LOCALLY COMPLETE INTERSECTION MULTIPLE STRUCTURES  
on smooth algebraic curves  

M. BORATYNISKI  

(Communicated by Louis J. Ratliff, Jr.)  

Abstract. The aim of this paper is to characterize the class of the locally complete intersection multiple structures on smooth curves contained in a smooth three-dimensional variety.  

INTRODUCTION  

Let $C$ be a smooth (connected) curve contained in a smooth three-dimensional variety $X$. In [1] Banica and Forster described all the multiple structures on $C$, i.e., the locally Cohen Macaulay curves $\overline{C} \subset X$ such that scheme theoretically $\overline{C} \supset C$ and $|\overline{C}|$—the underlying space of $\overline{C} = |C|$. $C$ is called a quasi-primitive multiple structure on $C$ if for almost all points $x \in C \text{ em dim}_x C = 2$.  

The aim of this note is to characterize the quasi-primitive multiple structures that are locally complete intersections (lcio). In order to make this paper self-contained, we present first the above-quoted results of Banica and Forster, which we shall need in the formulation of our result.  

In the sequel $C$ will be a smooth curve of a three-dimensional smooth (algebraic) variety $X$.  

Definition. A locally Cohen Macaulay (LCM) curve $\overline{C} \subset X$ is called a multiple structure on $C$ if $\overline{C} \supset C$ scheme theoretically and $|\overline{C}| = |C|$.  

Let $I$ and $J$ denote the ideal sheaves of $C$ and $\overline{C}$ respectively. For any $i \geq 1$ we define $J_i$ as the minimal ideal sheaf containing $J + I^i$, which defines a LCM curve of $X$. Since $J_i$ is obtained by removing all the embedded components of $J + I^i$, we infer that both considered ideals are [A generically equal. We have $J_i = I$ and $J_i = J$ for $i \geq t + 1$ where $t + 1$ is the least $i$ such that $J \supset I^i$. Moreover $J_i \supset J_{i+1}$ for $i \geq 1$.  

We claim that $J_i \cdot J_j \subset J_{i+j}$. This is of course true in all the points of $C$ where $J_i = J + I^i$. So the ideal $J_i \cdot J_j + J_{i+j}/J_{i+j} \subset O_X/J_{i+j}$ has a zero-dimensional support. Since $O_X/J_{i+j}$ is LCM, this ideal is zero and therefore
\( J_i \cdot J_j \subset J_{i+j} \). In particular \( I J_i \subset J_{i+1} \). So for all \( i \geq 1 \), \( J_i/J_{i+1} \) is a sheaf of \( O_X/I \)-modules. Since \( O_X/J_{i+1} \) is ICM and \( C \) is smooth, the \( O_X/I \)-modules \( J_i/J_{i+1} \) are locally free.

The multiplication map \( J_i \times J_j \to J_{i+j} \) induces a generically surjective map \( E_i \otimes E_j \to E_{i+j} \) where \( E_i = J_i/J_{i+1} \). In particular, one has morphisms \( E_i \otimes E_j \to E_i \), which are also generically surjective.

**Definition.** With the above notation, a multiple structure on \( C \subset X \) is called quasi-primitive if \( \text{rank} E_1 = 1 \). Note that this implies that \( \text{rank} E_i \leq 1 \) for \( i \geq 1 \). Actually \( \text{rank} E_i = 1 \) for \( 1 \leq i \leq t \) and \( E_i = 0 \) for \( i > t \).

Now we can formulate our

**Theorem.** Let \( \overline{C} \) be a quasi-primitive multiple structure on \( C \subset X \). Then \( \overline{C} \) is lci (i.e., \( J \) is locally generated by 2 elements) if and only if the morphisms \( E_i \otimes E_{t-i} \to E_t \) are isomorphisms for \( 1 \leq i \leq t - 1 \).

**Remark.** If \( C \) is projective then \( E_i \otimes E_{t-i} \to E_t \) is an isomorphism if and only if \( \deg E_i + \deg E_{t-i} = \deg E_t \).

