are irrational. The \(p\)-image of any constant sequence is an endpoint of \(L(A)\). The points of \(L(A)\) can also be partitioned [7] into those of finite character and those of infinite character; That is, \(\alpha\) has finite character only if some expansion \(a_1a_2\cdots\) of \(\alpha\) has only finitely many symbols—\(\{a_1, a_2, \ldots\}\) is finite. Every rational point of \(L(A)\) has finite character while irrational points appear in both varieties.

Let \(\bigcup\{I(a) | a \in A\}\) be a system of unit segments \([0, 1]\). By identifying all zeros in \(\bigcup\{I(a) | a \in A\}\) we get a star-shaped set \(S(A)\). Defining a metric \(d_S\) in \(S(A)\) by

\[
d_S(x, y) = \begin{cases} |x-y| & \text{if } x, y \text{ belong to the same segment } I(a), \\ x+y & \text{if } x, y \text{ belong to distinct segments,} \end{cases}
\]

we obtain a metric space called star-shaped with index \(A\). The space \(S(A)\) was used by Nagata [13] and Kowalsky [3] to construct general imbedding theorems [7].

Turning to Hilbert's Space, for any set \(A\) let \(E^A\) be the cartesian product of \(|A|\) copies of the real line \(E^1\). The metric space \(l^2(A)\) is that having (1) elements: every \(x = \{x_a\} \in E^A\) such that \(x_a = 0\) for all but at most countably many \(a \in A\) and \(\sum x_a^2\) converges; and (2) topology: that induced by the metric \(d(x, y) = \sqrt{\sum (x_a - y_a)^2}\).

Generalized Euclidean space \(\mathbb{R}^A\) is the subspace of \(l^2(A)\) consisting of those \(\{x_a\}\) with \(x_a = 0\) for all but at most finitely many \(a \in A\). If \(\mathcal{V}(A)\) denotes the set containing all the standard basis vectors \(u_a, a \in A\), and the zero vector \(0\), then the infinite-dimensional simplex \(\Delta^A\) (in \(l^2(A)\)) is the convex hull (an intersection of closed convex sets) of \(\mathcal{V}(A)\).

If \(A = \{0, 1, \ldots, n\}\), then, as we shall see from the imbedding (given below) of \(L(A)\) into \(l^2(A)\), the (one-dimensional when \(n > 0\)) space \(p(N(A)) = L(A)\).
LIPSCOMB'S $L(A)$ SPACE

is a closed subspace of the $n$-simplex $\sigma^n_0$. In this case, the vertices of $\sigma^n_0$ are $u_i = (1, 0, \ldots, 0), \ldots, u_n = (0, \ldots, 0, 1)$ and $\emptyset = (0, 0, \ldots, 0)$. For $A = \{0, 1, \ldots, n\}$ an $n$-web $\omega^n$ is either the subspace $L(A)$ of $\sigma^n_0$ or any image of $L(A)$ in any $n$-simplex $\sigma^n$ under a linear homeomorphism [10] from $\sigma^n_0$ onto $\sigma^n$. Any simplicial complex $K$ induces a web complex; for every $n$ replace each $\sigma^n$ in $K$ with its subspace $\omega^n$.

The term $n$-web is introduced here but was motivated by the title $A$ Fractal Skewed Web of plate 143 of Mandelbrot's book [8, p. 142]. The term web complex is introduced to increase the understanding of the dense subspace $M = \bigcup_n M_n$ of $L(A)$ where $M_0 \subset M_1 \subset \cdots$. Indeed, each $M_n$ is the web complex induced from the $n$-skeleton of the (possibly infinite-dimensional) simplex $\Delta^A$. One can also view $M_n$ as those points $\alpha$ in $L(A)$ whose expansions contain at most $n$ symbols of $A$. Thus, each $M_n$ contains both rational and irrational points of $L(A)$, and $M$ is the set of points of $L(A)$ that have finite character.

