ZERO-DIMENSIONALITY IN COMMUTATIVE RINGS

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Abstract. If \( \{ R_a \}_{a \in A} \) is a family of zero-dimensional subrings of a commutative ring \( T \), we show that \( \bigcap_{a \in A} R_a \) is also zero-dimensional. Thus, if \( R \) is a subring of a zero-dimensional subring \( T \) (a condition that is satisfied if and only if a power of \( rT \) is idempotent for each \( r \in R \)), then there exists a unique minimal zero-dimensional subring \( R^0 \) of \( T \) containing \( R \). We investigate properties of \( R^0 \) as an \( R \)-algebra, and we show that \( R^0 \) is unique, up to \( R \)-isomorphism, only if \( R \) itself is zero-dimensional.

1. Introduction

All rings considered in this paper are assumed to be commutative and unitary. If \( R \) is a subring of a ring \( S \), we assume that the unity element of \( S \) belongs to \( R \), and hence is the unity of \( R \). All allusions to the dimension of a ring refer to its Krull dimension; hence a zero-dimensional ring is one in which each proper prime ideal is maximal.

Arapovic in [A2] (see also [H, §3]) considered the problem of imbeddability of a ring in a zero-dimensional ring. The main result of [A2] is Theorem 7. We cite this result below, and in the remainder of this paper, we refer to it, for short, as AIT.

Arapovic’s Imbeddability Theorem. A commutative unitary ring \( R \) is imbeddable as a subring of a zero-dimensional ring if and only if there exists a family \( \{ Q_\lambda \}_{\lambda \in \Lambda} \) of primary ideals of \( R \) satisfying the following conditions (1) and (2):

1. \( \bigcap_{\lambda \in \Lambda} Q_\lambda = (0) \).
2. For each \( a \in R \), there exists a positive integer \( n_a \) such that \( a^{n_a} \notin \bigcup_{\lambda}(P_\lambda - Q_\lambda) \), where \( P_\lambda = \text{rad}(Q_\lambda) \) is the prime ideal associated with \( Q_\lambda \).

Given a family \( \{ Q_\lambda \} \) satisfying (1) and (2), a first step in the proof of the “if” part of AIT is the imbedding of \( R \) in \( T = \prod_{\lambda \in \Lambda}(R/Q_\lambda)(P_\lambda/Q_\lambda) \), a product of zero-dimensional quasilocal rings. In this connection we have considered several problems related to AIT, imbeddability, direct products of rings, and the family of Artinian subrings of a commutative ring [GH1, GH2, GH3].
current paper also has close connections with AIT, with its main focus being the families of zero-dimensional subrings and zero-dimensional extension rings of a given ring $R$. The main results of the paper are

1. Theorem 2.4, which shows that an arbitrary intersection of zero-dimensional subrings of a given ring is also zero-dimensional;

2. Theorem 3.1, which (a) provides equivalent conditions for a subring $R$ of a ring $T$ to be contained in a zero-dimensional subring of $T$ and (b) exhibits a set of $R$-algebra generators for the unique minimal zero-dimensional subring $R^0$ of $T$ containing $R$ in the case where the conditions of (a) are satisfied;

3. Theorem 3.3, which describes properties of $R^0$ as an algebra over $R$; and

4. Theorem 4.1, which shows that if $R$ is imbeddable in a zero-dimensional ring, then $R^0$ is unique up to $R$-isomorphism only if $R$ itself is zero-dimensional.

In §4, besides the question of uniqueness, we also investigate certain minimal zero-dimensional extension rings of $R$ that arise from canonical imbeddings of $R$ in a zero-dimensional ring via AIT.

Let $\{R_\alpha\}_{\alpha\in A}$ be a family of zero-dimensional subrings of a ring $T$ and let $R = \bigcap_{\alpha\in A} R_\alpha$. We show in Theorem 2.4 that $R$ is zero-dimensional. Consequently, $R$ is von Neumann regular if some $R_\alpha$ is von Neumann regular, and $R$ is Artinian if $R$ is reduced and some $R_\alpha$ is Artinian. Lemmas 2.1 and 2.3 are the key results in the proof of Theorem 2.4. A proof of Lemma 2.1 can easily be obtained from [O] or from [Gl], but because of the importance of Lemma 2.1 in the subsequent development and the brevity of its proof, a proof is included here.

**Lemma 2.1** ([O, Lemma 2; Gl, Lemma 4.3.9]). Suppose $x$ and $z$ are elements of the ring $R$ such that $x = x^2z$. If $y = z^2x$, then $x = x^2y$ and $y = y^2x$. Moreover, if $w \in R$ is such that $x = x^2w$ and $w = w^2x$, then $w = y$.

**Proof.** To prove the first assertions, note that $x^2y = x^3z^2 = x(x^2z)z = xxz = x^2z = x$ and $y^2x = z^4x^3 = z^3(zx^2)x = z^3x^2 = z^2 \cdot zx^2 = z^2x = y$. For the statement concerning uniqueness, we have $y^2w^2x^3 = y^2 \cdot w^2x \cdot x^2 = y^2 \cdot w \cdot x^2 = y^2 \cdot x = y$, and a similar argument shows that $y^2w^2x^3 = w$.

Suppose $x \in R$, a ring. If there exists an element $y \in R$ such that $x = x^2y$ and $y = y^2x$ (that is, if $xR$ is idempotent), then following [Gl, p. 137], we call $y$ the **pointwise inverse** of $x$. We note that in this case $x$ and $y$ generate the same ideal of $R$, and $xy$ is the unique idempotent generator of that ideal.

**Lemma 2.2.** Suppose $R$ is a subring of the ring $S$. If $e$ is an idempotent element of $R$, then the ideal $eR$ is contracted from $S$.

**Proof.** It suffices to show that $eS \cap R \subseteq eR$. Thus, suppose $r = es \in eS \cap R$, where $s \in S$. Then $er = es = r \in eR$, as desired.
The hypothesis that $R$ is unitary was not used in the proof of Lemma 2.1 or 2.2, but the unitary hypothesis is used in the proof of the next result.

**Lemma 2.3.** Suppose $R$ is a subring of the ring $S$ and $x \in R$ is such that $x^nR = x^{n+1}R$ for some $n \in \mathbb{Z}^+$. Then $x^nS = x^{n+1}S$, and if $m \in \mathbb{Z}^+$ is chosen minimal so that $x^mS = x^{m+1}S$, then $x^mR = x^{m+1}R$.

