THE FOURIER-BESSEL SERIES REPRESENTATION OF THE PSEUDO-DIFFERENTIAL OPERATOR \((-x^{-1}D)^\nu\)

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Abstract. For a certain Fréchet space \(F\) consisting of complex-valued \(C^\infty\) functions defined on \(I = (0, \infty)\) and characterized by their asymptotic behaviour near the boundaries, we show that:

I. The pseudo-differential operator \((-x^{-1}D)^\nu, \nu \in \mathbb{R}, D = d/dx\), is an automorphism (in the topological sense) on \(F\);

II. \((-x^{-1}D)^\nu\) is almost an inverse of the Hankel transform \(h_\nu\) in the sense that

\[
h_\nu \circ (x^{-1}D)^\nu (\varphi) = h_0(\varphi), \quad \forall \varphi \in F, \forall \nu \in \mathbb{R};
\]

III. \((-x^{-1}D)^\nu\) has a Fourier-Bessel series representation on a subspace \(F_b \subset F\) and also on its dual \(F_b^*\).

1. Introduction

Let \(F\) be the space of all \(C^\infty\) complex-valued function \(\varphi(x)\) defined on \(I = (0, \infty)\) such that

\[
\varphi(x) = \sum_{i=0}^{k} a_i x^{2i} + o(x^{2k})
\]

near the origin and is rapidly decreasing as \(x \to \infty\).

For \(\nu > -\frac{1}{2}\), we define a \(\nu\)th order Hankel transform \(h_\nu\) on \(F\) by

\[
\Phi(y) = [h_\nu \varphi(x)](y) = \int_0^\infty \varphi(x) J_\nu(xy) \, dm(x),
\]

where

\[
dm(x) = m'(x) \, dx = [2^\nu \Gamma(\nu + 1)]^{-1} x^{2\nu+1} \, dx,
\]

and \(J_\nu(x)\) is the Bessel function of order \(\nu\). The inversion formula for (1.2) is given by [1, 3, 4],

\[
\varphi(x) = \int_0^\infty \Phi(y) J_\nu(xy) \, dm(y).
\]

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In this paper we will show that for every real $\nu$:

(I) The pseudodifferential operator $(-x^{-1}D)^\nu$ is a topological automorphism on $F$.

(II) The Hankel transform $h_\nu$ is also an automorphism on $F$.

(III) On $F$, $(-x^{-1}D)^\nu$ is almost an inverse of $h_\nu$ in the sense that

$$[h_\nu \circ (-x^{-1}D)^\nu](\varphi) = h_0(\varphi), \quad \varphi \in F.$$ 

(IV) On a certain subspace $F_b \subset F$ and on its dual $F'_b$, $(-x^{-1}D)^\nu$ has Fourier-Bessel series representations.

In the sequel all automorphisms are topological automorphisms.

2. Preliminaries

For any real $\nu \neq -\frac{1}{2}$, $F_\nu$ is the space of all $C^\infty$ complex-valued function $\varphi(x)$ defined on $I$ such that

$$\gamma_{m,k}^\nu(\varphi) = \sup_{x \in I} |x^m \Delta_{\nu,x}^k \varphi(x)| < \infty,$$

for each $m, k = 0, 1, 2, \ldots$, where

$$\Delta_{\nu,x} = D^2 + x^{-1}(2\nu + 1)D.$$ 

$F_\nu$ is a Fréchet space. Its topology is generated by the countable family of separating seminorms $\{\gamma_{m,k}^\nu\}_{m,k=0,1,2,\ldots}$, [5; 7, p. 8].

Theorem 2.1(i) of Lee [3, p. 429] shows that $F_\nu = F_\mu = F$ (as a set) for each $\nu, \mu(\neq -\frac{1}{2}) \in \mathbb{R}$. Hence for each $\nu \neq -\frac{1}{2}$, we have a topology $T_\nu$ on $F$ generated by the countable family of seminorms $\gamma_{m,k}^\nu$. Hence $(F, T_\nu)$ is a Fréchet space. When $\nu = -\frac{1}{2}$, $F_{-1/2} \neq F$, since the factor $x^{-1}(2\nu + 1)D$ in $\Delta_{\nu,x}$, responsible for the even nature of $\varphi(x) \in F_\nu(x)$ near the origin, vanishes. For example $e^{-x} \in F_{-1/2}$ but $e^{-x} \notin F_{\nu}; \nu \neq -\frac{1}{2}$.

