A NOTE ON A COMMON FIXED POINT THEOREM
OF BRODSKII AND MILMAN AND A LEMMA OF DAY

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Abstract. We should show that we may use a lemma of Day to prove, among others, a generalization of a common fixed point theorem of Brodskii and Milman when restricted to normed linear spaces that are uniformly convex in every direction.

Introduction

Throughout this note $N$ will denote a normed linear space with dual $N^*$. The norms in $N$ and $N^*$ are denoted by $\| \cdot \|$.

We denote by $I$ the identity mapping of $N$; by $T: D(T) \subseteq N \to N$ a mapping $T$ with domain $D(T) \subseteq N$ into $N$ and with range of $T$, $R(T) = \{ T(u) : u \in D(T) \}$; and for scalars $r$ and $s$, by $(rI + sT)$ the mapping with domain $D(T)$ such that $(rI + sT)u = rIu + sTu$ for $u \in D(T)$, where $Tu = T(u)$. For scalar $r$, subsets $R, S \subseteq N$, $rS$ is the set of all $ry$ with $y \in S$ and $R + S$ denotes the set of all $x + y$ with $x \in R$ and $y \in S$. We also write $y + S$ for $\{ y \} + S$.

Let $T: D(T) \subseteq N \to N$. $T$ is said to be nonexpansive if $\| Tu - Tv \| \leq \| u - v \|$ for all $u, v \in D(T)$; noncontractive if $\| Tu - Tv \| \geq \| u - v \|$ for all $u, v \in D(T)$; pseudocontractive if $\|(1 + r)I - rT\)u - ((1 + r)I - rT)v\| \geq \| u - v \|$ for all $u, v \in D(T)$ and all $r > 0$; accretive if $\|(I + rT)u - (I + rT)v\| \geq \| u - v \|$ for all $u, v \in D(T)$ and all $r > 0$.

Remark 1. It is known that a single-valued nonexpansive map is pseudocontractive.

The importance of pseudocontractive mappings is established by the following characterization in Browder [2]:

$T: D(T) \subseteq N \to N$ is pseudocontractive if and only if $I - T$ is accretive.

In Kato [7] a multiple-valued mapping $T: D(T) \subseteq N \to N$, that is, $Tu$ is a subset of $N$ for each $u \in D(T)$, is called accretive if for each $r > 0$ and $u, v \in D(T)$,

$$\| x - y \| \leq \| u - v \|$$

whenever $x \in (I + rT)u$, $y \in (I + rT)v$.

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If $T: D(T) \subseteq N \to N$ is multiple-valued and we define $T$ to be pseudocontractive if for each $r > 0$ and $u, v \in D(T)$,

$$\|x - y\| \geq \|u - v\|$$

whenever $x \in ((1 + r)I - rT)u$, $y \in ((1 + r)I - rT)v$,

then Browder’s characterization of single-valued pseudocontractive mapping still holds for a multiple-valued mapping.

**Proposition.** Let $T: D(T) \subseteq N \to N$ be multiple-valued. $T$ is pseudocontractive if and only if $I - T$ is accretive.

**Proof.** Given $r > 0; u, v \in D(T); w \in Tu$, and $z \in Tv$, we have

$$\|((1 + r)u - rw) - ((1 + r)v - rz)\| = \|(u + r(u - w)) - (v + r(v - z))\|$$

from which the truth of the proposition is easily deduced.

If $T$ is multiple-valued and pseudocontractive, then $((1 + r)I - rT)u$ are disjoint for different $u$ and all $r > 0$, so we can define, for each $r > 0$, a single-valued mapping $J_r = ((1 + r)I - rT)^{-1}$, with $D(J_r) = R((1 + r)I - rT)$ and $R(J_r) = D(T)$, by $J_rx = u$ if and only if $x \in ((1 + r)I - rT)u$.

**Lemma 1.** For each $r > 0$,

(i) $J_r$ is nonexpansive;

(ii) $J_r$ and $T$ have the same fixed point(s).

**Proof.** (i) This follows directly from the definitions of pseudocontractive mapping and $J_r$.

(ii) From the definition of $J_r$, we have

$$J_ru = u \quad \text{if and only if} \quad u \in ((1 + r)I - rT)u.$$ 

Since $u \in Tu$ if and only if $u \in ((1 + r)I - rT)u$, (ii) follows.

**Remark 2.** The definition of a multiple-valued accretive mapping that we use here is that of Kato [7], where it is shown that if $N$ is a real Banach space then $T$ is accretive if and only if for each $u, v \in D(T)$ and each $x \in Tu$ and $y \in Tv$, there exists $f \in F(u - v)$ such that $(x - y, f) \geq 0$. Here $F$ is the duality map of $N$ into $N^*$; it is by definition the unique multiple-valued mapping from $N$ into $N^*$ with domain $D(F) = N$ such that $f \in Fx$ if and only if $(x, f) = \|x\|^2 = \|f\|^2$, where $(x, f)$ denotes the pairing between $x \in N$ and $f \in N^*$.

