LOCALLY COMPLETE INTERSECTION MULTIPLE STRUCTURES ON SMOOTH ALGEBRAIC CURVES

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Abstract. The aim of this paper is to characterize the class of the locally complete intersection multiple structures on smooth curves contained in a smooth three-dimensional variety.

Introduction

Let \( C \) be a smooth (connected) curve contained in a smooth three-dimensional variety \( X \). In [1] Banica and Forster described all the multiple structures on \( C \), i.e., the locally Cohen Macaulay curves \( \overline{C} \subset X \) such that scheme theoretically \( \overline{C} \supset C \) and \( |\overline{C}| \) — the underlying space of \( \overline{C} = |C| \). \( \overline{C} \) is called a quasi-primitive multiple structure on \( C \) if for almost all points \( x \in C \) \( \dim_x \overline{C} = 2 \).

The aim of this note is to characterize the quasi-primitive multiple structures that are locally complete intersections (lci). In order to make this paper self-contained, we present first the above-quoted results of Banica and Forster, which we shall need in the formulation of our result.

In the sequel \( C \) will be a smooth curve of a three-dimensional smooth (algebraic) variety \( X \).

Definition. A locally Cohen Macaulay (lCM) curve \( \overline{C} \subset X \) is called a multiple structure on \( C \) if \( \overline{C} \supset C \) scheme theoretically and \( |\overline{C}| = |C| \).

Let \( I \) and \( J \) denote the ideal sheaves of \( C \) and \( \overline{C} \) respectively. For any \( i > 1 \) we define \( J_i \) as the minimal ideal sheaf containing \( J + I^i \), which defines an lCM curve of \( X \). Since \( J_i \) is obtained by removing all the embedded components of \( J + I^i \), we infer that both considered ideals are [A generically equal. We have \( J_1 = I \) and \( J_i = J \) for \( i \geq i + 1 \) where \( i + 1 \) is the least \( i \) such that \( J \supset I \). Moreover \( J_i \supset J_{i+1} \) for \( i \geq 1 \).

We claim that \( J_i \cdot J_i \subset J_{i+j} \). This is of course true in all the points of \( C \) where \( J_i = J + I^i \). So the ideal \( J_i \cdot J_i + J_{i+j}/J_{i+j} \subset O_X/J_{i+j} \) has a zero-dimensional support. Since \( O_X/J_{i+j} \) is lCM, this ideal is zero and therefore...
$J_i \cdot J_j \subset J_{i+j}$. In particular $IJ_i \subset J_{i+1}$. So for all $i \geq 1$, $J_i/J_{i+1}$ is a sheaf of $O_X/I$-modules. Since $O_X/J_{i+1}$ is ICM and $C$ is smooth, the $O_X/I$-modules $J_i/J_{i+1}$ are locally free.

The multiplication map $J_i \times J_j \rightarrow J_{i+j}$ induces a generically surjective map $E_i \otimes E_j \rightarrow E_{i+j}$ where $E_i = J_i/J_{i+1}$. In particular, one has morphisms $E_i^i \rightarrow E_i$, which are also generically surjective.

**Definition.** With the above notation, a multiple structure on $C \subset X$ is called quasi-primitive if $\text{rank } E_i = 1$. Note that this implies that $\text{rank } E_i \leq 1$ for $i \geq 1$. Actually $\text{rank } E_i = 1$ for $1 \leq i \leq t$ and $E_i = 0$ for $i > t$.

Now we can formulate our

**Theorem.** Let $\overline{C}$ be a quasi-primitive multiple structure on $C \subset X$. Then $\overline{C}$ is lci (i.e., $J$ is locally generated by 2 elements) if and only if the morphisms $E_i \otimes E_{t-i} \rightarrow E_t$ are isomorphisms for $1 \leq i \leq t - 1$.

**Remark.** If $C$ is projective then $E_i \otimes E_{t-i} \rightarrow E_t$ is an isomorphism if and only if $\deg E_i + \deg E_{t-i} = \deg E_t$.

