

K_0 OF CERTAIN SUBDIAGONAL SUBALGEBRAS OF VON NEUMANN ALGEBRAS

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ABSTRACT. We show that K_0 of any finite maximal subdiagonal subalgebra of a separably acting finite von Neumann algebra is isomorphic to K_0 of the diagonal of the subalgebra. It results that K_0 of any finite, σ -weakly closed, maximal triangular subalgebra of a separably acting finite von Neumann algebra is isomorphic to K_0 of the diagonal of the subalgebra, provided that the diagonal of the subalgebra is a Cartan subalgebra of the von Neumann algebra. In addition, given any separably acting type II_1 factor \mathcal{M} , we explicitly compute K_0 of those triangular subalgebras \mathcal{T} of \mathcal{M} that have the property that there exists a UHF subalgebra \mathcal{A} of \mathcal{M} and a standard triangular UHF algebra \mathcal{S} in \mathcal{A} such that \mathcal{A} is σ -weakly dense in \mathcal{M} and \mathcal{T} is the σ -weak closure of \mathcal{S} .

1. INTRODUCTION

Let \mathcal{N} be an arbitrary finite von Neumann algebra, and let \mathcal{B} be any finite maximal subdiagonal subalgebra of \mathcal{N} . In this paper we use a factorization theorem of W. B. Arveson to show that $K_0(\mathcal{B})$ is isomorphic to $K_0(\mathcal{E})$, where $\mathcal{E} = \mathcal{B} \cap \mathcal{B}^*$ is the diagonal of \mathcal{B} . We then apply this result to the following two cases. First, we show that if \mathcal{T} is any finite σ -weakly closed maximal triangular subalgebra of a given finite von Neumann algebra \mathcal{M} such that the diagonal of \mathcal{T} is a Cartan subalgebra of \mathcal{M} , then $K_0(\mathcal{T})$ is isomorphic to K_0 of the diagonal of \mathcal{T} . Secondly, let \mathcal{S} be any standard triangular UHF subalgebra of a UHF algebra \mathcal{A} , and suppose that \mathcal{A} is σ -weakly dense in a type II_1 hyperfinite factor \mathcal{M} . Define \mathcal{T} to be the σ -weak closure of \mathcal{S} in \mathcal{M} . Then we prove that \mathcal{T} is a finite maximal subdiagonal subalgebra of \mathcal{M} such that $K_0(\mathcal{T})$ is isomorphic to $L_\infty(X, \mathbf{Z}, \nu)$, where ν is some bounded measure on the spectrum X of the C^* -algebra $\mathcal{E} = \mathcal{S} \cap \mathcal{S}^*$ and $L_\infty(X, \mathbf{Z}, \nu)$ is the algebra of all ν -measurable, essentially bounded \mathbf{Z} -valued functions on X .

Previous work on the K -theory of nonselfadjoint operator algebras can be found in the works of D. Pitts, J. Peters and B. Wagner, and C. Qiu. In [6] Pitts computes the K_0 -groups of nest algebras. Our method of calculating the

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K_0 -groups of finite maximal subdiagonal subalgebras of finite von Neumann algebras resembles the method he used. In [8] Peters and Wagner compute the K -groups of TAF algebras and nest subalgebras of C^* -algebras. In [9] Qiu computes the K -theory of analytic crossed products. Because relatively little research has been done on the K -theory of nonselfadjoint operator algebras, it is hoped that this paper will contribute to the exploration of this domain.

Throughout this paper we will assume that all of our von Neumann algebras act on separable Hilbert spaces.

2. SUBDIAGONAL SUBALGEBRAS OF VON NEUMANN ALGEBRAS

In this section we prove that K_0 of any finite maximal subdiagonal subalgebra \mathcal{B} of a finite von Neumann algebra is isomorphic to K_0 of the diagonal of \mathcal{B} . This is the content of Theorem 2.9.

