RANGE TRANSFORMATIONS
ON A BANACH FUNCTION ALGEBRA. IV

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Abstract. Functions in $\text{Op}(I_D, \text{Re} \, A + L)$ are harmonic on $D$ for a closed subalgebra $A$ of $C_0(Y)$, an ideal $I$ of $A$ and a linear subspace $L$ of finite dimension in $C_{0, R}(Y)$ unless the uniform closure of $I$ is selfadjoint.

Introduction

Let $Y$ be a locally compact Hausdorff space, and $A$ be a uniformly closed subalgebra of $C_0(Y)$. We have shown in [5, Theorem 2] that functions in $\text{Op}(I_D, \text{Re} \, A)$ for certain ideals $I$ of a closed subalgebra $A$ of $C_0(Y)$, and a plane domain $D$ containing the origin, are harmonic near the origin. In this paper we show that the functions are harmonic on the whole of $D$ for more general range transformations. The proof of Theorem 2 in [5] depends on analysis of the behavior of functions near each point of $Y - \text{Ker} \, I$. In this paper we show that it is enough to analyze near the strong boundary points.

Notations and terminologies are essentially due to [5], while in this paper, $\| \cdot \|_{\infty(K)}$ denotes the uniform norm on a set $K$, which is simply denoted by $\| \cdot \|_{\infty}$ in [5].

Lemma 1. Let $X$ be a compact Hausdorff space, $A$ a closed subalgebra of $\{ f \in C(X) : f|K = 0 \}$ for a (possibly empty) compact subset $K$ of $X$ which satisfies that for any pair of different points $x$ in $X - K$ and $y$ in $X$, there exists $f$ in $A$ with $f(x) \neq f(y)$. Suppose that each $x$ in the Choquet boundary $\text{Ch}(A)$ for $A$, with at most finitely many exceptions, has a compact neighborhood $G_x$ such that $A|G_x = C(G_x)$. Then $A$ coincides with $\{ f \in C(X) : f|K = 0 \}$.

Proof. We may assume $K \neq X$. If $K$ is empty, then we add $\infty$ as an isolated point and $X'$ denotes $X \cup \{ \infty \}$ and $A'$ denotes $\{ f \in C(X') : f(\infty) = 0, f|X \in A \}$. If $K$ is not empty, then $X'$ denotes the quotient space of $X$ obtained by identifying points in $K$, and we denote the point in $X'$ which corresponds to $K$ by $\infty$. We may suppose that $A$ is a closed subalgebra of $\{ f \in C(X') : f(\infty) = 0 \}$. We write $A'$ instead of $A$ if we view $A$ as an algebra on $X'$. In any case $A'$ is a point separating uniformly closed subalgebra of $\{ f \in C(X') : f(\infty) = 0 \}$.
Let $B$ denote the function algebra \{ $f \in C(X') : f = g + c$ for $g \in A'$ and a constant function $c$ on $X'$ \}. If $B$ coincides with $C(X')$, then we see that $A' = \{ f \in C(X') : f(\infty) = 0 \}$, in particular, $A = \{ f \in C(X) : f|K = 0 \}$. Thus we will show that $B = C(X')$. First we show that the family of peak sets for $A'$ coincides with the family of peak sets for $B$ which do not contain $\infty$. Every peak set for $A'$ is trivially a peak set for $B$ without $\infty$, so we prove the inverse implication. Let $P$ be a peak set for $B$ with $\infty$ off $P$ and $g$ a peaking function for $P$, that is, $g$ is a function in $B$ with $\|g\|_{\infty}(X') = 1$ such that $g = 1$ on $P$ and $|g| < 1$ off $P$. Without loss of generality we may suppose that $0 \leq |g(x)| < \frac{1}{2}$. Put $h = (g - g(\infty))/(1 - g(\infty))$. We see that $h$ is a function in $A'$ with $h = 1$ on $P$ and $\|h\|_{\infty} \leq 2$. For every nonnegative integer $n$ put $G_n = \{ x \in X' : |h(x)| \leq 1 + 2^{-n} \}$. Then $G_n$ is a compact neighborhood of $P$ with $G_n \supset G_{n+1}$ for every $n$. Since each Int$G_n$ includes $P$, we have $\sup \{|g(x)| : x \in X' - G_n\} = \delta_n < 1$ for every positive integer $n$. Choose positive integers $m(n)$ with $\delta_n^{m(n)} < 2^{-(n+1)}$. We see that $f_N = \sum_{n=1}^{N} 2^{-n} h^{m(n)}$ is in $A'$ for every positive integer $N$ since $B \cdot A' \subset A'$ and that $f_N$ converges to a peaking function $f$ for $P$ in $A'$. We conclude that the two families of peak sets coincide. It follows that $\text{Ch}(B) = \{ \infty \} = \text{Ch}(A')$. We may suppose that $\text{Ch}(A) = \text{Ch}(A')$. Let $\{X_\alpha\}$ be the maximal antisymmetric decomposition of $X'$ for $B$. We see that $\text{Ch}(B|X_\alpha) = \text{Ch}(B) \cap X_\alpha$ for every $X_\alpha$ since $X_\alpha$ is an intersection of peak sets. We show that every $X_\alpha$ is singleton. Suppose not. Choose $X_\alpha$ which is not singleton. Then $X_\alpha$ contains an infinite number of points in $\text{Ch}(B|X_\alpha)$. By the condition on $\text{Ch}(A)$ there is a point $x$ in $\text{Ch}(B|X_\alpha)$ which has a compact neighborhood $G_x$ in $X'$ with $B|G_x \subset A|G_x = C(G_x)$. Thus $X_\alpha \cap G_x$ is an interpolating compact neighborhood of $x$ for $B|X_\alpha$. It follows by Lemma 2.5 in [2] that $X_\alpha$ is a single point, which is a contradiction. Thus we see that every $X_\alpha$ is a finite set, so every $X_\alpha$ is a one point set. We conclude that $B = C(X')$ by Bishop’s theorem.