**Proof.** It is clear that $x^nS = x^{n+1}S$; if $m = n$, the statement that $x^mR = x^{m+1}R$ is also clear. If $m < n$, then to prove that $x^mR = x^{m+1}R$, it suffices to show that $x^m \in x^nR$. Let $e$ be the idempotent generator for $x^nR$. Then $x^mS = eS$, so $x^m \in eS \cap R$, and Lemma 2.2 shows that $eS \cap R = eR = x^nR$.

**Theorem 2.4.** Suppose $\{R_\alpha\}_{\alpha \in A}$ is a family of zero-dimensional subrings of the ring $T$.

(a) $T = \bigcap_{\alpha \in A} R_\alpha$ is zero-dimensional.
(b) $|\text{Spec}(R)| \leq |\text{Spec}(R_\alpha)|$ for each $\alpha$.
(c) If $x \in R$ and if $x$ is a unit in some $R_\alpha$, then $x$ is a unit of each $R_\alpha$, and hence is a unit of $R$.

**Proof.** (a) To show that $T$ is zero-dimensional, it suffices to show that for each $r \in R$, some power of $rR$ is idempotent [W, Proposition 4.2; M, Proposition 2.3; H, Theorem 3.1]. Thus, take $r \in R$ and take $\alpha \in A$. Some power of $rR_\alpha$ is idempotent since $R_\alpha$ is zero-dimensional, so some power of $rT$ is idempotent. Choose $m \in \mathbb{Z}^+$ minimal so that $r^mT$ is idempotent. Then Lemma 2.3 shows that $r^mR_\beta$ is idempotent for each $\beta \in A$. Let $x = r^m$. By Lemma 2.1, $x$ has a pointwise inverse $y_\beta \in R_\beta$. Moreover, since each $R_\beta$ is a subring of $T$, Lemma 2.1 also shows that $y_\beta = y_\gamma$ for all $\beta, \gamma \in A$. Hence if $y = y_\beta$, then $y \in R$ and $x = x^2y$, so the ideal $xR = r^mR$ is idempotent. This completes the proof of (a).

Statement (b) follows from the fact that minimal primes are contracted from an arbitrary extension ring.

(c) Suppose $x$ is a unit of $R_\beta$ and a nonunit of $R_\alpha$. Then $x$ belongs to a maximal ideal $M$ of $R_\alpha$, and hence $x$ belongs to $M \cap R$, a minimal prime of $R$. It follows [K, Theorem 84] that $x$ is a zero divisor of $R$, contrary to the assumption that $x$ is a unit of $R_\beta$. We conclude that $x$ is a unit of each $R_\alpha$ and of $R$, as asserted.

**Corollary 2.5.** Suppose $R$ is a subring of a zero-dimensional subring of a ring $T$. Then there exists a unique minimal zero-dimensional subring $R^0$ of $T$ containing $R$.

**Proof.** We take $R^0$ to be the intersection of the family of all zero-dimensional subring of $T$ containing $R$. Theorem 2.4(a) shows that $R^0$ is zero-dimensional, and it is clear that $R^0$ is the unique minimal zero-dimensional subring of $T$ containing $R$.

We note that in the case where $T$ is zero-dimensional, Corollary 2.5 is the same as Theorem 7 of [A1]. We will continue to denote the minimal zero-dimensional $T$-overring of $R$, when it exists, by $R^0$, but we observe in §4 that the ring $R^0$ is not determined up to isomorphism by $R$ alone; $R^0$ also depends upon $T$, so notation such as $R^0(T)$ would more accurately reflect the situation that exists. In §4 we consider what we call canonical minimal zero-dimensional...
extensions of a ring $R$ that is imbeddable in a zero-dimensional ring. These extensions are determined by a certain type of representation of the zero ideal of $R$ as an intersection of primary ideals. We learned of the next result from Wiegand; this result was also known to Popescu and Vraciu [PV, p. 271].

**Corollary 2.6.** If $\{R_\alpha\}_{\alpha \in A}$ is a nonempty family of von Neumann regular subrings of a ring $S$, then $R = \bigcap_{\alpha \in A} R_\alpha$ is also von Neumann regular.

**Proof.** Theorem 2.4 shows that $R$ is zero-dimensional, and clearly $R$ is reduced. Hence $R$ is von Neumann regular [K, Exercise 12, p. 63; G. Exercise 16, p. 111].

**Remark 2.7.** If $R$ is a ring, Olivier in [O] shows that there exists a Von Neumann regular ring $R^*$ and a homomorphism $f : R \to R^*$ such that if $g : R \to S$ is any homomorphism with $S$ von Neumann regular, then there exists a unique homomorphism $g^* : R \to S$ such that $g = g^* \circ f$. The ring $R^*$ is called the regular ring associated to $R$. Popescu and Vraciu in [PV, §3] observe that $R^*$ can be realized in the form of a minimal zero-dimensional extension ring as follows. Let Spec$(R) = \{P_i\}_{i \in I}$ and let $K_i$ be the quotient field of $R/P_i$. Let $f_i$ be the canonical map from $R$ onto $R/P_i$, and let $f : R \to T = \prod_{i \in I} K_i$ denote the product of the maps $f_i$; we note that $f$ is an imbedding of $R$ if and only if $R$ is reduced. Popescu and Vraciu show that $R^* \cong (f(R))^0$, the minimal zero-dimensional subring of $T$ containing $f(R)$.

The proof of the next result follows easily from Corollary 2.6.

**Corollary 2.8.** Suppose $\{R_\alpha\}_{\alpha \in A}$ is a family of zero-dimensional subrings of a reduced ring $S$ and let $R = \bigcap_{\alpha \in A} R_\alpha$. If some $R_\alpha$ is Artinian, then $R$ is also Artinian.

**Proof.** The ring $R$ is von Neumann regular by Corollary 2.6. If $R_\alpha$ is Artinian, then $R_\alpha$ contains only finitely many idempotents, [ZS, Chapter IV, §3] and hence $R$ has only finitely many idempotents. Therefore $R$ is a finite direct sum of fields, and $R$ is Artinian.

We remark that, in general, the intersection of two or more Artinian subrings of a ring $S$ need not be Artinian [GH3, §4].

3. The algebra structure of a minimal zero-dimensional extension

Suppose $R$ is a subring of a ring $T$. Theorem 3.1 gives equivalent conditions for $R$ to admit a zero-dimensional $T$-overring, and if these conditions are satisfied, exhibits a set of generators for $R^0$ as an algebra over $R$. Theorem 3.3 gives some properties of $R^0$ as an algebra over $R$.