2. Projecting $N(A)$ onto Cantor subspaces

Let $z \in A$ be fixed and define $A' = A - \{z\}$. For each element $b$ of $A'$, $N(A)$ has a subspace $\mathcal{E}(z, b)$ of sequences whose values lie in $\{z, b\}$: $\mathcal{E}(z, b)$ is a copy of Cantor's space. For each $b \in A'$ define a continuous projection $\pi_b: N(A) \to \mathcal{E}(z, b)$ as follows:

$$\pi_b: a_1 a_2 \cdots \mapsto a_1^b a_2^b \cdots \quad \text{where } a_k^b = \begin{cases} b & \text{if } b = a_k, \\ z & \text{otherwise}. \end{cases}$$

The map $\pi_b$ is an open map since basic open sets $\langle a_1, \ldots, a_n \rangle = \{a_1\} \times \cdots \times \{a_n\} \times A \times A \times \cdots$ are mapped to open sets $\pi_b(\langle a_1, \ldots, a_n \rangle) = \langle a_1^b, \ldots, a_n^b \rangle \subset \mathcal{E}(z, b)$.

The map $\pi_b$ is continuous since the $\pi_b$-inverse image of $\langle x_1, \ldots, x_n \rangle \subset \mathcal{E}(z, b)$ is an open set of the form

$$\left\langle \bigtimes_j (A - \{b\}), \times_k \{b\}_k \right\rangle,$$

where $J \subset \{1, 2, \ldots, n\}$ is the set of indices $j$ for which $x_j \neq b$ and $K$ is the other set of indices, i.e., those for which $x_k = b$. However, when $A$ is infinite, then $N(A)$ is not compact and the projection $\pi_b$ is not a closed map: Let $\{a_1, a_2, \ldots\}$ be an infinite subset of $A$. Then the set

$$(2) \quad F = \{a_1 b a_1, a_2 a_2 b a_2, a_3 a_3 a_3 b a_3, \ldots\}$$

is closed in $N(A)$ while $\pi_b(F)$ is not closed in $\mathcal{E}(z, b)$. These observations are summarized in the following theorem.

Theorem 1. The projection $\pi_b: N(A) \to \mathcal{E}(z, b)$ is a continuous open map. Also, $\pi_b$ is not closed $\iff A$ is infinite.

3. Projecting $L(A)$ onto unit interval subspaces

Again, let $z \in A$ be fixed and define $A' = A - \{z\}$. Call $p(\overline{z}) = \zeta$ the zero of $L(A)$. Let $\beta$ be any other endpoint of $L(A)$, i.e., $\beta = p(\overline{b})$ for some
b ∈ A'. Using projection πₜ and identification p, we can induce a projection within L(A) that takes L(A) onto the subspace \( p(\mathcal{C}(z, b)) = I(\zeta, \beta) \) of L(A). Diagram (3) is illustrative: \( p_b \) is the restriction of \( p \), the induced projection is \( \pi_\beta \), and the subspace \( I(\zeta, \beta) \) of L(A) is a unit interval.

\[
\begin{array}{ccc}
N(A) & \xrightarrow{\pi_b} & \mathcal{C}(z, b) \\
\downarrow p & & \downarrow p_b \\
L(A) & \xrightarrow{\pi_\beta} & I(\zeta, \beta).
\end{array}
\]

Since \( \pi_b \) maps adjacent endpoints in \( N(A) \) onto adjacent endpoints in \( \mathcal{C}(z, b) \), \( p_b \circ \pi_b \) is constant on the fibers of \( p \). Thus, \( \pi_\beta \) is well defined. Because the diagram (3) is commutative and the map \( p \) is closed, \( \pi_\beta \) pulls closed sets back to closed sets and is therefore continuous.

**Lemma 2.** The projection \( \pi_\beta : L(A) \to I(\zeta, \beta) \) is a continuous map. Also, \( \pi_\beta \) is not closed \( \iff \) \( A \) is infinite.