Lemma 2. Let $F$ be a compact Hausdorff space and $Q$ a compact subset (possibly empty) of $F$. Let $S$ be a Banach algebra contained in $\{ f \in C(F) : f|Q = 0 \}$ with the norm $\| \cdot \|_S$ such that $S$ separates a point $x$ in $F$ and a point $y$ in $F - Q$ if $x \neq y$. Suppose that $Q = \{ x \in F : f(x) = 0 \text{ for every } f \text{ in } S \}$. Suppose that $\tilde{S}^\Lambda$ (with respect to $\| \cdot \|_S$) separates the different points in $F_x^\Lambda$ for every $x \in F - Q$ and a discrete space $\Lambda$ with cardinality not less than that of an open base for the topology of $Y$ at $x$. Let $\varepsilon$ be a positive number and $h_0$ a continuous real valued function on $\{ z \in C : |z| \leq \varepsilon \}$ which is not harmonic on any open neighborhood of the origin. Let $T$ be a Banach space with the norm $\| \cdot \|_T$ continuously embedded in $C_R(F)$, that is, the identity map from $T$ into $C_R(F)$ is continuous. Let $S_0$ be a dense (with respect to the topology induced by $\| \cdot \|_S$) subset of $\{ f \in S : \|f\|_S < \delta \}$ for a positive number $\delta$ with $\delta < \varepsilon$. Suppose that the function $h_0(f)$ is in $T$ for every $f \in S_0$. Then for every point $x$ in $F - Q$ there exists a compact neighborhood $G$ of $x$ with $G \subset F - Q$ which satisfies $\{ f \in T : f|Q = 0 \} \mid G = C_R(G)$.

Proof. We denote $T_0 = \{ f \in T : f|Q = 0 \}$ and $T_1 = \{ f \in T : f|Q \text{ is constant} \}$ with the norm $\| \cdot \|_T$ restricted to $T_0$ and $T_1$ respectively. We will
show that \( \text{cl}(\tilde{T}_0^A|F_x^A) = C_R(F_x^A) \) for a point \( x \) in \( F - Q \) and a discrete space \( A \) with cardinality not less than that of an open base for the topology of \( Y \) at \( x \). It follows that \( T_0|G = C_R(G) \) for a compact neighborhood \( G \) of \( x \) with \( G \subset F - Q \) by (2) of Theorem 1 in [5]. (The theorem is stated for a Banach space included (see [5, p. 89] for the definition) in \( C_0(Y) \) or \( C_{0,r}(Y) \) in [5]. Every Banach space continuously embedded in \( C_0(Y) \) or \( C_{0,r}(Y) \) has a norm equivalent to that of a Banach space included in \( C_0(Y) \) or \( C_{0,r}(Y) \). So the theorem is also true for Banach spaces continuously embedded in \( C_0(Y) \) or \( C_{0,r}(Y) \).) Take a point \( x \) in \( F - Q \). Then there is \( f \) in \( S \) with \( f(x) = 1 \). For some \( \eta \) with \( 0 < \eta < e/(2||f||_s) \), and some smoothing operator \( \sigma_\eta(w) \) of class \( C^\infty \) supported in \( \{ z \in C : |z| < \eta \} \), we have \( \Delta_1(h_\eta(0, 1)) \neq 0 \) (see [3, p. 566; 4, pp. 634, 635; 5, p. 115], where

\[
    h_\eta(z_1, z_2) = \int \int h_0(z_1 - z_2 w)\sigma_\eta(w)dxdy.
\]

Choose a discrete space \( A \) with cardinality not less than that of an open base for the topology of \( Y \) at \( x \). Let \( \tilde{g} \) be a function in \( \tilde{S}^A \). For a complex number \( \beta \) with sufficiently small absolute value and a complex number \( w \) with \( |w| < \eta \),

\[
    h_0(\langle f \rangle^2 \beta - \langle f \rangle w) \in \text{cl} \tilde{T}_1^A \quad \text{since} \quad S_0 \quad \text{is dense (with respect to the topology induced by} \| \cdot \|_S) \quad \text{in} \{ f \in S : \| f \|_S \leq \delta \} \quad \text{and} \quad \| \cdot \|_\infty(\tilde{F}_A) \leq \| \cdot \|_\infty(\bar{S}^A). \]