**Theorem 3.1.** Suppose $R$ is a subring of a ring $T$.

1. If $R$ admits a zero-dimensional $T$-overring, then a power of $rT$ is idempotent for each $r$ in $R$.

2. Conversely, if a power of $rT$ is idempotent for each $r \in R$, then $R$ admits a zero-dimensional $T$-overring. In fact, if for $r$ in $R$, $m(r)$ is the smallest positive integer $k$ such that $r^kT$ is idempotent and $t_r$ is the pointwise inverse of $r^{m(r)}$ in $T$, then $R[\{t_r : r \in R\}]$ is the unique minimal zero-dimensional $T$-overring of $R$. 


Proof. (1) If $S$ is a zero-dimensional $T$-overring of $R$, then a power of $rS$, and hence of $rT$, is idempotent for each $r$ in $R$.

(2) Let $S = R[\{t_r : r \in R\}]$. To prove that $S$ is zero-dimensional, let $P$ be a proper ideal of $S$, let $\bar{R} = (R+P)/P \cong R/(P \cap R)$, let $\bar{S} = S/P$, and for $s \in S$, let $\bar{s} = s + P$. We note that $\bar{S} = \bar{R}[\{t_r : r \in R\}]$. If $\bar{r}$ is a nonzero element of $\bar{R}$, where $r \in R$, and if $m = m(r)$, then $\bar{r}^m = \bar{t}_r^{2m}$, and since $\bar{S}$ is an integral domain, $\bar{1} = \bar{r}^{m(r)}$. It follows that $\bar{r}$ is a unit of $\bar{S}$, and since $\bar{r}$ is an arbitrary nonzero element of $\bar{R}$, the quotient field $K$ of $\bar{R}$ is a subfield of $\bar{S}$. On the other hand, if $\bar{t}_a$ is a nonzero element of $\bar{S}$, the equation $\bar{t}_a(\bar{t}_a\bar{u}(a) - \bar{1}) = 0$ implies that $\bar{t}_a = (\bar{a}^{m(a)})^{-1} \in K$, so $\bar{S} = \bar{R}[[\{t_r\}] \subseteq K$ and equality holds: $\bar{S} = K$. This proves that $S/P$ is a field, so $S$ is zero-dimensional. If $U$ is any zero-dimensional $T$-overring of $R$ and if $r \in R$, then Lemma 2.3 shows that $r^{m(r)}U$ is idempotent. Hence $r^{m(r)}$ has a pointwise inverse $u_r$ in $U$, and by uniqueness of pointwise inverses in $T$, $t_r = u_r \in U$. We conclude that $S = R[\{t_r\}] \subseteq U$, and this completes the proof of Theorem 3.1.

Proposition 3.2. Suppose $R$ is a subring of a ring $T$. Assume that $\{e_i\}_{i=1}^n$ is a set of mutually orthogonal nonzero idempotents of $R$ with sum 1. Let $R_i = Re_i$ and $T_i = Te_i$ for each $i$, so that $R = R_1 \oplus \cdots \oplus R_n$ and $T = T_1 \oplus \cdots \oplus T_n$. Then $R$ admits a zero-dimensional $T$-overring if and only if $R_i$ admits a zero-dimensional $T_i$-overring for each $i$; moreover, if these conditions are satisfied, then the minimal zero-dimensional $T$-overring $R^0$ of $R$ is $R_1^0 \oplus \cdots \oplus R_n^0$.

Proof. If $S$ is a zero-dimensional $T$-overring of $R$, then $Se_i$ is a zero-dimensional $T_e_i$-overring of $Re_i$ for each $i$, and conversely, if $S_i$ is a $T_e_i$-overring of $Re_i$ for each $i$, then $S_1 \oplus \cdots \oplus S_n$ is a zero-dimensional $T$-overring of $R$. This establishes the existence assertion in Proposition 3.2. If $R$ admits a zero-dimensional $T$-overring, then inclusion $R^0 \subseteq R_1^0 \oplus \cdots \oplus R_n^0$ is clear, and the reverse inclusion holds because $R^0 = R_1^0e_1 \oplus \cdots \oplus R_n^0e_n$, where $R_1^0e_1$ is a zero-dimensional $T_1$-overring of $R_1$.

Theorem 3.3. Suppose $R$ is a subring of a zero-dimensional subring of a ring $T$ and let $R^0$ be the minimal zero-dimensional $T$-overring of $R$.

(a) If $P$ is a prime ideal of $R^0$, then $R^0/P$ is the quotient field of its subring $(R + P)/P \cong R/(P \cap R)$.

(b) ([A1, Proposition 8]) A primary ideal of $R^0$ is uniquely determined by its contraction to $R$.

(c) $|\text{Spec}(R^0)| \leq |\text{Spec}(R)|$; in particular, $\text{Spec}(R^0)$ is finite if $\text{Spec}(R)$ is finite. If $H \in \text{Spec}(R)$, then $H$ is contracted from $R^0$ if and only if $H$ is contracted from $T$.

(d) If $\text{Spec}(R)$ is finite, then $R^0$ is a finitely generated $R$-algebra.

Proof. Throughout the proof we use the notation $m(r)$ and $t_r$, for $r \in R$, as in the statement of (2) of Theorem 3.1; thus $R^0 = R[\{t_r : r \in R\}]$. The assertion in (a) is established in the course of the proof of Theorem 3.1. To prove (b), let $Q_1$ and $Q_2$ be primary ideals of $R^0$ with the same contraction $Q$ to $R$. Let $P_1 = \text{rad}(Q_1)$ and let $P = \text{rad}(Q)$; we have $P = P_1 \cap R$. For $r \in R$, we first show that $t_r \in Q_1$ if and only if $t_r \in Q_2$. Thus, assume that $t_r \in Q_1$. Since $r^{m(r)}$ and $t_r$ generate the same ideal of $R^0$, it follows that $r \in P_1 \cap R = P$, and hence $r \in P_2$. We have $t_r(1 - t_r r^{m(r)}) = 0 \in Q_2$ and $1 - t_r r^{m(r)} \notin P_2$, so...
\[ t_r \in Q_2 \]. Therefore \( Q_1 \) and \( Q_2 \) contain exactly the same elements \( t_r \) for \( r \in R \).