Definition. Zemanian [7, 8] defined a Hankel transform $h_\nu (\nu \geq -\frac{1}{2}) by

$$\Psi(y) = \left[h_\nu \psi(x)\right](y) = \int_0^\infty \psi(x) \sqrt{x} J_\nu(xy) dx.$$ 

He proved that $h_\nu$ is an automorphism on the space $H_\nu$ that consists of complex-valued $C^\infty$ functions defined on $I$ and satisfies the relation

$$\gamma_{m,k}^\nu(\psi) = \sup_{x \in I} |x^m (x^{-1}D)^k [x^{-\nu-1/2}\psi(x)]| < \infty,$$

for each $m, k = 0, 1, 2, \ldots$, where $D = d/dx$.

The following theorem is a key result for the latter development of our theory.

Theorem 2.1. Let $\nu, \mu$ be real number $\neq -\frac{1}{2}$. Then

(I) The operation $\varphi \rightarrow x^{\nu+1/2}\varphi$ is an homeomorphism from $F$ onto $H_\nu$.

(II) $(x^{-1}D)^\nu: F \rightarrow F$ is an automorphism on $F$.

(III) $(F, T_\nu)$ and $(F, T_\mu)$ are equivalent topological spaces.

(IV) $h_\nu(\varphi) = (-1)^n [h_{\nu+n}(x^{-1}D)^n] \varphi$, for $\varphi \in F, \nu \geq -\frac{1}{2}$, and $n = 0, 1, 2, \ldots$. 

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Notation. In view of the Theorem 2.1(I, III), we will always write $x^{\nu+1/2}\varphi(x) = \varphi(x) \in H_\nu$ for $\varphi \in F$ and drop the suffix $\nu$ from $T_\nu$. So henceforth the topological linear space $(F, T)$ will be denoted by $F$.

Proof. (I) By induction on $n$ and noting that
\begin{equation}
\Delta_{\nu, x} = x^2(x^{-1}D)^2 + 2(\nu + 1)(x^{-1}D),
\end{equation}

it can be proved that
\begin{equation}
\Delta^n_{\nu, x} = x^{2n}(x^{-1}D)^{2n} + a_1x^{2(n-1)}(x^{-1}D)^{2n-1} + \cdots + a_n(x^{-1}D)^n,
\end{equation}

where $a_i$'s are the constants depending on $\nu$. Now $\varphi \in F$ iff $\varphi \in H_\nu$ (follows from Remark II of Lee [3]) and taking $a_0 = 1$, it follows from (2.5) that
\begin{equation}
\gamma_{m,k}^\nu(\varphi) \leq \sum_{i=k}^{2k} a_{2k-i} \gamma_{m+2(i-k), i+n}^{\nu}(\varphi),
\end{equation}

proving the continuity of the inverse operation $\varphi \rightarrow x^{-\nu-1/2}\varphi$. Invoking the Open Mapping Theorem [6, p. 172], $F$ being Fréchet space, we complete the proof.

(II) Let $\varphi_j$ be a sequence tending to zero in $F$. Then $\varphi_j \rightarrow 0$ in $H_\nu$ for arbitrary $\nu \neq -\frac{1}{2}$. Hence
\begin{equation}
\gamma_{m,k}^\nu[(x^{-1}D)^n\varphi_j(x)] \leq \sum_{i=k}^{2k} a_{2k-i} \gamma_{m+2(i-k), i+n}^{\nu}(\varphi_j) \quad \text{(from (2.6))}
\end{equation}

\rightarrow 0 \quad \text{as } j \rightarrow \infty.

It remains to be shown that $(x^{-1}D)^n$ is bijective. It is enough to prove this for $n = 1$. So, let $x^{-1}D\varphi_1(x) = x^{-1}D\varphi_2(x)$ for $\varphi_1, \varphi_2 \in F$. Hence $\varphi_1(x) - \varphi_2(x) = \text{constant. But } \varphi_1(x)$ and $\varphi_2(x)$ are of rapid descent as $x \rightarrow \infty \Rightarrow \varphi_1(x) = \varphi_2(x)$. Now let $\psi(x) \in F$. Then $\varphi(x) = -\int_x^\infty t\psi(t) dt$, defined uniquely (since $\psi$ is of rapid descent as $x \rightarrow \infty$) in $F$, is such that $x^{-1}D\varphi(x) = \psi(x)$. So we see that $(x^{-1}D)^n$ is a continuous bijection on $F$. The space $F$ being a Fréchet space, the Open Mapping Theorem shows that $(x^{-1}D)^n$ is a bicontinuous bijection on $(F, T_\nu)$ for each $\nu \in \mathbb{R} - \{\frac{1}{2}\}$.