Thus we have by Lemma 5 in [4], that

\[
    |\tilde{g}|^2 \Delta_1(h_\eta(0, \langle f \rangle) \cdot |\langle f \rangle|^4) \in \{ \tilde{\psi} \in \text{cl} \tilde{T}_1^A : \tilde{\psi}|\tilde{Q}^A = 0 \}
\]

since \( \tilde{g} = 0 \) on \( \tilde{Q}^A \). We show that \( \tilde{T}_0^A = \{ \tilde{\psi} \in \text{cl} \tilde{T}_1^A : \tilde{\psi}|\tilde{Q}^A = 0 \} \). When \( T_1 = T_0 \) it is trivial. Suppose that \( T_1 - T_0 \neq \phi \). Let \( \tilde{\psi} = \langle \psi_\lambda \rangle \) be in \( \tilde{T}_1^A \) with \( \tilde{\psi}|\tilde{Q}^A = 0 \). For every \( \xi > 0 \) there is \( \langle \psi_\lambda^{(\xi)} \rangle \) in \( \tilde{T}_1^A \) such that

\[
    \| \langle \psi_\lambda \rangle - \langle \psi_\lambda^{(\xi)} \rangle \|_\infty(\bar{F}_A) < \xi.
\]

Let \( \psi \) be a function in \( T_1 - T_0 \). We may assume that \( \psi|Q = 1 \). So

\[
    \langle \psi_\lambda^{(\xi)}(Q) \psi \rangle \in \tilde{T}_0^A
\]

and

\[
    \| \langle \psi_\lambda \rangle - \langle \psi_\lambda^{(\xi)}(Q) \psi \rangle \|_\infty(\bar{F}_A) < \xi(1 + \| \psi \|_{(F)}),
\]

since \( \psi_\lambda^{(\xi)}(Q) \) is a constant with \( |\psi_\lambda^{(\xi)}(Q)| \leq \| \langle \psi_\lambda \rangle - \langle \psi_\lambda^{(\xi)} \rangle \|_\infty(\bar{F}_A) \). Thus the conclusion follows. We see that \( |\tilde{g}|^2 \Delta_1(h_\eta(0, \langle f \rangle) \cdot |\langle f \rangle|^4) \) is in \( \text{cl} \tilde{T}_0^A \), so we have

\[
    |\tilde{g}|^2 \Delta_1(h_\eta(0, 1))|F_x^A \in \text{cl} \tilde{T}_0^A|F_x^A \subset \text{cl}(\tilde{T}_0^A|F_x^A),
\]

since \( \langle f \rangle = 1 \) on \( F_x^A \) by Lemma 4 in [5]. So

\[
    |\tilde{g}|^2|F_x^A \in \text{cl}(\tilde{T}_0^A|F_x^A) \quad \text{since} \quad \Delta_1(h_\eta(0, 1)) \neq 0.
\]

It follows that the algebra generated by \( |\tilde{g}|^2|F_x^A \) for \( \tilde{g} \in \tilde{S}^A \) is contained in \( \text{cl}(\tilde{T}_0^A|F_x^A) \) since \( \tilde{S}^A \) is an algebra. Let \( p \) and \( q \) be different points in \( F_x^A \). There is \( \tilde{u} \) in \( \tilde{S}^A \) with \( \tilde{u}(p) \neq \tilde{u}(q) \) and there is \( v \) in \( S \) with \( u(x) = \tilde{u}(p) \). So \( \tilde{u} - \langle v \rangle \) is in \( \tilde{S}^A \) and \( |\tilde{u} - \langle v \rangle|^2 \) separates \( p \) and \( q \) since \( \langle v \rangle(p) = \langle v \rangle(q) = v(x) \) by Lemma 4 in [5]. It follows by the Stone-Weierstrass theorem that \( \text{cl}(\tilde{T}_0^A|F_x^A) = C_R(F_x^A) \).

**Lemma 3.** Let \( E \) be a complex (resp. real) Banach space continuously embedded in \( C(X) \) (resp. \( C_R(X) \)), and let \( L \) be a finite-dimensional complex (resp. real) subspace of \( C(X) \) (resp. \( C_R(X) \)) such that \( E \cap L = \{ 0 \} \). Let \( A \) be a discrete space. Then \( E + L \) is a Banach space with respect to the norm defined by

\[
    \| u + v \|_{E + L} = \| u \|_E + \| v \|_\infty,
\]
where \( \| \cdot \|_E \) is the norm on \( E \). Suppose that \( x \) is a point in \( X \) with \( E_x \neq E \). Then

\[
(E + L)^\sim A |F_x^A = \tilde{E}_x^A |F_x^A.
\]