Definition 2.1. Let \mathcal{N} be a von Neumann algebra, and let \mathcal{B} be a given σ -weakly closed unital subalgebra of \mathcal{N} . Let $\mathcal{E} = \mathcal{B} \cap \mathcal{B}^*$ be the diagonal of \mathcal{B} . The subalgebra \mathcal{B} is said to be *subdiagonal* if there exists a faithful normal expectation Φ from \mathcal{N} onto \mathcal{E} such that the following conditions hold:

- (i) $\mathcal{B} + \mathcal{B}^*$ is σ -weakly dense in \mathcal{N} .
- (ii) $\Phi(xy) = \Phi(x)\Phi(y)$, $x, y \in \mathcal{B}$.

The subdiagonal subalgebra \mathcal{B} is said to be *finite* if \mathcal{N} is a finite von Neumann algebra and there exists a faithful normal finite trace ρ on \mathcal{N} such that $\rho \circ \Phi = \rho$. Finally, we say that \mathcal{B} is a *maximal* subdiagonal subalgebra of \mathcal{N} (with respect to Φ) if \mathcal{B} is not contained in any larger subalgebra of M , with the same diagonal \mathcal{E} , on which Φ is multiplicative.

Throughout the rest of this section we will assume that \mathcal{N} is a given finite von Neumann algebra acting on the Hilbert space \mathcal{H} and \mathcal{B} is a given finite subdiagonal subalgebra of \mathcal{N} with respect to an expectation Φ onto $\mathcal{E} = \mathcal{B} \cap \mathcal{B}^*$. For each $n \in \mathbb{N}$, $M_n(\mathcal{N})$ will act on $\mathcal{H}^{\oplus n}$, and Φ will be extended to $M_n(\mathcal{N})$ by $\Phi_n(x) = (\Phi(x_{ij}))$, $x = (x_{ij}) \in M_n(\mathcal{N})$.

Our first goal is to prove that Φ_n makes $M_n(\mathcal{B})$ into a finite subdiagonal subalgebra of $M_n(\mathcal{M})$ such that if \mathcal{B} is a maximal subdiagonal subalgebra of \mathcal{N} , then $M_n(\mathcal{B})$ is a maximal subdiagonal subalgebra of $M_n(\mathcal{N})$ with respect to Φ_n .

We state the following two lemmas without proof.

Lemma 2.2 [1, 6.1.1(v)]. *Let \mathcal{R} be a von Neumann algebra, and let \mathcal{F} be a von Neumann subalgebra of \mathcal{R} . Let Ψ be an expectation from \mathcal{N} onto \mathcal{F} . Assume that there exists a faithful normal state ρ such that $\rho \circ \Psi = \rho$. Then Ψ is faithful and normal.*

Lemma 2.3 [10]. *Let \mathcal{F} , \mathcal{R} , and Ψ be as in Lemma 2.2. Then Ψ is completely positive, i.e., for each n , the map $\Psi_n: M_n(\mathcal{R}) \rightarrow M_n(\mathcal{F})$ is positive.*

Lemma 2.4. *For each $n \in \mathbb{N}$, the map $\Phi_n: M_n(\mathcal{N}) \rightarrow M_n(\mathcal{E})$ is a faithful normal expectation onto $M_n(\mathcal{E})$. Moreover, the algebra $M_n(\mathcal{B})$ is a finite subdiagonal subalgebra of $M_n(\mathcal{N})$ with respect to Φ_n .*

Proof. Fix $n \in \mathbb{N}$. By Lemma 2.3, the map Φ_n is an expectation on $M_n(\mathcal{E})$ and Φ_n is clearly multiplicative on $M_n(\mathcal{B})$. To complete the proof, we need to prove only that Φ_n is faithful and normal. It suffices, by Lemma 2.2, to

show that there exists a faithful normal finite trace ρ on $M_n(\mathcal{N})$ such that $\rho \circ \Phi_n = \rho$. Such a trace is given by $\rho(x) = \sum_{i=1}^n \tau(x_{ii})$, where τ is any faithful normal finite trace on \mathcal{N} such that $\tau \circ \Phi_n = \tau$. \square

In order to apply Arveson's factorization theorem to prove Theorem 2.9, we need the following definition and lemmas.