**Proof.** Only the characterization of \( \pi_\beta \) being not closed remains. On the one hand, if \( F \) is the closed subset of \( N(A) \) defined by (2) then \( p(F) \) is a closed subset of \( L(A) \). It follows that

\[
F' = p^{-1}p(F) = F \cup \{a_1a_1b, a_2a_2b, a_3a_3b, \ldots\}
\]

is closed in \( N(A) \). But \( p_b(\pi_b(F')) = \pi_\beta(p(F)) \) is not closed in \( I(\zeta, \beta) \). On the other hand, if \( A \) is finite then \( N(A) \) is compact. In this case, \( \pi_b \) is closed, and it follows that the \( \pi_\beta \) pushes closed subsets of \( L(A) \) to closed sets. \( \square \)

**Lemma 3.** The projection \( \pi_\beta : L(A) \to I(\zeta, \beta) \) is not open \( \iff \) \( A \) has at least three members.

**Proof.** (\( \Leftarrow \)) Let \( B = (b, z) \) be a basic open set in \( N(A) \) where \( b \neq z \). Let \( E_B \) be the set of points \( a \) in \( B \) such that \( a_3 = a_4 = \ldots \). Then \( H = B - E_B \) is an open set in \( N(A) \) and \( p^{-1}p(H) = H \). Thus \( p(H) \) is open in \( L(A) \). However, since \( A \) has at least three members,

\[
\pi_b(H) = (b, z) - \{bzb\}
\]

and \( bzb \in \pi_b(H) \). But then \( p_b \circ \pi_b(p(H)) \) is the half-closed interval

\[ [p_b(bzb), p_b(bzb)]. \]

(\( \Rightarrow \)) Since \( \zeta \neq \beta \) and \( A \) has two members, the projection is a homeomorphism. \( \square \)

The following theorem is the analogue of Theorem 1 and is a combination of Lemmas 2, 3.

**Theorem 4.** The projection \( \pi_\beta : L(A) \to L(\zeta, \beta) \) is a continuous map. Also, \( \pi_\beta \) is not closed \( \iff \) \( A \) is infinite. Furthermore, \( \pi_\beta \) is not open \( \iff \) \( A \) has at least three members.

If \( \pi_\beta(\alpha) \neq \zeta \) then we say that \( \alpha \) has a nonzero projection into \( I(\zeta, \beta) \). The following lemma is immediate.
Lemma 5. Each member of $L(A)$ has a nonzero projection into at most a countable number of the subspaces $I(\zeta, \beta)$.

4. The Star in $L(A)$ and $l^2(A)$

The subspace $\bigcup_{\beta} I(\zeta, \beta)$ of $L(A)$ is homeomorphic to a star-space $S(A)$ with index set $A$ [7] whenever $A$ is infinite. There is also a copy of $S(A)$ in Hilbert’s $l^2(A)$ space: Let $u_b$, $b \in A$, denote the unit vectors in $l^2(A)$, i.e.,

$$u_b = \{x_a\} \quad \text{where} \quad x_b = 1 \quad \text{and} \quad x_a = 0 \quad \text{otherwise}.$$  

Then for $b \in A'$ the subspace

$$[0, u_b] = \{t u_b | 0 \leq t \leq 1\}$$

of $l^2(A)$ is a copy of the unit interval. Furthermore, when $A$ is infinite, $S(A)$ is homeomorphic to the metric subspace $\bigcup_{b \in A'} [0, u_b]$ of $l^2(A)$.

The next lemma exhibits a homeomorphism $\psi_\beta$ from the “$\beta$th arm,” $I(\zeta, \beta)$ of the star in $L(A)$ to the unit interval. The induced homeomorphism $\alpha \mapsto \psi_\beta(\alpha) u_b$ from the $\beta$th arm of the star in $L(A)$ to the $b$th arm of the star in $l^2(A)$ is fundamental to the embedding of $L(A)$ into $l^2(A)$.