**Proof.** It is trivial that \( \| \cdot \|_{E+L} \) is a complete norm since \( L \) is finite dimensional. \((E + L)^\sim A \supset \tilde{E}_x^A \) is also trivial, so \((E + L)^\sim A |F_x^A \supset \tilde{E}_x^A |F_x^A \). We prove the inverse inclusion. Suppose that \( \{k_1, \ldots, k_n\} \) is a base for \( L \). Let \( \langle u_\lambda + v_\lambda \rangle \in (E + L)^\sim A \), where \( u_\lambda \in E \) and \( v_\lambda = \sum_{i=1}^n a_i^{(k)} k_i \). Then \( \langle u_\lambda \rangle \in \tilde{E}_x^A \) and \( \sup \|v_\lambda\|_{\infty(X)} < \infty \) by the definition of \( \| \cdot \|_{E+L} \). Let \( u \) be a function in \( E \) with \( u(x) = 1 \). Since the linear functional defined by \( \sum_{i=1}^n a_i k_i \to a_j \), is continuous for every \( j \), we see that \( \langle u_\lambda \rangle + \sum_{i=1}^n (a_i^{(k)}) k_i \) is in \( \tilde{E}_x^A \). We see that

\[
\langle u_\lambda \rangle |F_x^A + \sum_{i=1}^n (a_i^{(k)}) k_i |F_x^A = \langle u_\lambda \rangle |F_x^A + \sum_{i=1}^n (a_i^{(k)}) k_i |F_x^A
\]

by Lemma 4 of [5].

**Lemma 4.** Let \( E \) be an ultraseparating complex (resp. real) Banach space continuously embedded in \( C(X) \) (resp. \( C_0(X) \)) for a compact Hausdorff space \( X \), i.e., \( \| f \|_E \leq M \| f \|_E \) for every \( f \) in \( E \). Let \( x \) be a point in \( X \). Let \( \Lambda \) and \( \Lambda' \) be infinite discrete spaces. If \( p \in F_x^A \) and \( q \in F_x^A - \{(x) \times \Lambda \} \) are different points, then \( \tilde{E}_x^A \) separates \( p \) and \( q \). If \( q \in F_x^A - \{(x) \times \Lambda \} \), then \( (\tilde{E}_x^A)^\sim A' \) separates the different points in \( F_q^{A'} \).

**Proof.** If \( p \in \{(x) \times \Lambda \} \) and \( q \in F_x^A - \{(x) \times \Lambda \} \), then there is \( g \in \tilde{E}_x^A \) such that \( g(q) \neq g(p) = 0 \) by Proposition 2 in [5]. Suppose that \( p \) and \( q \) are different points in \( F_x^A - \{(x) \times \Lambda \} \). Since \( E \) is ultraseparating, there is \( \tilde{f} = \langle f_\lambda \rangle \) in \( \tilde{E}_x^A \) with \( \tilde{f}(p) \neq \tilde{f}(q) \). Put \( h_\lambda = f_\lambda - f_\lambda(x) \cdot u \), where \( u \in E \) with \( u(x) = 1 \). (Such a function \( u \) exists since \( \tilde{E}_x^A \) separates \( (x, \lambda) \) and \( (x, \lambda') \) for different \( \lambda \) and \( \lambda' \).) So \( \langle h_\lambda \rangle \in \tilde{E}_x^A \). Take \( g \in \tilde{E}_x^A \) with \( g(p) \neq 0 \). Then \( \langle h_\lambda \rangle \) or \( \langle f_\lambda \rangle \cdot g \) or \( \langle f_\lambda(x) \rangle \cdot g \) separates \( p \) and \( q \). Let \( q \) be a point in \( F_x^A - \{(x) \times \Lambda \} \). By Corollary 1 of [5], \( (\tilde{E}_x^A)^\sim A' \) separates the different points of \( (\tilde{X}^A)^\sim A' \), in particular, if \( \alpha \) and \( \beta \) are different points in \( F_q^{A'} \), then there is \( \tilde{f} = \langle (f_\lambda, \lambda) \rangle \in (\tilde{E}_x^A)^\sim A' \) with \( \tilde{f}(\alpha) \neq \tilde{f}(\beta) \). Put \( h_{\lambda, \lambda'} = f_{\lambda, \lambda'} - f_{\lambda, \lambda'}(x) \cdot u \), so \( \langle h_{\lambda, \lambda'} \rangle \in (\tilde{E}_x^A)^\sim A' \). Then \( \langle h_{\lambda, \lambda'} \rangle \) or \( \langle (f_{\lambda, \lambda'}(x)) \cdot g \rangle \) separates \( \alpha \) and \( \beta \) for \( g \in \tilde{E}_x^A \) with \( g(q) \neq 0 \).

**Theorem.** Let \( A \) be a uniformly closed subalgebra of \( C_0(Y) \) for a locally compact Hausdorff space \( Y \), and \( I \) be a subalgebra of \( A \) such that there are a finite number of subalgebras \( I_0, I_1, \ldots, I_n \) of \( A \) which satisfy the condition that \( I_k \) is an ideal of \( \text{cl} I_{k-1} \) for every \( k = 1, 2, \ldots, n \), where \( I_0 = A \) and \( I_n = I \). Let \( D \) be a plane domain containing the origin. Let \( L \) be a linear subspace of finite dimension in \( C_{0,r}(Y) \). Suppose that \( \text{Op}(I_D, \text{Re} A + L) \) contains a function which is not harmonic on \( D \). Then \( I|K \) is uniformly closed and selfadjoint for every compact subset \( K \) of \( Y - \text{Ker} I \) and \( \text{cl} I \) is selfadjoint.