Let $N$ and $M$ be real Banach spaces, $M^*$ the conjugate space of $M$. Let $\phi$ be a mapping of $N$ into $M^*$ such that $\phi(N)$ is dense in $M^*$ with $\|\phi(u)\|_{M^*} = \|u\|_N$, $\phi(ru) = r\phi(u)$, for all $u$ in $N$, $r \geq 0$.

Browder [3] introduced $\phi$-accretive mappings, generalizing the concept of a monotone mapping from $N$ to $N^*$ and of an accretive mapping from $N$ to $N$.

In Browder [3] a map $f$ of $N$ into $M$ is said to be strongly $\phi$-accretive if there exists $c > 0$ such that for all $u$ and $v$ in $N$, $(f(u) - f(v), \phi(u - v)) \geq c\|u - v\|^2$.

It is also mentioned there that similar definitions may be formulated for maps of $N$ into $2^M$, and in particular for single-valued mappings $f$ defined only on a subset $D(f)$ of $N$.

Let $f$ be a multiple-valued mapping from a subset of $N$ into $M$, that is, $f: D(f) \subseteq N \to 2^M$. The map $f$ is said to be strongly weak-$\phi$-accretive with
constant \( c \) if there exists \( c > 0 \) such that for all \( u, v \) in \( N \) and \( x \) in \( f(u) \) and \( y \) in \( f(v) \),
\[
(y - x, \phi(v - u)) \geq c\|v - u\|^2,
\]
where \( \phi \) is only required to satisfy \( \|\phi(u)\|_{\mathcal{M}} \leq \|u\|_N \) for all \( u \) in \( N \).

\[
c\|v - u\|^2 \leq (y - x, \phi(v - u)) \leq \|y - x\| \cdot \|\phi(v - u)\|
\]
\[
\leq \|y - x\| \cdot \|v - u\|
\]
so \( f(u) \) are disjoint for different \( u \), and we can define a single-valued mapping \( f^{-1} \), with \( D(f^{-1}) = R(f) \) and \( R(f^{-1}) = D(f) \), by \( f^{-1}(x) = u \) if and only if \( x \in f(u) \).

**Lemma 2.** Let \( f: D(f) \subseteq N \to M \) be a multiple-valued strongly weak-\( \phi \)-accretive mapping with constant \( c \). If \( c \geq 1 \), then

(i) \( f^{-1} \) is nonexpansive;

(ii) \( f^{-1} \) and \( f \) have the same fixed point(s).

**Proof.** (i) and (ii) follow directly from the definitions of \( f^{-1} \) and strongly weak \( \phi \)-accretive mapping with constant \( c \).

The radius \( R_p(A) \) of a bounded set \( A \subseteq N \) from a point \( p \in N \) is \( \sup\{\|p - x\|: x \in A\} \). The diameter of \( A \), \( \text{diam } A \), is \( \sup\{\|x - y\|: x, y \in A\} \). If \( C \) is another set in \( N \), then the \( \text{Chebyshev radius} \) for \( A \) in \( C \), \( R(A, C) \), is \( \inf\{R_c(A): c \in C\} \), and the \( \text{Chebyshev centers of } A \) in \( C \), \( C(A, C) \), is \( \{c \in C: R_c(A) = R(A, C)\} \).

A convex set \( A \subseteq N \) is said to have normal structure if for each closed convex bounded set \( W \) in \( A \) with more than one point there is a point \( p \) in \( W \) such that \( R_p(W) < \text{diam } W \).

Brodskii and Milman [1] introduced the notion of normal structure and proved their well-known theorem:

**Theorem 1** (Brodskii and Milman). If \( K \subseteq N \), where \( N \) is complete, is a convex weakly compact set with normal structure, then there is a common fixed point for the set of all isometries of \( K \) onto \( K \).

A normed linear space \( N \) is said to be uniformly convex or rotund in every direction if and only if, for every nonzero member \( z \) of \( N \) and \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that \( |\lambda| < \varepsilon \) if \( \|x\| = \|y\| = 1 \), \( x - y = \lambda z \) and \( \|x + y\| > 2(1 - \delta) \).

**Theorem 2** (Day, James, and Swaminathan). Let \( N \) be a normed linear space that is uniformly convex in every direction, and let \( H \) be a nonempty bounded subset of a convex subset \( S \) of \( N \). Then \( C(H, S) \) has at most one member.

**Lemma 3.** If \( E \subseteq N \) is a set that has in it exactly one \( \text{Chebyshev center } c \), then \( c \) is a fixed point of every isometry of \( E \) onto \( E \).

It so happened that with an equality sign changed to an inequality sign, the proof of this lemma remains valid for onto nonexpansive maps, so we call the following lemma.
Day’s Lemma. If $K \subseteq N$ is a set that has in it exactly one Čebyšev center $c$, then $c$ is a common fixed point for all those mappings $T$ of $K$ onto $K$ that are either nonexpansive or noncontractive.

Proof. For mappings $T$ that are nonexpansive, please refer to the proof of Lemma 3.i.c. of Day [4].

If $T$ of $K$ onto $K$ is noncontractive, then $T$ is one-to-one, so its inverse $T^{-1}$ exists. Moreover $T^{-1}$ is from $K$ onto $K$ and nonexpansive, so $T^{-1}c = c$ and hence $Tc = c$.