(III) Let $\nu = \mu + a$, $a \in \mathbb{R}$, and $\varphi_n$ be a sequence tending to zero in $(F, T_\nu)$. Then
\begin{align*}
\gamma_{m,k}^\nu(\varphi_n) &= \sup_{x \in I} |x^m[\Delta_{\mu, x} + 2a(x^{-1}D)]^k \varphi_n(x)| \\
&\leq \sup_{x \in I} x^m \left[ \sum_{i=0}^{k} \Delta_{\mu, x}^{k-i}(2ax^{-1}D)^i \varphi_n(x) \right] + \sum_{i=0}^{k} |(2ax^{-1}D)^{k-i}\Delta_{\mu, x}^{i}\varphi_n(x)| \\
&\quad \quad + \text{terms of the type } |\Delta_{\mu, x}^{i_1}(2ax^{-1}D)^{j_1}\Delta_{\mu, x}^{i_2}(2ax^{-1}D)^{j_2} \cdots \varphi_n(x)|
\end{align*}

(\text{where } i_1 + i_2 + \cdots = j_1 + j_2 + j_3 \cdots = k)

\rightarrow 0 \text{ as } n \rightarrow \infty, \quad \text{for each } m, k = 0, 1, 2 \ldots,
since $\Delta_{\mu,x}^i$ and $(x^{-1}D)^i$ are continuous on $(F, T_\mu)$.

(IV) follows from integration by parts and induction on $n$.

Remark 1. It can be shown that on $F$

$$\Delta_{\nu,x}^k (x^{-1}D)^n = (x^{-1}D)^n \circ \Delta_{\nu-n,x}^k.$$

The proof follows by induction on $k$.

Definition. In view of Theorem 2.1(IV), we define the Hankel transform $h_\nu$ formally for any $\nu \in \mathbb{R}$, as

$$h_\nu(\varphi) = h_{\nu+n} \circ (-x^{-1}D)^n \varphi, \quad \varphi \in F,$$

where $n$ is so chosen that $\nu + n > -\frac{1}{2}$.

This is a well-defined definition as $(x^{-1}D)^n$ is an automorphism.

Definition. Let $F'$ be the dual space of $F$. Then for $f \in F'$, define the generalized Hankel transform $h_\nu f(= \tilde{f})$ of $f$ by

$$\langle h_\nu f, h_\nu \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in F, \; \nu \in \mathbb{R}.$$

Theorem 2.2. For $\nu \in \mathbb{R}$, $h_\nu$ is an automorphism on $F$ and hence on $F'$.

Proof. Let $\varphi(x) \in F$. Then

$$h_\nu(\varphi) = \Phi(y) = \int_0^\infty (x^{-1}D)^{2n} \varphi(x) \mathcal{J}_{\nu+2n}(xy) \, dm(x)$$

$$= y^{-\mu-1/2} h_{\mu}(\psi(x))(y) \quad \text{where } \mu = \nu + 2n > -\frac{1}{2},$$

where

$$\psi(x) = x^{\mu+1/2} \varphi(x) = x^{\nu+1/2}(x^{-1}D)^{2n} \varphi(x).$$

Let

$$\varphi_m(x) \to 0 \quad \text{in } F \quad \Rightarrow \quad \psi_m(x) \to 0 \quad \text{in } H_\mu \quad \text{(Theorem 2.1(I))},$$

$$\Rightarrow \quad h_{\mu}(\psi_m) \to 0 \quad \text{in } H_\mu,$$

$$\Rightarrow \quad h_\nu(\varphi_m) \to 0 \quad \text{in } F.$$

Now $\tilde{h}_\mu$, the Zemanian Hankel transform, being bijective, (2.8) shows that $h_\nu$ is a bijection. Hence use of the Open Mapping Theorem completes the proof.