**Remark.** If \( I \) is an ideal of \( A \), then the condition on \( I \) and \( A \) is satisfied with \( n = 1 \). An ideal of an ideal need not be an ideal. Let \( \text{Ad} \) be the disk
algebra on the closed unit disk $\mathcal{D}$. Let $I = \{ f \in A(\mathcal{D}) : f(0) = f'(0) = 0 \}$ and $J = \{ f \in I : f''(0) = 0 \}$. Then $I$ is an ideal of $A(\mathcal{D})$ and $J$ is an ideal of $I$ while $J$ does not satisfy the condition $J \cdot A(\mathcal{D}) \subset J$.

**Proof of Theorem.** The notations $\bar{Y}$, $\infty$, $\bar{Y}_1$, $p$, $\bar{Y}_0$, and $I'$ are the same as in the proof of Theorem 2 in [5]. If $Y$ is not compact, then $\bar{Y}$ denotes the one point compactification of $Y$ and $\infty$ denotes the point in $\bar{Y} - Y$. If $Y$ is compact, then we add $\infty$ as an isolated point and $\bar{Y}$ denotes $Y \cup \{ \infty \}$. We may suppose that $A$ is a closed subalgebra of $C(\bar{Y})$ such that $\langle f(\infty) = 0 \rangle$ for every $f \in A$. Let $\bar{Y}_1$ be the quotient space obtained by identifying the points in $\bar{Y}$ which cannot be separated by $A$. Let $\bar{Y}_0$ be the quotient space obtained by identifying the points in $\bar{Y}$ which cannot be separated by $I$. Let $p$ be the point in $\bar{Y}_0$ which corresponds to the equivalence class in $\bar{Y}$ containing $\infty$. We may suppose that $\bar{Y}_0$ is the quotient space obtained by identifying points in $\bar{Y}_1$ which cannot be separated by $I$ and that $p$ corresponds to $\text{Ker } I$. We may also suppose that each point in $\bar{Y}_0 - \{ p \}$ corresponds to a point in $\bar{Y}_1 - \text{Ker } I$, that is, we may suppose that $\bar{Y}_0 - \{ p \} = \bar{Y}_1 - \text{Ker } I$. Let $I' = cl I + C$ be the sum of the uniform closure of $I$ and the space of constant functions $C$. Then $I'$ is a function algebra on $\bar{Y}_0$. Let $\text{Ch}(I')$ be the Choquet boundary for $I'$. We consider two cases. They are different from those of the proof of Theorem 2 in [5]: (1) There is no accumulation point of $\text{Ch}(I')$ which is a point in $\text{Ch}(I')$, or $p$ is the only accumulation point of $\text{Ch}(I')$ which is a point in $\text{Ch}(I')$. (2) There is an accumulation point of $\text{Ch}(I')$ which is also a point in $\text{Ch}(I')$ and is not $p$.

**Case (1).** Let $\Gamma$ be the Shilov boundary for $I'$. We may suppose that $I'$ is a function algebra on $\Gamma$. We show that $\{ x \}$ is itself a compact neighborhood of $x$ for every $x$ in $\text{Ch}(I') - \{ p \}$. Since $x$ is not an accumulation point of $\text{Ch}(I')$, there is an open neighborhood $U_x$ in $\Gamma$ of $x$ with $U_x \cap \text{Ch}(I') = \{ x \}$. If a point $y$ is in $U_x - \{ x \}$, then $y$ is in $\Gamma$. So there is an open neighborhood $V_y$ in $\Gamma$ of $y$ such that $V_y \subset U_x$ and $x$ is not contained in $V_y$. Since $V_y \cap \text{Ch}(I') \subset U_x \cap \text{Ch}(I') = \{ x \}$ and $x$ is not contained in $V_y$ we have $V_y \cap \text{Ch}(I') = \emptyset$, which is a contradiction since $y \in \Gamma$ and $\text{Ch}(I')$ is dense in $\Gamma$. We conclude that $U_x = \{ x \}$, so $\{ x \}$ is a compact neighborhood of $x$. We see by Lemma 1 that $I'|\Gamma = C(\Gamma)$ since $I'|U_x = C(U_x)$ for every $x$ in $\text{Ch}(I') - \{ p \}$. It follows that $\bar{Y}_0 = \Gamma$ and $I' = C(\bar{Y}_0)$. The rest of the proof is the same as in case (1) of the proof of Theorem 2 in [5].