Now consider an arbitrary element \( w \) of \( Q_1 \). We have \( w \in R[t_{r_1}, \ldots, t_{r_s}] \) for some finite subset \( \{r_j\}_{j=1}^s \) of \( R \). Thus \( w = \sum_{i=1}^k a_i t_{r_{i_1}} \cdots t_{r_{i_t}} \), where each \( a_i \in R \). We wish to show that \( w \in Q_2 \), and because \( Q_1 \) and \( Q_2 \) contain exactly the same elements \( t_r \), we may assume without loss of generality that each \( t_r \) belongs to neither \( Q_1 \) nor \( Q_2 \). To simplify the notation, let \( b_i = r^{m(r)} \) and \( u_i = t_{r_i} \) for each \( i \). Since \( b_i \) and \( u_i \) generate the same ideal of \( R^0 \), no \( b_i \) is in \( Q_1 \) or \( Q_2 \). The equation \( b_i(1 - u_i b_i) = 0 \) then implies that \( 1 - u_i b_i \in P_1 \cap P_2 \), and since \( u_i b_i \) and \( 1 - u_i b_i \) are idempotent, it follows that \( 1 - u_i b_i \in Q_1 \cap Q_2 \). Let \( e_i = \max\{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\} \) for \( 1 \leq i \leq s \) and let \( b = b_1^{e_1} \cdots b_s^{e_s} \). Since \( b_i u_i \equiv 1 (\mod Q_1 \cap Q_2) \) for each \( i \), we have \( bw = \sum_{i=1}^k a_i b_i ^{e_{i_1}} u_i^{e_{i_2}} \cdots b_s^{e_s} u_s^{e_{s_2}} \equiv c = \sum_{i=1}^k a_i b_i ^{e_{i_1} - e_{i_2}} \cdots b_s^{e_s - e_{s_2}} \) (mod \( Q_1 \cap Q_2 \)). Now \( w \in Q_1 \), so \( bw \equiv 0 (\mod Q_1) \), and hence \( c \in Q_1 \cap R = Q \). Consequently, \( c \in Q_2 \), \( bw \in Q_2 \), and thus \( w \in Q_2 \) since \( b \notin P_2 \). We have proved that \( Q_1 \subseteq Q_2 \), and by symmetry, it follows that \( Q_1 = Q_2 \).

The first assertion in (c) follows immediately from (b). For the second, it is clear that \( H \) is contracted from \( R^0 \) if it is contracted from \( T \), and the converse holds because each prime of \( R^0 \) is contracted from \( T \).

(d) If \( \text{Spec}(R) \) is finite, then so is \( \text{Spec}(R^0) \), and hence the set of idempotents of \( R^0 \) is finite. Express \( R^0 \) as a direct sum \( R^0 e_1 \oplus \cdots \oplus R^0 e_n \), where each \( e_i \) is a nonzero idempotent, each \( R^0 e_i \) is indecomposable as a ring, and hence each \( R^0 e_i \) is quasilocal. Replacing \( R \) by \( R[e_1, \ldots, e_n] = Re_1 \oplus \cdots \oplus Re_n \), we see that in view of Proposition 3.2, it suffices to prove (d) under the assumption that \( R^0 \) is quasilocal. In this case let \( M \) be the maximal ideal of \( R^0 \) and let \( P = M \cap R \). Then \( P \) is the unique minimal prime of \( R \). If \( x \in R \) is chosen in each prime ideal of \( R \) except \( P \), then \( x \) is a unit of \( R^0 \) and \( R[x^{-1}] \) is a zero-dimensional subring of \( R^0 \). By definition of \( R^0 \), it follows that \( R^0 = R[x^{-1}] \), so \( R^0 \) is finitely generated over \( R \), as we wished to show.

Remark 3.4. In connection with part (b) of Theorem 3.3, we remark that if \( Q \) is a primary ideal of \( R^0 \), then \( Q \) need not be the extension to \( R^0 \) of \( Q \cap R \). In fact, if \( P \) is a prime ideal of \( R^0 \), \( P \) need not be contained in \( \text{rad}(P \cap R) \). For example, let \( (R, M) \) be a one-dimensional local domain and let \( T = K \oplus (R/M) \), where \( K \) is the quotient field of \( R \) and where \( R \) is imbedded in \( T \) by the inclusion map on the first component and the natural homomorphism of \( R \) onto \( R/M \) in the second component. If \( m \in M \), \( m \neq 0 \), then the pointwise inverse of \( m \) in \( T \) is \((m^{-1}, \bar{0})\) and this element is in \( R^0 \). If \( x \) is any element of \( K \), then \( x \) is of the form \( y/n \), where \( y \in R \) and \( n \in M \). Hence \((x, \bar{0}) = (y, \bar{0})(n^{-1}, \bar{0}) \in R^0 \), and from this it follows easily that \( R^0 = T \). Let \( P \) be the prime ideal \(((0), R/M) \) of \( T \). Then \( P \cap R = (0) \), and \( P \notin \text{rad}((0)R^0) = (0) \).

Remark 3.5. Let the notation and hypothesis be as in the statement of Theorem 3.3. In the case where \( T \) is zero-dimensional, Arapovic proves that \( R^0 \) is the total quotient ring of an integral extension of \( R \) [A1, Theorem 6]. The proof of Theorem 3.3(d) implicitly proves this result in the case where \( \text{Spec}(R) \) is finite; we can see that the result holds in general (without the assumption that \( T \) is zero-dimensional) as follows: for \( r \in R \), the element \( f_r = r^{m(r)} t_r \) of \( R^0 \) is idempotent, so \( S = R[\{f_r : r \in R\}] \) is an integral extension of \( R \). The element
s_r = r^{m(r)} + (1 - f_r) of S belongs to no maximal ideal of R^0 because r^{m(r)} and f_r generate the same ideal of R^0, and hence s_r is a unit of R^0. It follows that S[[s_r^{-1} : r \in R]] \subseteq R^0 = R[[t_r : r \in R]], but since t_r = f_r/s_r, the reverse inclusion also holds. Thus R^0 = S[[s_r^{-1} : r \in R]] is the total quotient ring of S. (Since S[[s_r^{-1}]] is zero-dimensional, it is equal to its own total quotient ring.)

4. Uniqueness of minimal zero-dimensional extensions

and some canonical minimal extensions

related to Arapovic's Imbeddability Theorem

If the ring R is imbeddable in a zero-dimensional ring, then R admits a minimal zero-dimensional extension R^0. Theorem 4.1 shows, however, that R^0 is uniquely determined up to R-isomorphism only if R is zero-dimensional, and hence R = R^0. Proposition 4.4 deals with calculation of the ring R^0 in case the zero ideal of R is a finite intersection of primary ideals. After proving Proposition 4.4, we turn our attention to some canonical minimal extensions of R that arise from AIT. (Recall that AIT designates the theorem of Arapovic cited in the introduction.) These extensions are determined by a family of primary ideals of R satisfying conditions (1) and (2) of AIT; in general there may be many such families in R, and this is another factor that lends credence to the frequent lack of uniqueness of R^0.