Lemma 6. If we identify $z$ with 0 and $b$ with 1, then we can induce a homeomorphism $\psi_b: \mathcal{C}(z, b) \to \mathcal{C}(0, 1)$. Then, from $\psi_b$ and the identifications ($p_b$ and $p_1$) of adjacent endpoints, we induce a unique topological correspondence $\psi_\beta$ from $I(\zeta, \beta)$ onto the unit interval $[0, 1]$ that makes the following diagram commute.

$$\mathcal{C}(z, b) \xrightarrow{\psi_b} \mathcal{C}(0, 1)$$

$$\downarrow p_b \quad \downarrow p_1$$

$$I(\zeta, \beta) \xrightarrow{\psi_\beta} [0, 1].$$

For each $\alpha \in L(A)$ and each $b \in A'$, define

$$\alpha^b = \psi_\beta \circ \pi_\beta(\alpha).$$

Also, for $b = z$ define $\alpha^z = 0$. Combining diagrams (3) and (4), we see $0 \leq \alpha^b \leq 1$ for each $b \in A$ and, from Lemma 5, $0 < \alpha^b$ for at most a countable number of $b \in A$. We can say more:

Lemma 7. For each $\alpha \in L(A)$ we can choose a binary expansion $x_1^b x_2^b \cdots$ of each $\alpha^b$ such that for each subscript $i \in \{1, 2, \ldots\}$ there is at most one $b \in A$ with $x_i^b = 1$. From this it follows that

$$\sum_{b \in A} \alpha^b \leq 1.$$  

Proof. For each $\alpha \in L(A)$ choose $\phi(\alpha) = a_1 a_2 \cdots \in \alpha$. Then for $b \neq z$ use the choice function $\phi$ to calculate

$$\alpha^b = \psi_\beta \circ \pi_\beta(\alpha) = p_1 \circ \psi_b \circ \pi_b(a_1 a_2 \cdots )$$

$$= p_1 \circ \psi_b(a_1^b a_2^b \cdots ) = \sum_{i=1}^{\infty} \frac{x_i^b}{2^i},$$
where
\[
    x_i^b = 1 \iff a_i^b = b \iff a_i = b.
\]
Now let \( i \in \{1, 2, \ldots \} \) be fixed. Then \( a_i \) is a fixed member of \( A \). Thus, 
\( a_i = b \) for exactly one member \( b \in A \). But then, via (6), 
\( x_i^b = 1 \) for exactly one \( b \in A \). To see that the inequality in (5) holds, note that for 
\( i = 1 \) there is at most one of \( b \in A \) such that \( x_1^b = 1 \), i.e., at most one of the \( a_i^b \)'s contribute 
1/2 to the sum in (5). An induction argument finishes the proof. \( \square \)

**Corollary 8.** For each \( \alpha \in L(A) \),
\[
    \sum_{b \in A} (\alpha_{b})^2 \leq 1.
\]

5. The imbedding map

For each point \( \alpha \in L(A) \), map \( \alpha \) to the point \( f(\alpha) = \{\alpha_b\} \in l^2(A) \). Corollary 8 above shows that \( f \) is into \( l^2(A) \).