**Case (2).** Choose a point $q$ in $\text{Ch}(I')$ which is also an accumulation point of $\text{Ch}(I')$ other than $p$. Suppose that $h$ is a function in $\text{Op}(I_D, \text{Re } A + L)$ which is not harmonic on $D$. First we show that $h$ is continuous on $D$. Suppose not. There is a point $a$ in $D$ such that $h$ is not continuous at $a$. There is a function $k$ in $I$ such that $k(a) = d > 0$. $\|k\|_\infty \leq 1$. By the proof of Lemma 1, $\text{Ch}(I') - \{ p \} = \text{Ch}(cl I)$, so $q$ is a point in $\text{Ch}(cl I)$. Thus there is a function $u$ in $cl I$ with $u(q) = 1$, $\|u\|_\infty \leq 1$. Choose an analytic function $H$ defined on the open unit disk $\mathcal{D}$ with range in $D$, such that $H(0) = 0$ and $a \in H(\mathcal{D})$. Such a function exists since $D$ is connected. Let $re^{i\theta}$ be a point in $\mathcal{D}$ with $H(re^{i\theta}) = a$. So $H(re^{i\theta}u)$ is a function in $cl I$ such that $H(re^{i\theta}u)(q) = a$ and $H(re^{i\theta}u)(\bar{Y}_0)$ is a compact subset of $D$. In particular, there is a positive $\varepsilon$ with $\bigcup_{y \in \bar{Y}_0} \{ z \in C : |z - H(re^{i\theta}u)(y)| \leq 2\varepsilon \} \subset D$. There
is a function \( \psi \) in \( I \) with \( \psi(q) = 1 \). Choose a function \( \psi_1 \) in \( I \) with
\[
\|\psi_1 - H(re^{i\theta}u)\|_{\infty(Y_0)} < \varepsilon/(2\|\psi\|_{\infty(Y_0)}).
\]
Put
\[
\psi_q = \psi_1 - (\psi_1(q) - a)\psi,
\]
so \( \psi_q(q) = a \). For a point \( y \) in \( Y_0 \), suppose that \( z \) is a complex number with
\[
|z - \psi_q(y)| < \varepsilon.
\]
Then we have \( |z - H(re^{i\theta}u)(y)| \leq 2\varepsilon \). We conclude that
\[
\bigcup_{y \in Y_0} \{ z \in C : |z - \psi_q(y)| < \varepsilon \} \subset D.
\]

As in the proof of Theorem 2 in [5] we can choose a sequence \( \{q_n\} \) in \( \text{Ch}(I') \) with an accumulation point \( q''_0 \) and a function \( f \) in \( \text{cl} I \) such that
\[
h(fk + \psi_q) \in \text{Re} A + L
\]
and that \( h(fk + \psi_q(q_n)) \) does not converge to 
\( h(fk + \psi_q(q''_0)) \), which is a contradiction. Thus we may assume that \( h \) is continuous on \( D \) in case (2). Choose a point \( a \) in \( D \) such that \( h \) is not harmonic on any open neighborhood of \( a \). We show that for every \( x \) in \( \text{Ch}(\text{cl} I) \) there exists a compact neighborhood \( G_x \) of \( x \) with \( \text{cl} I \backslash G_x = C(G_x) \). If so, \( \text{cl} I \) is selfadjoint by Lemma 1. Let \( x \) be a point in \( \text{Ch}(\text{cl} I) \). Then there is a function \( \phi \) in \( I \) with \( \phi(x) = a \) and \( \phi(Y_0) \subset D \) in the same way as above. Let \( \phi(A) = 1 \) with \( \phi(x) = 1 \). Put \( f_0 = f''_{l+1} \). Since \( \text{cl} I_k \) is an ideal of \( \text{cl} I_{k-1} \) by the condition on \( I_k \) we have \( f^k_{l+1} \cdot A \subset \text{cl} I_k \). Thus \( f_0 \cdot A \subset I \) and \( f_0 \cdot A \) is a Banach algebra contained in \( C(Y_0) \) with respect to the norm defined by
\[
\|u\|_{f_0 \cdot A} = \inf\{\|f_0\|_{\infty} \cdot \|g\|_{\infty} : g \in A, u = f_0 g\}
\]
for \( u \) in \( f_0 \cdot A \). Without loss of generality we may assume \( (\text{Re} A) \cap L = \{0\} \). \( \text{Re} A + L \) is a Banach space with respect to the norm defined by
\[
\|u + v\|_{\text{Re} A + L} = \|u\|_{\text{Re} A} + \|v\|_{\infty}
\]
for every \( u \in \text{Re} A \) and \( v \in L \) by Lemma 3. We show that \( f_0 \cdot A \) is ultraseparating near \( x \). Let \( \varepsilon \) be a positive number such that \( d(\phi(Y_0), D^c) > \varepsilon \), where \( d(\cdot, \cdot) \) is the usual euclidian distance and \( D^c \) is the complement of \( D \) in the complex plane \( C \). Since \( h \) is not harmonic near \( a \), we see that
\[
|\Delta_1(h_\eta(z, z'))| \geq (1/2)|\Delta_1(h_\eta(a, 1))| \neq 0
\]
on \( \{ (z, z') \in C^2 : |z - a| < \varepsilon'' \}, \{ |z' - 1| < \varepsilon'' \} \) for a suitably chosen \( \eta \) with \( \varepsilon/(2\|f_0\|_{\infty}) > \eta > 0 \) and a suitably chosen smoothing operator \( \sigma_\eta \) and an \( \varepsilon > \varepsilon'' > 0 \), where
\[
h_\eta(z, z') = \iint h(z - z'w)\sigma_\eta(w)dx dy
\]
in the same way as in [4, pp. 634, 635]. Since \( L \) is finite dimensional, we have \( \text{cl}(\text{Re} A + L) = (\text{cl} \text{Re} A) + L \). So we see that
\[
h_\eta(\phi + g_1f_0^2t, f_0) \in (\text{cl} \text{Re} A) + L
\]
for \( g_1 \in A + C \) and a complex number \( t \) with sufficiently small absolute value. Thus by Lemma 5 in [4]
\[
|g_1|^2|\Delta_1(h_\eta(\phi, f_0) \cdot |f_0|^4) \in (\text{cl} \text{Re} A) + L.
\]
So by the Stone-Weierstrass theorem we see that
\[ C(\overline{Y}_1) \setminus A_1(h_\eta(\phi, f_0) \cdot |f_0|^4) \subset (\cl \Re A) + L. \]