Writing $\nu = 0$ in (2.7) we get

$$h_0(\varphi) = h_n \circ (-x^{-1}D)^n(\varphi), \quad \varphi \in F.$$

The above equation motivates us to propose the following

Definition. For $\nu \in \mathbb{R}$, define $(-x^{-1}D)\nu$ by

$$(-x^{-1}D)^\nu(\varphi) = h_{\nu-1} \circ h_0(\varphi), \quad \varphi \in F.$$

Then $(-x^{-1}D)^\nu$ is clearly an automorphism on $F$ for each real $\nu$. From equation (2.9) we get

$$(-x^{-1}D)^\nu \varphi(x) = \int_0^\infty dm(y) \mathcal{J}_\nu(xy) \int_0^\infty dm(x) \varphi(x) \mathcal{J}_0(xy).$$
For distributions \( f \in F' \), define \((-x^-1D)^\nu\) by

\[(2.11) \quad \langle (-x^-1D)^\nu f, \phi \rangle = \langle f, (-x^-1D)^\nu \phi \rangle, \quad \phi \in F.\]

So we modify Theorem 2.1(II) to give our main result.

**Theorem 2.3.** The pseudodifferential operator \((-x^-1D)^\nu\) is an automorphism on \(F\) and hence on \(F'\) for each \(\nu \in \mathbb{R}\).

### 3. The Fourier-Bessel series expansion of \((-x^-1D)^\nu\)

Equation (2.10) gives the integral representation of the operator \((-x^-1D)^\nu\). To get the Fourier-Bessel series expansion, we modify our leading function space \(F\) suitably as follows (similar to the ones as in Zemanian [7, 9]).

For \(b > 0\), define

\[(3.1) \quad F_b = \{ \phi \in F | \phi \equiv 0 \text{ for } x > b \}.

The topology of \(F_b\) is generated by a countable family of seminorms

\[(3.2) \quad \gamma_k^v(\phi) = \sup_{0 < x < b} |\Delta^k_{\nu, x} \phi(x)| < \infty, \quad k = 0, 1, 2, \ldots.

Clearly all the topologies obtained by choosing different \(\nu\)'s are equivalent.

**Remark 2.** Without loss of generality, we may take \(\nu > -\frac{1}{2}\).

**Definition.** We define finite Hankel transform \(h_{\nu}\) by

\[(3.3) \quad \Phi(z) = [h_{\nu} \phi](z) = \int_0^b \phi(x) J_{\nu}(xz) \, dm(x).

Then \(\Phi(z)\) is an even entire function by Griffith's Theorem [2, 9]. Let \(z = y + iw\) and \(G_b = \{ \Phi(z) | \Phi(z) \text{ is an even entire function satisfying (3.4)} \}\).

\[(3.4) \quad \alpha^k_b(\Phi) = \sup_{z \in \mathbb{C}} |e^{-b|w|} z^{2k} \Phi(z)| < \infty,

for \(k = 0, 1, 2, \ldots\). Then \(G_b\) is a linear topological space with \(\alpha^k_b\) as seminorms.

Both the spaces \(F_b\) and \(G_b\) are Hausdorff, locally convex topological linear spaces satisfying the axiom of first countability. They are sequentially complete spaces.

**Theorem 3.1.** \(h_{\nu}\) is an homeomorphism from \(F_b\) onto \(G_b\).

**Proof.** Let \(\phi \in F_b\). Then

\[\Phi(z) = h_{\nu+2m}[(x^-1D)^{2m}\phi(x)], \quad \text{for } m \in \mathbb{N}.

Hence

\[z^{2m} \Phi(z) = \int_0^b x^{2\nu+2m+1}[(x^-1D)^{2m}\phi(x)](x z)^{-\nu} J_{\nu+2m}(x z) \, dz.

From the asymptotic formula

\[J_{\nu}(z) \sim \sqrt{2/\pi z} \cos \left(z - \frac{\nu \pi}{2} - \frac{\pi}{4}\right), \quad |z| \to \infty, \quad |\arg z| < \pi,

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and from the fact that \( z^{-\nu}J_{\nu+m}(z) \) is an entire function, it follows that for all \( x \) and \( z \),
\[
|e^{-b|x|}(xz)^{\nu}J_{\nu+2m}(xz)| < C_{mn} \quad \text{(a constant)}.
\]

Hence
\[
(3.5) \quad a_{b}^{m}(\Phi) \leq C_{mn}b^{2(m+\nu+1)}\gamma_{0}^{m}[(x^{-1}D)^{2m}\varphi(x)] < \infty.
\]

(\( x^{-1}D \)) being an automorphism (also on \( F_{b} \)), (3.5) implies the continuity of \( h_{\nu} \). \( h_{\nu} \) is clearly injective. For any \( \Phi(z) \in G_{b} \), take
\[
\varphi(x) = \int_{0}^{\infty} \Phi(y)J_{\nu}(xy) \, dm(y).
\]