**Lemma 9.** The map \( f: L(A) \to l^2(A) \) is injective.

**Proof.** Let \( \alpha \) and \( \gamma \) be distinct members of \( L(A) \) and suppose that \( a \in \alpha \) and 
\( c \in \gamma \). It follows that
\[
    a \neq c \quad \text{and} \quad a \sim c.
\]
From the first condition in (7), we can choose \( j \) as the smallest index such that
\[
    a = a_j \neq c_j = c.
\]
(Without loss of generality, we assume that \( a \neq z \).) Now evaluate the projection
\( \pi_a \) at the points \( a \) and \( c \). It follows from (8) that
\[
    \pi_a(a) \neq \pi_a(c).
\]
Statement (9) coupled with the fact that \( \psi_a \) is a homeomorphism show that
either \( f(\alpha) \neq f(\gamma) \) (in which case we are finished) or
\[
    \pi_a(a) \text{ is adjacent to } \pi_a(c).
\]
Thus, for the rest of this proof we assume (10) to be true. Now from (8) and the definition of \( j \), both points in (10) have tail-index \( j \). Furthermore, (10) and \( (\pi_a(a))_j = a \neq z \), imply \( a = c_{j+1} = c_{j+2} = \cdots \). If \( c \neq z \) then similarly we conclude 
\( c = a_{j+1} = a_{j+2} = \cdots \). But then \( a \sim c \), which contradicts the second 
condition in (7). Thus, only the case \( c = z \) remains: In this case, we have
\[
    c = a_1 \cdots a_{j-1} za \cdots \quad \text{and} \quad a = a_1 \cdots a_{j-1} aa_{j+1}a_{j+2} \cdots.
\]
From this representation of \( c \), and the second condition in (7), there is an index
\( k > j \) such that \( a_k \neq z \). If \( a_k = a \) then we contradict (10). Thus, for this
index \( k \) we can suppose \( a \neq a_k \neq z \). Then
\[
    \pi_{a_k}(a) \neq \pi_{a_k}(c) \quad \text{and} \quad \pi_{a_k}(a) \sim \pi_{a_k}(c).
\]
Therefore, in this final case, and consequently in every case, we have \( f(\alpha) \neq f(\gamma) \). \( \square \)
6. THE EMBEDDING MAP FACTORED THROUGH \( N(A) \)

To see that the embedding map \( f \) is both continuous and open, we first combine diagrams (3) and (4). That is, for each \( b \in A' \), define \( g_b \) as the diagonal map.

\[
\begin{array}{ccccccccccc}
N(A) & \xrightarrow{\pi_b} & \mathbb{C}(z,b) & \xrightarrow{\psi_b} & \mathbb{C}(0,1) \\
\downarrow p & & \downarrow g_b & & \downarrow p_1 \\
L(A) & \xrightarrow{\pi_\beta} & I(\zeta,\beta) & \xrightarrow{\psi_\beta} & [0,1]
\end{array}
\]

(11)

Then define \( g : N(A) \to l^2(A) \) as

\[
g(a) = \{g_b(a)\},
\]

where \( g_z \) is the zero map. From the definition of \( f \) we have

\[
g(a) = f \circ p(a) = \{\psi_\beta \circ \pi_\beta \circ p(a)\}.
\]

That is, diagram (12) is commutative \( (g = f \circ p) \).

\[
\begin{array}{ccccccccccc}
N(A) & \xrightarrow{g} & l^2(A) \\
\downarrow p & & \downarrow f \\
L(A)
\end{array}
\]

(12)

Since \( f \) is injective and \( g = f \circ p \), we calculate that \( g \) and \( p \) have the same fibers,

\[
g^{-1}(f(\alpha)) = p^{-1}(f^{-1}(f(\alpha))) = p^{-1}(\alpha).
\]

Lemma 10. The map \( g : N(A) \to l^2(A) \) is continuous.

Proof. It suffices to show that if \( a^n \to a \) in \( N(A) \), then

\[
d(g(a^n), g(a)) \to 0.
\]

Let \( k > 0 \) be given and define \( A_k = \{a_1, \ldots, a_k\} \), i.e., \( A_k \) contains the first \( k \) coordinates of \( a \), \( A'_k \) the other coordinates of \( a \), and \( A''_k \) the complementary set, relative to the coordinates of \( a^n \), of the first \( k \) coordinates of \( a^n \).

Since \( a^n \to a \), there is an \( N \) such that whenever \( n > N \),

\[
a_i^n = a_i \quad \text{for } 1 \leq i \leq k.
\]

Then, when \( n > N \),

\[
(d(g(a^n), g(a)))^2 = \sum_{b \in A} |g_b(a^n) - g_b(a)|^2 
\]

\[
\leq \sum_{b \in A_k} |\cdots| + \sum_{b \in A'_k} |\cdots| + \sum_{b \in A''_k} |\cdots| 
\]

\[
\leq \frac{k}{2^k} + \frac{1}{2^k} + \frac{1}{2^k} = \frac{k + 2}{2^k}.
\]

\[\square\]

Lemma 11. The map \( f : L(A) \to l^2(A) \) is continuous.