Theorem 4.1. Suppose the ring R is imbeddable in a zero-dimensional ring S and that R^0 is unique up to R-isomorphism.

(1) If Q is a P-primary ideal of R, then there exists a prime ideal P* of R^0 and a P*-primary ideal Q* of R^0 such that P* and Q* lie over P and Q, respectively.

(2) If P is a prime ideal of R, then there exists a unique minimal P-primary ideal of R.

(3) R is zero-dimensional.

Proof. (1) Let T be the zero-dimensional ring S ⊕ R_P/Q_R_P. The ideal I = S ⊕ (0) of T is primary for the ideal J = S ⊕ P_R_P/Q_R_P. Moreover, if R is considered as a subring of T via the diagonal imbedding, then P and Q are the contractions to R of J and I, respectively. Let U be the minimal zero-dimensional T-overring of R. Then I ∩ U is J ∩ U-primary in U, and these ideals lie over Q and P in R. Because of uniqueness of R^0 up to R-isomorphism, the conclusion of (1) then follows.

(2) Let P* be a prime ideal of R^0 lying over P in R. Since R^0 is zero-dimensional, the P*-primary component Q* of (0) in R^0 is the unique minimal P*-primary ideal of R^0. Let Q = R ∩ Q*. We claim that each P-primary ideal Q_0 of R contains Q. Thus, by (1), there exists a P*-primary ideal Q_0* of R^0 lying over Q_0 in R. Since Q_0* contains Q*, it follows that Q_0 contains Q, as claimed.

(3) Suppose that dim(R) > 0, and choose prime ideals P_0, P_2 of R with P_0 < P_2. Choose x ∈ P_2 - P_0, let P be a minimal prime ideal of P_0 + (x), and let P_1 be a prime of R that contains P_0, is contained in P, and is maximal with respect to failure to contain x. Then P_1 < P, and there is no prime ideal of R properly between these two. Thus [G, (17.4)], the intersection of the set of
P-primary ideals of $R$ that contain $P_1$ is $P_1$. Therefore if $Q$ is the minimal $P$-primary ideal of $R$, then $Q$ is contained in $P_1$, and this contradicts the fact that $\text{rad}(Q) = P$. This completes the proof.

**Remark 4.2.** If a ring $R$ is a subring of a zero-dimensional quasilocal ring $(T, N)$, then $N \cap R = P$ is the unique minimal prime of $R$. Moreover, if $x \in R - P$, then $x$ is a unit of $T$ and hence is a regular element of $R$. Therefore the total quotient ring of $R$ is the localization $R_P$, and $R_P$ is the minimal zero-dimensional subring of $T$ containing $R$.

We note that Remark 4.2 is the special case of the next result where $n = 1$.

**Proposition 4.3.** Suppose $n$ is a positive integer, $T$ is a zero-dimensional ring with exactly $n$ minimal prime ideals, and $R$ is a subring of $T$ with exactly $n$ minimal primes. Then the total quotient ring of $R$ is the minimal zero-dimensional extension of $R$ in $T$.

**Proof.** Let $\{P_i\}_{i=1}^n$ be the set of minimal primes of $T$. Since each minimal prime of $R$ is contracted from $T$, it follows that $\{P_i \cap R\}_{i=1}^n$ is the set of (distinct) minimal primes of $R$. Let $S$ be any zero-dimensional $T$-overring of $R$. Then $\{P_i \cap S\}_{i=1}^n$ is the set of prime ideals of $S$ by the argument just given, and these ideals are distinct since their contractions to $R$ are distinct. If $r \in R - (\bigcup_{i=1}^n (P_i \cap R))$, then $r \in S - (\bigcup_{i=1}^n (P_i \cap S))$, and hence $r$ is a unit of $S$. Therefore $r$ is a regular element of $R$ and $\bigcup_{i=1}^n (P_i \cap R)$ is the set of zero divisors of $R$. Thus the total quotient ring of $R$ is zero-dimensional and is contained in $S$. This completes the proof.

**Proposition 4.4.** Let $R$ be a ring in which $(0)$ has a finite primary decomposition. Let $(0) = Q_1 \cap \cdots \cap Q_n$ be a shortest primary decomposition of $(0)$, where $Q_i$ is a $P_i$-primary ideal in $R$. Consider the ring $T = (R_{P_1}/Q_1 R_{P_1}) \oplus \cdots \oplus (R_{P_n}/Q_n R_{P_n})$ and let $\phi: R \to T$ be the canonical imbedding of $R$ in $T$. Then any proper subring of $T$ containing $\phi(R)$ is of positive dimension, and hence $\phi(R)^0 = T$.

**Proof.** We may identify $R$ with $\phi(R)$ and regard $R$ as a subring of $T$. Let $S$ be a zero-dimensional $T$-overring of $R$. Because the zero ideal of $T$ admits a finite primary decomposition, the same is true for the zero ideal of $S$, and any such primary decomposition of $(0)$ in $S$ contracts to $R$ to give a primary decomposition of $(0)$ in $R$. Therefore $S$ must have at least $n$ prime ideals. It follows that the map from $\text{Spec}(T)$ to $\text{Spec}(S)$ induced by the inclusion map $S \subseteq T$ is a bijection, and $S$ has exactly $n$ prime ideals. Let $W_i$ denote the prime ideal of $T = (R_{P_1}/Q_1 R_{P_1}) \oplus \cdots \oplus (R_{P_n}/Q_n R_{P_n})$ consisting of all tuples $(t_1, \ldots, t_n)$ for which $t_i = 0$. Note that $T_{W_i} \cong R_{P_i}/Q_i R_{P_i}$. Let $\phi_i$ denote the canonical map of $R$ into $R_{P_i}/Q_i R_{P_i} \cong T_{W_i}$. By permutability of localization and residue class ring formation, we have $R_{P_i}/Q_i R_{P_i} \cong (R/\{Q_i\}_P)^{P_i}$ and $\phi_i(R) = R/\{Q_i\}_P \subseteq T_{W_i}$. Moreover, if $W_i \cap S = U_i$, then $R/\{Q_i\}_P \subseteq S_{U_i} \subseteq T_{W_i}$. Since $T_{W_i}$ is the total quotient ring of $R/\{Q_i\}_P$ and since $S_{U_i}$ is zero-dimensional, we have $S_{U_i} = T_{W_i}$. Because this is true for each prime $U_i$ of $S$, it follows that $S = T$.