We are fortunate enough to have noticed this lemma, as it enables us to prove, among others, a generalization of the above-mentioned theorem of Brodskii and Milman when the underlying linear space is uniformly convex in every direction.

Theorem 3. If $K$ is a convex, weakly compact set in a normed linear space $N$ that is uniformly rotund in every direction, then there is a common fixed point (the unique Čebyšev center $c$ of $K$) for the set of all those mappings $T$ of $K$ onto $K$ that are either nonexpansive or noncontractive.

Proof. Since $C(K, K)$ is not empty and a weakly compact set in a normed linear space is bounded (see, e.g., Day [5]), Theorem 3 follows from Theorem 2 and Day’s Lemma.

Theorem 4. If $c$ is the center of a closed or open ball $B$ of radius $r$ in a normed linear space $N$, then $c$ is a common fixed point for the set of all those mappings $T$ of $B$ onto $B$ that are either nonexpansive or noncontractive.

Proof. It is easy to see that $c$ is the unique Čebyšev center of $B$ in $B$, so Theorem 4 follows from Day’s Lemma.

Theorem 5. If $K$ is a convex, weakly compact set in a normed linear space $N$ that is uniformly rotund in every direction, then there is a common fixed point (the unique Čebyšev center $c$ of $K$) for the set $P$ of all those multiple-valued pseudo-contractive mappings $T: D(T) = K \to N$, of $K$ into $N$ satisfying the following conditions. For each $T \in P$,

(i) there exists a positive number $r(T)$ such that $M_{r(T)}(u) \cap K$ is not empty for each $u \in K$, where, if we set $r(T) = r$,

$$M_{r(T)} = ((1 + r(T))I - r(T)T)$$

$$= ((1 + r)I - rT): D(T) = K \to N,$$

(ii) range of $M_{r(T)}$ contains $K$.

Proof. In view of Lemma 1 and Theorem 3, it suffices to show that for each $T \in P$, the restriction of $J_r = M_{r(T)}^{-1}$ to $K$, $J_r$, maps $K$ onto $K$.

Since the range of $M_{r(T)}$ contains $K$, $J_r$ maps $K$ into $K$. To show that $J_r$ is onto, let $u \in K$, it follows from condition (i) that there is a $y \in M_{r(T)}u \cap K$. Clearly $J_r(y) = u$. Thus $J_r$ maps $K$ onto $K$.

Theorem 6. Let $c$ be a point in a normed linear space $N$ and $B_R$ be a closed or open ball of radius $R$ with center $c$. Let $P_R$ be the set of all those multiple-valued pseudocontractive mappings $T: D(T) \subseteq N \to N$, whose domain, $D(T)$, contains $B_R$. Suppose that for each $T \in P_R$, conditions (i) and (ii) of Theorem 5 are satisfied if $K$ is changed to $B_R$ and $T$ to the restriction of $T$ to $B_R$. 

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If \( P \) is the union of \( P_R \) for all \( R > 0 \), then \( c \) is a common fixed point of the members of \( P \).

Proof. The truth of Theorem 6 can be seen from Lemma 1, Theorem 4, and the proof of Theorem 5.

**Theorem 7.** If \( K \) is a convex, weakly compact set in a normed linear space \( N \) that is uniformly rotund in every direction, then there is a common fixed point (the unique Čebyšev center \( c \) of \( K \)) for the set \( P \) of all those multiple-valued strongly weak-\( \phi \)-accretive mappings \( f : D(f) = K \to N \) with constant \( c \geq 1 \) satisfying the following conditions. For each \( f \) in \( P \),

1. \( f(u) \cap K \) is not empty for each \( u \) in \( K \);
2. range of \( f \) contains \( K \).

Proof. In view of Lemma 2 and Theorem 3, it suffices to show that for each \( f \) in \( P \), the restriction of \( f^{-1} \) to \( K \), \( f^{-1} \), maps \( K \) onto \( K \).

Since the range of \( f \) contains \( K \), \( f^{-1} \) maps \( K \) into \( K \). Let \( u \) in \( K \), it follows from condition (i) that there is a \( x \) in \( f(u) \cap K \). Clearly \( f^{-1}(x) = u \), so \( f^{-1} \) maps \( K \) onto \( K \).

**Theorem 8.** Let \( c \) be a point in a normed linear space \( N \) and \( B_R \) be a closed or open ball of radius \( R \) with center \( c \). Let \( P_R \) be the set of all those multiple-valued strongly weak-\( \phi \)-accretive mappings \( f : D(f) \subseteq N \to N \) with constant \( c \geq 1 \), whose domain, \( D(f) \), contains \( B_R \). Suppose that for each \( f \) in \( P_R \), conditions (i) and (ii) of Theorem 7 are satisfied if \( K \) is changed to \( B_R \) and \( f \) to the restriction of \( f \) to \( B_R \). If \( P \) is the union of \( P_R \) for all \( R > 0 \), then \( c \) is a common fixed point of the members of \( P \).

Proof. The truth of Theorem 8 can be seen from Lemma 2, Theorem 4, and the proof of Theorem 7.

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**References**