Proof. Since \( g : N(A) \to l^2(A) \) is continuous, \( g \) pulls back open sets in \( l^2(A) \) to open \( g\)-inverse sets in \( N(A) \). But a subset of \( N(A) \) is a \( g\)-inverse set \( \Rightarrow \)
it is a $p$-inverse set. Since $p$ is a quotient map, $p$ maps open $p$-inverse sets to open sets in $L(A)$. The continuity of $f$ follows from the fact that for any subset $S$ of $L^2(A)$, $f^{-1}(S) = p(g^{-1}(S))$. □

**Lemma 12.** The map $f^{-1} : f(L(A)) \rightarrow L(A)$ is continuous.

**Proof.** Suppose $g(a^n) \rightarrow g(a)$ in $L^2(A)$. Then define sets

$$R = g^{-1}g(a) \quad \text{and} \quad R_n = g^{-1}g(a^n) \quad (n = 1, 2, \ldots).$$

Because $g$ and $p$ have the same fibers and because $p$ is a closed map (open $p$-inverse sets form a local base at $R$), it suffices to show that $R_n \rightarrow R$, i.e., for any open set $G$ with $R \subseteq G$ there is an $N$ such that when $n > N$ then $R_n \subseteq G$. Suppose this is not the case. Then there is an infinite subset $M$ of $N$ and a sequence $\{r^n\} \subseteq R$ none of which are members of $G$. There are two possibilities: Case I. The sequence $\{r^n\}$ has a convergent subsequence, the limit of which must be a point $p \notin R$. But this would contradict the fact that $g$ is continuous. Thus, this case is impossible. Case II. The sequence $\{r^n\}$ has no convergent subsequence. In this case there is some $i \geq 1$ such that the $i$th components $r^m_i$, $m \in M$, of the members of $\{r^n\}$ form an infinite set. (Otherwise, we could obtain a subsequence of $\{r^n\}$ that converges.) It follows that $\{r^n\}$ has a subsequence $\{s^k\}$ whose $i$th components form an infinite set containing neither $z$ nor any of the first $i + 1$ components of the (at most two) members of $R$. But then, using the notation (11), for any $k$ and any $b = s^k_i$,

$$1/2^{i+1} \leq |g_b(s^k) - g_b(a)| \leq d(g(s^k), g(a)).$$

This contradicts $g(a^n) \rightarrow g(a)$. □

The following theorem is a combination of Lemmas 9, 11, and 12.

**Theorem 13.** The map $\alpha \mapsto \{\alpha^b\}$ is a homeomorphism from $L(A)$ into $L^2(A)$.

### 7. The web complexes

The imbedding $\alpha \mapsto \{\alpha^b\}$ maps the endpoints of $L(A)$ to the origin and unit basis vectors while expansions of points of $L(A)$ convey position analogous to the binary expansions of points in $I$.

Fractals appear: Web complexes $M_0 \subset M_1 \subset \cdots$ exist where $M_n$ is the union of all webs in the web complex induced from the $n$-skeleton of the (possibly infinite-dimensional) simplex $\Delta^A$. One can also think of $M_n$ as those points $\alpha$ in $L(A)$ with some expansion $a_1a_2\cdots$ where the set $\{a_1, a_2, \ldots\}$ has at most $n + 1$ members.

The union $M$ of the webs in $\bigcup_n M_n$ is a dense subspace of $L(A)$ since it contains all rational points of $L(A)$. Inside $L^2(A)$ each rational point of $L(A)$ has only finitely many nonzero coordinates, i.e., each rational point is actually in the subspace $E^A$ of $L^2(A)$. Indeed, on the other hand, since $M$ consists of those points $\alpha$ in $L(A)$ of finite character, we see, inside $L^2(A)$, that $M \subset E^A$. On the other hand, since $M' = L(A) - M$ consists of those points in $L(A)$ with infinite character, we see $M' \subset (L^2(A) - E^A)$. This subspace $M'$ is also dense in $L(A) = M \cup M'$.  

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