The proof of Theorem 4.1 uses the fact that the isomorphisms of minimal zero-dimensional extensions under consideration are $R$-algebra isomorphisms.
With additional hypothesis on \( R \), we show in Proposition 4.5 that \( R \) admits two minimal zero-dimensional extensions that are not isomorphic as rings; in particular, the hypothesis of Proposition 4.5 is satisfied if \( R \) is Noetherian.

**Proposition 4.5.** Suppose \( R \) is a ring of positive dimension and assume that \((0)\) in \( R \) is a finite intersection of primary ideals. Let \((0) = Q_1 \cap \cdots \cap Q_n\) be a shortest primary decomposition of \((0)\) in \( R \), where rad\((Q_i)\) = \( P_i \), \( i = 1, \ldots, n \). Assume that either

1. there exists \( P \in \text{Spec}(R) - \{P_i\}_{i=1}^n \), or
2. each of the primary ideals \( Q_i \) contains a power of its radical \( P_i \).

Then there exist two minimal zero-dimensional extensions of \( R \) that are not isomorphic as rings.

**Proof.** If \( T_i = R_{P_i}/Q_i R_{P_i} \oplus \cdots \oplus R_{P_n}/Q_n R_{P_n} \), then Proposition 4.4 shows that \( T_i \) is minimal zero-dimensional extension of \( R \); moreover, \( T_i \) has exactly \( n \) prime ideals, and these primes lie over \( P_1, \ldots, P_n \) in \( R \). Suppose there exists \( P \in \text{Spec}(R) - \{P_i\}_{i=1}^n \), let \( K \) be the quotient field of \( R/P \), and let \( T \) be the zero-dimensional ring \( T_i \oplus K \). If \( R \) is considered as a subring of \( T \) under the diagonal imbedding, then each of the ideals \( P, P_1, \ldots, P_n \) is contracted from \( T \), and hence, by Theorem 3.3(c), from the minimal zero-dimensional extension \( R^0 \) of \( R \) in \( T \). Consequently, \( |\text{Spec}(R^0)| > n \) and \( R^0 \) is not isomorphic to \( T_i \).

Suppose \( \text{Spec}(R) = \{P_i\}_{i=1}^n \). If \( P_i^{k_i} \subseteq Q_i \), and if \( k \) is the maximum of the integers \( k_1, \ldots, k_n \), it follows that \( T_i \) has the property that each primary ideal of \( T_i \) contains the \( k \)th power of its radical; we exhibit a minimal zero-dimensional extension of \( R \) that does not share this property of \( T_i \). Thus, assume that the labeling is such that \( P_1 \) is maximal in \( R \) and \( P_1 > P_2 \). Choose \( x \in P_1 - \bigcup_{i=2}^n P_i \). Then \( xR \) is \( P_1 \)-primary. The powers of \( xR \) properly descend, for an equality \((x^t/R = (x^{t+1})R \) would lead to an equation \( x^t(1-rx) = 0 \), where neither \( x^t \) nor \( 1-rx \) is in \( P_2 \). If \( t > k \), then \( x^t \in Q_1 \), and it follows that \((0) = x^t R \cap Q_2 \cap \cdots \cap Q_n \) is a shortest primary decomposition of \((0)\). If \( T_2 \) is the minimal zero-dimensional extension of \( R \) arising from this decomposition as in Proposition 4.4, then \( x^tT_2 \) is a primary ideal of \( T_2 \) that does not contain the \( k \)th power of its radical.

We next turn our attention to what we call *canonical minimal zero-dimensional extensions* of a ring that arise from AIT. Thus, suppose \( R \) is imbeddable in a zero-dimensional ring. Then there exists a family \( \{Q_i\} \) of primary ideals of \( R \) satisfying conditions (1) and (2) of AIT. Now \( R \) is canonically imbedded in \( T = \prod_{\lambda}(R/Q_i)_{(P_i/Q_i)} \), and condition (2) of AIT implies that a power of \( rT \) is idempotent for each \( r \) in \( R \). Thus \( T \) contains a minimal zero-dimensional extension of \( R \), and we refer to a minimal zero-dimensional extension of \( R \) that arises in this way as a *canonical extension*. If \( S \) is any minimal zero-dimensional extension of \( R \), then \((0) \) in \( S \) is an intersection of a family of primary ideals of \( S \). The contraction to \( R \) of this family of primary ideals of \( S \) is a family \( \{Q_i\} \) of primary ideals of \( R \) satisfying conditions (1) and (2) of AIT. We remark that the canonical minimal zero-dimensional extension of \( R \) associated to the family \( \{Q_i\} \) is \( R \)-isomorphic to \( S \). In general, canonical extensions of \( R \) are not unique, but Remark 4.6 does allow us to make one significant simplifying assumption in regard to the family \( \{Q_i\} \).
Remark 4.6. If \( \{Q_\alpha\}_{\alpha \in A} \) is a family of \( P \)-primary ideals of a ring \( R \) and if \( Q = \bigcap_{\alpha \in A} Q_\alpha \), it is straightforward to show that \( Q \) is \( P \)-primary if and only if \( P \subseteq \text{rad}(Q) \). We observe that if \( \{Q_\alpha\}_{\alpha \in A} \) is a subfamily of a family \( \{Q_\lambda\}_{\lambda \in \Lambda} \) of primary ideals satisfying condition (2) of AIT, the \( Q_\alpha \) all having the same associated prime \( P \), then the inclusion \( P \subseteq \text{rad}(Q) \) holds. To see this, take \( a \in P \) and choose \( n_a \in \mathbb{Z}^+ \) so that \( a^{n_a} \notin \bigcup_{\lambda \in \Lambda} (P_\lambda - Q_\lambda) \); then \( a^{n_a} \in Q = \bigcap_{\alpha \in A} Q_\alpha \), so \( a \in \text{rad}(Q) \) as desired. This observation means that in considering families \( \{Q_\lambda\}_{\lambda \in \Lambda} \) of primary ideals satisfying conditions (1) and (2) of AIT, there is no loss of generality in assuming that \( \text{rad}(Q_\lambda) \neq \text{rad}(Q_\mu) \) for \( \lambda, \mu \in \Lambda \) and \( \lambda \neq \mu \).

Proposition 4.7. Assume that \( R \) is a ring imbeddable in a zero-dimensional ring, and let \( \mathcal{F} = \{Q_\lambda\}_{\lambda \in \Lambda} \) be a family of primary ideals of \( R \) satisfying conditions (1) and (2) of AIT. Let \( P_x = \text{rad}(Q_x) \) and consider \( R \) as a subring of \( T = \prod_{\lambda \in \Lambda} (R/\mathcal{Q}_\lambda)(P_x/Q_x) \) via the canonical imbedding. If \( \{P_x\}_{x \in \mathcal{X}} \) is infinite, then \( R^0 \) contains infinitely many idempotents.