Then it follows from Griffith’s Theorem [2] that \( \varphi \) is zero almost everywhere for \( x > b \). Also,
\[
\gamma_{k}^{\nu}([\Phi]) = \sup_{0<x<b} \left| \frac{1}{x} \int_{0}^{x} \Phi(y)J_{\nu}(xy) \, dm(y) \right| < \infty,
\]
where \( \Delta_{\nu}^{k}[(xy)^{-\nu}J_{\nu}(xy)]] = (-1)^{k}y^{2k}(xy)^{-\nu}J_{\nu}(xy), \) \( \Phi(y) \) is of rapid descent as \( y \rightarrow \infty \), and \( (xy)^{-\nu}J_{\nu}(xy) \) is bounded for \( 0 < y < \infty \). Therefore, \( \varphi \in F_{b} \). Hence \( h_{\nu} \) is surjective. Now the Open Mapping Theorem completes the proof.

**Theorem 3.2.** Let \( \varphi \in F_{b} \). Then
\[
(3.6) \quad \varphi(x) = \lim_{\varepsilon \to 0^{+}} \frac{2}{b^{2}} \sum_{n=1}^{\infty} \lambda_{\varepsilon}(x) \left( \frac{\lambda_{n}}{x} \right)^{\nu} \frac{J_{\nu}(x\lambda_{n})}{J_{\nu+1}^{2}(b\lambda_{n})} \Phi(\lambda_{n}),
\]

where the \( \lambda_{n} \)’s are the positive roots of \( J_{\nu}(bz) = 0 \) arranged in the ascending order and for \( 0 < \varepsilon < b/4 \),
\[
\lambda_{\varepsilon}(x) = \begin{cases} E\left(x/2\varepsilon\right), & 0 < x < 2\varepsilon, \\ 1, & 2\varepsilon \leq x \leq b - 2\varepsilon, \\ \frac{2\varepsilon - x - b}{2\varepsilon}, & b - 2\varepsilon < x < b, \\ 0, & x \geq b, \end{cases}
\]

and \( E(u) = \int_{0}^{u} \exp[1/x(x - 1)] \, dx / \int_{0}^{1} \exp[1/x(x - 1)] \, dx \).

**Proof.** Trivial. See also [5].

Theorem 3.2 gives the required Fourier-Bessel Series expansion for the pseudo-differential operator \( (-x^{-1}D)^{\nu} \), which we obtain in the following

**Theorem 3.3 (The Fourier-Bessel Series).** For \( \varphi \in F_{b} \), we have
\[
(3.7) \quad [(-x^{-1}D)^{\nu}]\varphi(x) = \lim_{\varepsilon \to 0^{+}} \frac{2}{b^{2}} \sum_{n=1}^{\infty} \lambda_{\varepsilon}(x) \left( \frac{\lambda_{n}}{x} \right)^{\nu} \frac{J_{\nu}(x\lambda_{n})}{J_{\nu+1}^{2}(b\lambda_{n})} \Phi(\lambda_{n}),
\]

where \( \Phi_{0}(y) = h_{0}[\varphi(x)](y) \).
Proof. Equation (2.9) along with Theorem 3.2 gives the required proof.

Note that

\[ |\lambda_n^{\nu+1/2} \Phi_0(\lambda_n)| \leq A_{\nu} \lambda_n^{(\nu+1/2)-2k}, \]

$A_{\nu}$ constants and $[\int_{\nu}(x\lambda_n)/x^{\nu+1/2}J_{\nu+1}^2(b\lambda_n)]$ is smooth and bounded on $0 < x < b$, $0 < \lambda_n < \infty$.

Hence the truncation error

\[ E_N = \lim_{\varepsilon \to 0^+} \frac{2}{b^2} \sum_{n=N+1}^{\infty} \lambda_n(x) \left( \frac{\lambda_n}{x} \right)^{\nu} J_{\nu+1}^2(b\lambda_n) \Phi_0(\lambda_n) \]

has exponential decay for large $N$.

The Theorem 3.3 gives the Fourier-Bessel series representation of the operator $(-x^{-1}D)^\nu$ on the testing function space $F_b$. We wish to investigate the nature of the Fourier-Bessel series for the pseudodifferential operator $(-x^{-1}D)^\nu$ on the distribution space $F_b'$.

