THE DUNFORD-PETTIS PROPERTY IN THE PREDUAL OF A VON NEUMANN ALGEBRA

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Abstract. The von-Neumann algebras whose predual has the Dunford-Pettis property are characterised as being Type I finite. This answers the question asked by Chu and Iochum in The Dunford Pettis property in C*-algebras, Studia Math. 97 (1990), 59–64.

The purpose of this paper is to characterise those von Neumann algebras $M$ for which the predual $M^*$ has the Dunford-Pettis property thereby settling the question raised in [1].

A Banach space $X$ is said to have the Dunford-Pettis property if for each Banach space $Y$ each weakly compact linear operator from $X$ to $Y$ sends weakly convergent sequences to norm convergent sequences. Classically, all $L^1$ spaces (Dunford and Pettis) and all $C(X)$ spaces (Gröthendieck) have the Dunford-Pettis property (see [2] for much more). A thorough study of the Dunford-Pettis property in C*-algebras and von Neumann algebras was undertaken in [1] and [3], to which we refer the reader for any unmentioned details. But one question remained unanswered. It was proved in [1] and [3] that if $M$ is a von Neumann algebra then

(a) if $M^*$ has the Dunford-Pettis property then $M$ is finite;
(b) if $M$ is Type I finite then $M^*$ has the Dunford-Pettis property.

An obstacle to a characterisation was that it was not known whether it was possible for the predual of a Type II$_1$ von Neumann algebra to have the Dunford-Pettis property. We show that is it not possible. Thus we establish the following.

Theorem. The following are equivalent for a von Neumann algebra $M$.

(i) $M^*$ has the Dunford-Pettis property.
(ii) $M$ is Type I finite.

Proof. In order to establish the validity of the converse of (b) let $M$ be a von Neumann algebra for which $M^*$ has the Dunford-Pettis property. In view of (a), and since the predual of every summand of $M$ clearly inherits the property, it can be supposed that $M$ is of Type II$_1$. At this point we appeal to the Jordan operator theory of spin factors contained in [4–6]. Since $M$ contains a...
Type II₁ subfactor it certainly contains a countably infinite spin system \( \{ s_n \} \), consisting of nontrivial symmetries \( s_n \) (in \( M_{sa} \)) satisfying \( s_n s_m + s_m s_n = 0 \) whenever \( n \neq m \). The real Banach subspace \( V \) of \( M_{sa} \) generated by \( \{ s_n \} \) is a JW-subalgebra of \( M_{sa} \) (called a spin factor). In addition, \( V \) is an infinite dimensional real Hilbert space in an equivalent norm. Evidently, then, there exists a sequence \( (x_n) \) in \( V \) such that \( ||x_n|| = 1 \) for all \( n \) and \( x_n \to 0 \) in the \( \sigma(V, V^*) \) topology. Clearly \( x_n \to 0 \) in the \( \sigma(M, M^*) \) topology. But as \( M_* \) has the Dunford-Pettis property this means that \( x_n^2 \to 0 \) in the \( \sigma(M, M^*) \) topology, by [3, Lemma 1] or [1, Corollary 5]. Hence \( x_n \to 0 \) in the strong operator topology. But as proved in [5, Theorem 7.1] the latter coincides on \( V \) with the norm topology. So \( x_n \to 0 \) in norm, a contradiction which completes the proof.

**References**


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