Proof. By definition, if \( a \in R \), then the idempotent generator \( e_a \) of the ideal \( aT \) is the tuple whose \( \lambda \)-th coordinate is 1 or 0, according as \( a \) is not, or is, in \( Q_\lambda \). Moreover, \( e_a \in \mathcal{F} \) by Theorem 3.1(2). Let \( n \in \mathbb{Z}^+ \), let \( \{P_1, P_2, \ldots, P_n\} \) be a set of \( n \) distinct elements of \( \{P_x\} \), and assume that the ordering is such that \( P_i \) is minimal in \( \{P_i, P_{i+1}, \ldots, P_n\} \) for \( 1 \leq i \leq n \). This means that if we choose \( a_1 \in Q_1 \cap \cdots \cap Q_n \), \( a_2 \in (Q_2 \cap \cdots \cap Q_n)\setminus Q_1 \), \ldots, \( a_n \in Q_n \setminus \bigcap_{i=1}^{n-1} Q_i \), then the idempotents \( e_{a_1}, e_{a_2}, \ldots, e_{a_n} \) in \( R^0 \) are distinct. Since \( n \) is arbitrary, it follows that \( R^0 \) has infinitely many idempotents.

Remark 4.8. Suppose \( R \) is a ring imbeddable in a zero-dimensional ring. If \( \text{Spec}(R) \) is infinite, then Proposition 4.7 can be used as follows to show that \( R \) has a minimal zero-dimensional extension \( R^0 \) with infinitely many idempotents: There exists a family \( \{Q_\lambda\} \) of primary ideals of \( R \) satisfying (1) and (2) of AIT, and the family \( \mathcal{F} = \{Q_\lambda\} \cup \text{Spec}(R) \) satisfies the same conditions. Using \( \mathcal{F} \) as the family \( \mathcal{F} \) in the statement of Proposition 4.7, we obtain an \( R^0 \) with infinitely many idempotents. On the other hand, if \( \text{Spec}(R) \) is finite, then Remark 4.6 shows that in any family \( \{Q_\lambda\} \) of primary ideals satisfying the conditions of AIT, we may assume without loss of generality that distinct ideals \( Q_\lambda \) have distinct radicals, and hence that \( \{Q_\lambda\} \) is finite. In this case the ring \( T = \prod_{\lambda}(R/\mathcal{Q}_\lambda)(P_x/Q_x) \) of Proposition 4.7 has only finitely many idempotents, and hence \( R^0 \) has only finitely many idempotents.

Theorem 4.9. Let \( R \) be a ring with finite spectrum of cardinality \( m \). The following conditions are equivalent.

1. \( R \) is imbeddable in a zero-dimensional ring.
2. The zero ideal of \( R \) is a finite intersection of primary ideals.
3. \( R \) admits a minimal zero-dimensional extension ring with at most \( m \) prime ideals.

Proof. That (1) implies (2) follows from AIT and Remark 4.6. Proposition 4.4 shows that (2) implies (3), and the implication (3) \( \Rightarrow \) (1) is obvious.

In relation to Theorem 4.9 it would be interesting to know whether a ring \( R \) which is a subring of a zero-dimensional ring and which has Noetherian
spectrum necessarily has the property that $(0)$ in $R$ is a finite intersection of primary ideals of $R$.

If $\{Q_i\}$ is a family of primary ideals of a ring $R$ satisfying the conditions of AIT, then the ring $T = \prod_i (R/Q_i)(P_i/Q_i)$ need not itself be zero-dimensional. For example, we show in Result 4.11 that there exist zero-dimensional rings not imbeddable in any zero-dimensional product of quasilocal rings. Theorem 4.10 gives equivalent conditions for a ring to be imbeddable in a zero-dimensional product of quasilocal rings. The statement of Theorem 4.10 uses the notation $n(\cdot)$, defined as follows (cf. [GH, p. 633]). Let $R$ be a ring with nilradical $N(R)$. If $x \in N(R)$, then $\eta(x)$ is the index of nilpotency of $x$—that is, $\eta(x) = k$ if $x^k = 0$ but $x^{k-1} \neq 0$. We define $\eta(R)$ to be $\sup\{\eta(x) : x \in N(R)\}$; if the set $\{\eta(x) : x \in N(R)\}$ is unbounded, then we write $\eta(R) = \infty$. For an ideal $I$ of $R$, $\eta(I)$ is defined to be $\eta(R/I)$; the definition amounts to the following: if $J = \text{rad}(I)$, then $\eta(I) = \inf\{n \in \mathbb{Z}^+ : x^n \in I \text{ for each } x \in J\}$, where it is understood that $\eta(I) = \infty$ if there is no positive integer $n$ such that $x^n \in I$ for each $x \in J$.

**Theorem 4.10.** For a ring $R$, the following conditions are equivalent.

1. $R$ is imbeddable in a zero-dimensional product of quasilocal rings.
2. The zero ideal of $R$ is representable as an intersection of a family $\{Q_i\}_{i \in I}$ of primary ideals such that, for some $k \in \mathbb{Z}^+$, $\{i \in I : \eta(Q_i) > k\}$ is finite.

**Proof.** $(1) \Rightarrow (2)$. Suppose $R$ is imbeddable in $S$, where $S$ is the product of a family $\{(R_i, M_i)\}_{i \in I}$ of quasilocal rings and $S$ is zero-dimensional. By [GH, Theorem 3.4], there exists $k \in \mathbb{Z}^+$ such that $\{i \in I : \eta(R_i) > k\}$ is finite. Let $U_i$ be the primary ideal of $S$ consisting of all tuples with $i$th coordinate 0; $\text{rad}(U_i) = W_i$ consists of all tuples whose $i$th coordinate is in $M_i$. Clearly $\bigcap_{i \in I} U_i = (0)$ and $\eta(U_i) = \eta(R_i)$. Let $Q_i$ and $P_i$ denote the contractions to $R$ of $U_i$ and $W_i$, respectively. Then $Q_i$ is $P_i$-primary, $\bigcap_{i \in I} Q_i = (0)$, and since it is clear that $\eta(Q_i) \leq \eta(U_i) = \eta(R_i)$, the required condition in $(2)$ is satisfied by the family $\{Q_i\}_{i \in I}$.