The spaces $F_b'$ and $G_b'$ are dual spaces of $F_b$ and $G_b$, respectively. They are assigned the weak topologies generated by the seminorms

\[ P_\phi(f) = (f, \phi), \quad \phi \in F_b, \quad f \in F_b', \]

and

\[ P_{\phi}(hvf) = (hvf, hv\phi), \quad hv\phi \in G_b, \quad hvf \in G_b', \]

respectively.

Both the spaces are sequentially complete.

**Definition.** For $f \in F_b'$, $\phi \in F_b$, we define the generalized finite Hankel transform $hv f$ by

\[ (3.8) \quad (hv f, hv\phi) = (f, \phi). \]

**Theorem 3.4.** For $\nu \in \mathbb{R}$, $hv$ is an homeomorphism from $F_b'$ onto $G_b'$.

**Theorem 3.5.** For every $\varepsilon \in (0, b/4)$ and each $f \in F_b'$, the function

\[ (3.9) \quad \hat{f}_\varepsilon(y) = (f(x), y^{-\nu-1/2}\lambda_\varepsilon(x)m'(y)\mathcal{F}_\varepsilon(xy)), \]

where $\lambda_\varepsilon(x)$ is defined as in Theorem 3.2, is a smooth function of slow growth, and defines a regular generalized function in $G_b'$.

**Proof.** Note that $(x^{-1}D)^k\lambda_\varepsilon(x)$ is bounded on $0 < x < b$ for each $k$. Using (2.6), it is easy to see that $y^{-\nu-1/2}\lambda_\varepsilon(x)m'(y)\mathcal{F}_\varepsilon(xy) \in F_b$. Hence (3.9) is well defined. The rest of the proof is similar to that of Zemanian [8, Lemma 12].

**Theorem 3.6.** The finite Hankel transform $hv f$ of a generalized function $f$ in $F_b'$ is the distributional limit, as $\varepsilon \to 0^+$, of the family $\hat{f}_\varepsilon(z)$ defined by (3.9).

**Proof.** Trivial.

**Theorem 3.7.** Let $f \in F_b'$ and $\hat{f} = hv f$. Then in the sense of convergence in $F_b'$, we have

\[ (3.10) \quad f(x) = \lim_{N \to \infty} \frac{2}{b^2} \sum_{n=1}^{N} x^{\nu+1} [\int_{\nu}(x\lambda_n)/J_{\nu+1}^2(b\lambda_n)] \cdot \hat{f}(\lambda_n). \]

**Proof.** The proof follows easily from Theorems 3.2 and 3.6.
Remark 3. For \( f \in F'_b \), such that either \( f \) is regular or \( \text{supp} \, f \subset (0, b] \), the limit of \( \tilde{f}_\varepsilon(z) \) as \( \varepsilon \to 0^+ \) exists as an ordinary function and is equivalent to the finite Hankel transform of \( f \) [5].

A consequence of the above theorem is the following

**Theorem 3.8.** Let \( f, g \in F'_b \). If \( (h_\nu f)(\lambda_n) = (h_\nu g)(\lambda_n) \), for \( n = 1, 2, 3, \ldots \), then \( f = g \) and \( h_\nu f = h_\nu g \).

**Definition.** For \( f \in F'_b \), define \((-x^{-1}D)^\nu f\) by

\[
\langle (-x^{-1}D)^\nu f, \varphi \rangle = \langle f, (-x^{-1}D)^\nu \varphi \rangle, \quad \varphi \in F_b, \quad \nu \in \mathbb{R}. \tag{3.11}
\]

From equations (2.9), (3.8), and (3.11), it follows that

\[
\langle (-x^{-1}D)^\nu f, \varphi \rangle = \langle f, (-x^{-1}D)^\nu \varphi \rangle = \langle h_0^{-1}h_\nu f, \varphi \rangle, \quad f \in F'_b, \quad \varphi \in F_b.
\]

Hence

\[
(-x^{-1}D)^\nu f = h_0^{-1}h_\nu f \quad \text{on} \quad F'_b. \tag{3.12}
\]

Applying Theorem 3.7 to equation (3.12) we get

**Theorem 3.10 (The Fourier-Bessel Series).** Let \( f \in F'_b \) and \( \hat{f} = h_\nu f \). Then in the sense of convergence in \( F'_b \), we have

\[
(-x^{-1}D)^\nu f(x) = \lim_{b \to \infty} \frac{2}{b^2} \sum_{n=1}^{N} \frac{x}{\sqrt{\lambda_n}} [J_0(x\lambda_n)/J_1^2(b\lambda_n)] \hat{f}(\lambda_n). \tag{3.13}
\]

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