**Proof.** The problem is local. So we can suppose that $O = O_X$ is a ring of a sufficiently small affine open neighborhood of a point of $C$. Since $C$ is smooth, the epimorphism $O/J \rightarrow O/I$ whose kernel is the nilpotent ideal $I/J$ splits. Therefore $O/J$ admits a certain $O/I$-module structure and $O/J \simeq O/I \oplus I/J$ as $O/I$-modules.

For every $i \geq 1$, $J_i/J$ becomes an $O/I$-submodule of $O/J$. The exact sequences of $O/I$-modules $0 \rightarrow J_{i+1}/J \rightarrow J_i/J \rightarrow E_i \rightarrow 0$ split since $E_i$ is free. Therefore $O/J \simeq O/I \oplus I/J \simeq O/I \oplus E_1 \oplus \cdots \oplus E_t$ as $O/I$-modules ($J_i = J$ for $i \geq t + 1$) $O/I \oplus E_1 \oplus \cdots \oplus E_t$ carries a multiplicative structure that corresponds to the multiplication on $O/J$. We have

$$E_i \cdot E_j \subset \bigoplus_{i+j \leq k \leq t} E_k$$

since $J_i/J \cdot J_i/J \subset J_{i+j}/J$ and $E_i \subset J_i/J$.

Moreover the composed maps

$$E_i \otimes E_j \rightarrow E_i \cdot E_j \rightarrow \bigoplus_{i+j \leq k \leq t} E_k \xrightarrow{\text{projection}} E_{i+j}$$

coincide with the previously defined morphisms $E_i \otimes E_j \rightarrow E_{i+j}$.

Suppose that $J$ is lci. This implies that $O/J$ is Gorenstein. Since $O/I$ is a finite module extension of $O/I$, there exists $\pi' \in \text{Hom}_{O/I}(O/J, E_i)$ such that the homomorphism $O/J \rightarrow \text{Hom}_{O/J}(O/J, E_i)$ induced by the bilinear form $(, ): O/J \times O/J \rightarrow E_i, (r, s) = \pi'(rs)$ is an isomorphism [2]. Note that $E_i$ is a rank 1 free $O/I$-module.

Let $\pi : O/J \rightarrow O/I$ be the projection induced by the decomposition $O/J \simeq O/I \oplus E_1 \oplus \cdots \oplus E_t$. There exists $s \in O/J$ such that $\pi(r) = \pi'(sr)$ for all $r \in O/J$. It is easy to see that $s$ is invertible. Therefore we can assume that $\pi' = \pi$.

Put $E_0 = O/I$ and let $\gamma$ denote the isomorphism $\bigoplus_{0 \leq i \leq t} E_i \rightarrow \text{Hom}(\bigoplus_{0 \leq i \leq t} E_i, E_i)$ induced by the bilinear form $(r, s) = \pi(r \cdot s)$. Let $\gamma_{ij} \in \text{Hom}(E_i, \text{Hom}(E_j, E_i))$ denote the $(i, j)$th entry of the corresponding matrix. The elements $\gamma_{i, t-i}, 0 \leq i \leq t$ are on its second diagonal. The elements below
the second diagonal are zero since $E_i \cdot E_j = 0$ if $i + j > t$. It follows that $\gamma$ is an isomorphism if and only if $\gamma_{i,t-i}$ is an isomorphism for $0 \leq i \leq t$. So the morphisms $E_i \otimes E_{t-i} \to E_t$ are isomorphisms since they are induced by $\gamma_{i,t-i}$.

Suppose now that for $1 \leq i \leq t - 1$ the maps $E_i \otimes E_{t-i} \to E_t$ are isomorphisms. Then $\gamma$ is an isomorphism since the maps $\gamma_{0t}: E_0 \to \text{Hom}(E_t, E_t)$ and $\gamma_{1,0}: E_t \to \text{Hom}(E_0, E_t)$ are obviously isomorphisms. To prove that $J$ is a lci it suffices to invoke that “in codimension 2 case Gorenstein implies complete intersection.”

REFERENCES


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