**Proof.** The problem is local. So we can suppose that \( O = O_X \) “is” a ring of a sufficiently small affine open neighborhood of a point of \( C \). Since \( C \) is smooth, the epimorphism \( O/J \to O/I \) whose kernel is the nilpotent ideal \( I/J \) splits. Therefore \( O/J \) admits a certain \( O/I \)-module structure and \( O/J \cong O/I \oplus I/J \) as \( O/J \)-modules.

For every \( i \geq 1 \), \( J_i/J \) becomes an \( O/I \)-submodule of \( O/J \). The exact sequences of \( O/I \)-modules \( 0 \to J_{i+1}/J \to J_i/J \to E_i \to 0 \) split since \( E_i \) is free. Therefore \( O/J \cong O/I \oplus J/I \cong O/I \oplus E_1 \oplus \cdots \oplus E_t \) as \( O/I \)-modules (\( J_i = J \) for \( i \geq t + 1 \)) \( O/I \oplus E_1 \oplus \cdots \oplus E_t \) carries a multiplicative structure that corresponds to the multiplication on \( O/J \). We have

\[
E_i \cdot E_j \subset \bigoplus_{i+j \leq k \leq t} E_k
\]

Moreover the composed maps

\[
E_i \otimes E_j \to E_i \cdot E_j \to \bigoplus_{i+j \leq k \leq t} E_k \to E_{i+j}
\]

coincide with the previously defined morphisms \( E_i \otimes E_j \to E_{i+j} \).

Suppose that \( J \) is lci. This implies that \( O/J \) is Gorenstein. Since \( O/J \) is a finite module extension of \( O/I \), there exists \( \pi' \in \text{Hom}_{O/I}(O/J, E_i) \) such that the homomorphism \( O/J \to \text{Hom}_{O/I}(O/J, E_i) \) induced by the bilinear form \( (,): O/J \times O/J \to E_i \) is an isomorphism [2]. Note that \( E_i \) is a rank 1 free \( O/I \)-module.

Let \( \pi: O/J \to O/I \) be the projection induced by the decomposition \( O/J \cong O/I \oplus E_1 \oplus \cdots \oplus E_t \). There exists \( s \in O/J \) such that \( \pi(r) = \pi'(sr) \) for all \( r \in O/J \). It is easy to see that \( s \) is invertible. Therefore we can assume that \( \pi' = \pi \).

Put \( E_0 = O/I \) and let \( \gamma \) denote the isomorphism \( \bigoplus_{0 \leq i \leq t} E_i \to \text{Hom}(\bigoplus_{0 \leq i \leq t} E_i, E_i) \) induced by the bilinear form \( (r, s) = \pi(r \cdot s) \). Let \( \gamma_{ij} \in \text{Hom}(E_i, \text{Hom}(E_j, E_i)) \) denote the \( (i, j) \)th entry of the corresponding matrix. The elements \( \gamma_{i,-i} \), \( 0 \leq i \leq t \), are on its second diagonal. The elements below
the second diagonal are zero since $E_i \cdot E_j = 0$ if $i + j > t$. It follows that $\gamma$ is an isomorphism if and only if $\gamma_{i, t-i}$ is an isomorphism for $0 \leq i \leq t$. So the morphisms $E_i \otimes E_{t-i} \to E_t$ are isomorphisms since they are induced by $\gamma_{i, t-i}$.

Suppose now that for $1 \leq i \leq t-1$ the maps $E_i \otimes E_{t-i} \to E_t$ are isomorphisms. Then $\gamma$ is an isomorphism since the maps $\gamma_0: E_0 \to \text{Hom}(E_t, E_t)$ and $\gamma_{t, 0}: E_t \to \text{Hom}(E_0, E_t)$ are obviously isomorphisms. To prove that $J$ is a lci it suffices to invoke that “in codimension 2 case Gorenstein implies complete intersection.”

REFERENCES