Let \( G_x \) be a compact neighborhood of \( x \) such that
\[ G_x = \{ y \in \overline{Y}_0 : |\phi(y) - a| \leq \varepsilon''/2, |f_0(y) - 1| \leq \varepsilon''/2 \} , \]
so \( A_1(h_\eta(\phi, f_0) \cdot |f_0|^4) \) never take zero on \( G_x \). Thus we have
\[ C(G_x) = (\cl \Re A + L)|G_x|. \]

Let \( \Lambda \) be a discrete space with cardinality not less than that of an open base for the topology of \( Y \) at \( x \). By Lemma 3 we see that \( C(F^\Lambda_x) = (\cl \Re A)^\Lambda|G_x| \). Thus for different points \( p \) and \( q \) in \( F^\Lambda_x \) there is a function \( \langle u_\lambda \rangle \) in \( (\cl \Re A)^\Lambda \) with \( \langle u_\lambda \rangle(p) \neq \langle u_\lambda \rangle(q) \), in particular, we may suppose that \( u_\lambda \in \Re A \). Then there exists a \( \nu_\lambda \in \Re A \) for every \( \lambda \) such that \( u_\lambda + i\nu_\lambda \in A \), so \( \exp(u_\lambda + i\nu_\lambda) \in A \) and since \( \| \exp(u_\lambda + i\nu_\lambda) \|_{\infty(\overline{Y}_1)} = \| \exp u_\lambda \|_{\infty(\overline{Y}_1)} \) we see that \( \exp(u_\lambda + i\nu_\lambda) \in A^\Lambda \).

By Lemma 4 in [5] we see
\[ \langle f_0 \cdot \exp(u_\lambda + i\nu_\lambda) \rangle = \langle f_0 \rangle \cdot \langle \exp(u_\lambda + i\nu_\lambda) \rangle = \langle \exp(u_\lambda + i\nu_\lambda) \rangle \]
on \( F^\Lambda_x \). By the definition of \( \| \cdot \|_{f_0^\Lambda} \) we have
\[ \langle f_0 \cdot \exp(u_\lambda + i\nu_\lambda) \rangle \in (f_0 \cdot A)^\Lambda \]
and it separates \( p \) and \( q \). So by Theorem 1(1) of [5], \( f_0 \cdot A \) is ultraseparating near \( x \). Put \( B = \{ g \in f_0 \cdot A : g(x) = 0, \| g \|_{f_0^\Lambda} \leq \varepsilon \} \). In the same way as in the proof of Lemma 1.2 in [4] (cf. [5, Lemma 11]) we see the following.

There are positive integers \( n_0 \) and a real number \( \varepsilon' \) with \( 0 < \varepsilon' < \varepsilon \), and a function \( g_0 \) in \( B \) such that
\[ \{ g \in f_0 \cdot A : g(x) = 0, \| g - g_0 \|_{f_0^\Lambda} < \varepsilon' \} \subset B , \]
and there is a dense (with respect to the topology induced by the norm \( \| \cdot \|_{f_0^\Lambda} \)) subset \( U \) in \( \{ g \in f_0 \cdot A : g(x) = 0, \| g - g_0 \|_{f_0^\Lambda} < \varepsilon' \} \) which satisfies that for every \( g \) in \( U \) we have
\[ h(g + \phi) \in \Re A + L \text{ and } \| h(g + \phi) \|_{\Re A + L} < n_0 . \]