$(2) \Rightarrow (1)$. Suppose the ideal $(0)$ of $R$ is an intersection of a family $\{Q_i\}_{i \in I}$ of primary ideals satisfying the condition in $(2)$. Then $R$ is imbedded in $S = \prod_{i \in I} (R/Q_i)(P_i/Q_i)$, each $(R/Q_i)(P_i/Q_i)$ is zero-dimensional and quasilocal, and $\eta((R/Q_i)(P_i/Q_i)) = \eta(Q_i)$ for each $i$. Again using Theorem 3.4 of [GH], we conclude that $S$ is zero-dimensional.

**Result 4.11.** Let $\{X_i\}_{i=1}^\infty$ be a set of indeterminates over a field $F$ and let $I$ be the ideal of $F[\{X_i\}_{i=1}^\infty]$ generated by the set $\{X_i(X_i + 1)^i\}_{i=1}^\infty$. Then $R = F[\{X_i\}_{i=1}^\infty]/I$ is zero-dimensional, but no family of primary ideals of $R$ satisfies condition $(2)$ of Theorem 4.9, and hence $R$ is not imbeddable in a zero-dimensional product of quasilocal rings.

**Proof.** If $A$ is any subset of $\mathbb{Z}^+$, let $A'$ denote the complement of $A$ in $\mathbb{Z}^+$. We note that $M_A = \{\{X_i : i \in A\} \cup \{X_i + 1 : i \in A'\}\}$ is a maximal ideal of $F[\{X_i\}_{i=1}^\infty]$ containing $I$. Any prime ideal $P$ of $F[\{X_i\}_{i=1}^\infty]$ containing $I$ is equal to some $M_A$, for since $X_i(X_i + 1)^i \in I \subseteq P$, then $P$ contains one and only one of $X_i$ or $X_i + 1$ for each $i \in \mathbb{Z}^+$. Hence if $B = \{i \in \mathbb{Z}^+ : X_i \in P\}$, then $M_B \subseteq P$, so $M_B = P$. Therefore $R$ is zero-dimensional.

Suppose $\{Q_i\}_{i \in A}$ is a family of primary ideals of $F[\{X_i\}_{i=1}^\infty]$ such that $I = \bigcap_i Q_i$. Let $P_\lambda = \text{rad}(Q_\lambda)$ and fix $\lambda \in \mathbb{Z}^+$. We partition $\Lambda$ into subsets $\Lambda_1$
and $\Lambda_2$, where $\Lambda_1 = \{ \lambda \in \Lambda : X_i \in P_\lambda \}$ and $\Lambda_2 = \{ \lambda \in \Lambda : X_i + 1 \in P_\lambda \}$.

We claim that there exists $\lambda \in \Lambda_2$ such that $(X_i + 1)^{i-1} \notin Q_i$. Suppose not. Then $(X_i + 1)^{i-1} \in \bigcap_{\lambda \in \Lambda_2} Q_\lambda$. Moreover, if $\lambda \in \Lambda_1$, then $X_i(X_i+1)^{i-1} \notin P_\lambda$, so $X_i \in \bigcap_{\lambda \in \Lambda_2} Q_\lambda$. Therefore $X_i(X_i+1)^{i-1} \in (\bigcap_{\lambda \in \Lambda_1} Q_\lambda) \cap (\bigcap_{\lambda \in \Lambda_2} Q_\lambda) = I$. This leads to a contradiction, as follows. If $X_i(X_i+1)^{i-1}$ were in $I$, then under the $F[X_i]$-homomorphism of $F\{\{X_i\}_{i=1}^\infty\}$ to $F[X_i]$ that maps $X_j$ to 0 if $j \neq i$, we would have $X_i(X_i+1)^{i-1} \in X_i(X_i+1)^{i}F[X_i]$, which is not the case. Therefore $(X_i + 1)^{i-1} \notin Q_i$ for some $\lambda \in \Lambda_2$, so $\eta(Q_i) \geq i$ for some $\lambda \in \Lambda_2$. Since $i$ is arbitrary, we conclude that $\{Q_i\}$ does not satisfy condition (2) of Theorem 4.10, as we wished to show.

If $R$ is as in the statement of Result 4.11, then the proof of that result shows that the prime ideals of $R$ are in one-to-one correspondence with the subsets of $\mathbb{Z}^+$, and hence Spec($R$) is uncountable. We can obtain an example of a ring with the same properties as described in (4.11) and with countable spectrum as follows. Let $F$ be a field, let $t$ be an indeterminate over $F$, and let $S$ be the subring of $T = \prod_{n=1}^\infty F[t]/(t^n)$ generated by $F$ and the direct sum ideal $J = \bigoplus_{n=1}^\infty F[t]/(t^n)$ of $T$. Since $S$ is integral over $F$, it is zero-dimensional. Moreover, Spec($S$) = $\{J\} \cup \{P_i\}_{i=1}^\infty$, where $P_i$ consists of all tuples with $i$th coordinate in $(t)/(t^i)$. If $Q_i$ is the $P_i$-primary ideal consisting of all tuples with 0 in the $i$th coordinate (hence $Q_i = P_i^\infty$), then as in Example 4 of [GH1], each $Q_i$ belongs to each family $\{Q_i\}$ of primary ideals of $S$ with intersection (0). Since $\eta(Q_i) = i$, it follows that $S$ too has the property that it is not imbeddable in a zero-dimensional product of quasilocal rings.

**Remark 4.12.** Even though AIT gives necessary and sufficient conditions in order that a ring $R$ be a subring of a zero-dimensional ring, it is sometimes difficult in practice to apply. For example, if $p$ is a fixed prime integer, $T = \prod_{n=1}^\infty \mathbb{Z}/p^n\mathbb{Z}$ and $I = \bigoplus_{n=1}^\infty \mathbb{Z}/p^n\mathbb{Z}$ is the direct sum ideal in $T$, then Roger Wiegand has asked whether the factor ring $R = T/I$ is a subring of a zero-dimensional ring. It is known that $T$ is not a subring of a zero-dimensional ring [GH1, Theorem 3], but the proof involves the irredundance of certain primary ideals in any representation of (0) in $T$ as an intersection of primary ideals, and does not apply to a ring such as $R$.

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**Added in proof**

Ken Goodearl has called to our attention the paper by W. D. Burgess and P. Menal, *On strongly π-regular rings and homomorphisms into them*, Comm. Algebra 16 (1988), 1701–1725. Theorem 1.3 of this paper implies that an intersection of zero-dimensional commutative rings is again zero-dimensional. Goodearl has also informed us that the result is in the (unpublished) 1977 dissertation of Dischinger.
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