Let \( U_0 = \{ g \in f_0 \cdot A : g + g_0 \in U \} \). Then \( U_0 \) is dense (with respect to the topology induced by \( \| \cdot \|_{f_0^\Lambda} \)) in \( \{ g \in f_0 \cdot A : g(x) = 0, \| g \|_{f_0^\Lambda} \leq \varepsilon' \} \). Put \( F = F^\Lambda_x, S = (f_0 \cdot A_x)^\Lambda|F^\Lambda_x, S_0 = \overline{U}_0^\Lambda|F^\Lambda_x, Q = \{ \{ x \} \times \Lambda \}, h_0(z) = h(z + a), \delta = \varepsilon' , \)
and \( T = \Re A^\Lambda|F^\Lambda_x \). Then the conditions of Lemma 2 are satisfied. We check this. \( F \) is trivially a compact Hausdorff space and \( Q \) is a compact subset of \( F \), and \( S \) is a Banach algebra contained in \( \{ \hat{f} \in C(F) : \hat{f}(Q) = 0 \} \). We see that the separation and ultraseparation conditions on \( S \) are satisfied by Lemma 4. We see that \( Q = \{ p \in F : \hat{f}(p) = 0 \text{ for every } \hat{f} \in S \} \) by Proposition 2 in [5]. Let \( \langle g_\lambda \rangle \in \overline{U}_0^\Lambda \). Then \( h(\langle g_\lambda \rangle + \langle g_0 \rangle + \langle \phi \rangle) \) is in \( (\Re A + L)^\Lambda \) by the definition of \( U_0 \). Thus we see that \( h(\hat{g} + a) \) is in \( \Re A^\Lambda|F^\Lambda_x \) for every \( \hat{g} \) in \( \overline{U}_0^\Lambda|F^\Lambda_x \) by Lemma 4 in [5] and Lemma 3. Equivalently \( h_0(\hat{g}) \in T \) for every \( \hat{g} \in S_0 \). Thus by Lemma 2 we see that for every point \( p \) in \( F^\Lambda_x - \{ \{ x \} \times \Lambda \} \) there is a compact neighborhood \( O_p \) of \( p \) in \( F^\Lambda_x \) with \( O_p \subset F^\Lambda_x - \{ \{ x \} \times \Lambda \} \) such that
\[ \{ \hat{u} \in \Re A^\Lambda|F^\Lambda_x : \hat{u}[(\{ x \} \times \Lambda) = 0] \} O_p = C_R(O_p) . \]
We will see that
\[ \{ \hat{u} \in \text{Re} \tilde{A}_x^\Lambda | F_x^\Lambda : \hat{u}\{\{x\} \times \Lambda\} = 0\} = \text{Re} \tilde{A}_x^\Lambda | F_x^\Lambda. \]
\[ \{ \hat{u} \in \text{Re} \tilde{A}_x^\Lambda | F_x^\Lambda : \hat{u}\{\{x\} \times \Lambda\} = 0\} \supset \text{Re} \tilde{A}_x^\Lambda | F_x^\Lambda \]
is trivial. We show the inverse inclusion. If \( \hat{u} \in \text{Re} \tilde{A}_x^\Lambda | F_x^\Lambda \) with \( \hat{u}\{\{x\} \times \Lambda\} = 0 \), then there is \( \langle u_\lambda + iv_\lambda \rangle \)
in \( \tilde{A}_x^\Lambda \) such that \( \langle u_\lambda \rangle | F_x^\Lambda = \hat{u} \). Put \( \langle u_\lambda + iv_\lambda \rangle - \langle iv_\lambda(x) \cdot f_0 \rangle \in A_x^\Lambda \). Since \( u_\lambda + iv_\lambda - iv_\lambda(x) \cdot f_0 \in A_x \), it follows that \( \langle u_\lambda + iv_\lambda \rangle - \langle iv_\lambda(x) \cdot f_0 \rangle \in \tilde{A}_x^\Lambda \) and \( \text{Re}(\langle u_\lambda + iv_\lambda \rangle - \langle iv_\lambda(x) \cdot f_0 \rangle)|F_x^\Lambda = \hat{u} \) by Lemma 4 in [5]. Thus we have
\[ \text{Re}(\tilde{A}_x^\Lambda)|O_p = C_R(O_p). \]

We may suppose that \( \text{cl}(\tilde{A}_x^\Lambda | F_x^\Lambda) + C \) is a function algebra on \( _0F_x^\Lambda \) (\( _0F_x^\Lambda \) is the quotient space of \( F_x^\Lambda \) obtained by identifying the points in \( \{\{x\} \times \Lambda\} \) by Proposition 2 in [5] since \( A \) is ultraseparating near \( x \). By a theorem of Hoffman and Wermer, and Bernard [1, 6] on the uniformly closed real part of a Banach function algebra we see that
\[ \text{cl}(\tilde{A}_x^\Lambda | F_x^\Lambda) + C|O_p = C(O_p), \]
so by Corollary 2.13 in [2] we see that
\[ \text{cl}(\tilde{A}_x^\Lambda | F_x^\Lambda) + C = C(_0F_x^\Lambda). \]
It follows that
\[ \text{cl}(\tilde{A}_x^\Lambda | F_x^\Lambda) = C(F_x^\Lambda). \]

We see that \( A|G = C(G) \) for a compact neighborhood \( G \) of \( x \) by Theorem 1 (2) in [5]. It follows that \( I|G' = C(G') \) for a compact neighborhood \( G' \) of \( x \) in \( \overline{Y_0} - \{ p \} \). The results follow.

As in Corollary 1.1 in [4] we see the following.

**Corollary.** Let \( A \) and \( I \) and \( L \) be the same as in the theorem. Let \( S \) be an interval of the real line. Suppose that one of the following holds.

1. \( \text{Op}(I_D, A) \) contains a nonanalytic function on \( D \).
2. \( \text{Op}(I_S, \text{Re} A + L) \) contains a nonaffine function on \( S \).

Then \( I|K \) is uniformly closed and selfadjoint for every compact subset \( K \) of \( Y - \text{Ker} I \) and \( \text{cl} I \) is selfadjoint.

**References**


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