Definition 2.5. Let \mathcal{R} be a von Neumann algebra, and let \mathcal{F} be a subdiagonal subalgebra of \mathcal{R} with respect to an expectation Ψ onto $\mathcal{F} \cap \mathcal{F}^*$. Define $\widetilde{\mathcal{F}} \subseteq \mathcal{F}$ and \mathcal{F}_{\max} by

$$\widetilde{\mathcal{F}} = \{x \in \mathcal{F} \mid \Psi(x) = 0\}, \quad \mathcal{F}_{\max} = \{x \in \mathcal{R} \mid \Psi(\widetilde{\mathcal{F}} x \mathcal{F}) = \Psi(\mathcal{F} x \widetilde{\mathcal{F}}) = 0\}.$$

Lemma 2.6 [1, 2.2.1]. *Let \mathcal{F} , \mathcal{R} , and Ψ be defined as in Definition 2.5. Then \mathcal{F}_{\max} is a maximal subdiagonal subalgebra of \mathcal{R} (with respect to Ψ) such that $\mathcal{F} \subseteq \mathcal{F}_{\max}$. Moreover, the latter algebra is σ -weakly closed.*

Lemma 2.7. *For each $n \geq 1$, we have $M_n(\mathcal{B}_{\max}) = (M_n(\mathcal{B}))_{\max}$.*

Proof. Fix $n \geq 1$. We claim that $(M_n(\mathcal{B}))_{\max} \subseteq M_n(\mathcal{B}_{\max})$. To prove this, define $\widetilde{\mathcal{B}}$ and $M_n(\widetilde{\mathcal{B}})$ as in Definition 2.5. Let $x \in (M_n(\mathcal{B}))_{\max}$ and fix $1 \leq i, j \leq n$. It is easy to see that every element of $\widetilde{\mathcal{B}} x_{ij} \mathcal{B}$ is the (i, j) -th entry of some element of $M_n(\widetilde{\mathcal{B}}) x M_n(\mathcal{B})$, and hence we have $\Phi(\widetilde{\mathcal{B}} x_{ij} \mathcal{B}) = 0$. A similar argument show that $\Phi(\mathcal{B} x_{ij} \widetilde{\mathcal{B}}) = 0$. Hence $x_{ij} \in \mathcal{B}_{\max}$. This proves that $x \in M_n(\mathcal{B}_{\max})$. Because the algebra $M_n(\mathcal{B}_{\max})$ is a subdiagonal subalgebra of $M_n(\mathcal{N})$, Lemma 2.6 implies that $(M_n(\mathcal{B}))_{\max} = M_n(\mathcal{B}_{\max})$. \square

The following factorization theorem is due to Arveson.

Lemma 2.8 [1, 4.2]. *Suppose that \mathcal{F} is a finite maximal subdiagonal subalgebra of a finite von Neumann algebra \mathcal{R} . Then every invertible operator in \mathcal{R} admits a factorization ua , where u is unitary and $a \in \mathcal{F} \cap \mathcal{F}^{-1}$.*

We are now ready to prove that $K_0(\mathcal{B})$ is isomorphic to $K_0(\mathcal{E})$. Let $S(\mathcal{E})$ denote the abelian semigroup of equivalence classes of idempotents in $\bigcup \mathcal{P}_n(\mathcal{B})$.

Theorem 2.9. *Suppose that \mathcal{B} is a finite maximal subdiagonal subalgebra of the finite von Neumann algebra \mathcal{N} . Then the inclusion map $i: \mathcal{E} \rightarrow \mathcal{B}$ induces an isomorphism $i_*: \mathcal{S}(\mathcal{E}) \rightarrow \mathcal{S}(\mathcal{B})$ of semigroups, and hence $K_0(\mathcal{B}) \cong K_0(\mathcal{E})$.*

Proof. The proof is modeled on the proof Proposition 6.8(b) in [7].

We prove only that i_* is onto. To this end, let $n \in \mathbb{N}$ and let $e \in M_n(\mathcal{B})$ be an idempotent. Define $s = (e^*e + (1 - e)^*(1 - e))^{1/2}$. Then s is positive and invertible. It is easy to see that ses^{-1} is a selfadjoint projection. Lemma 2.7 shows that $M_n(\mathcal{B}) = M_n(\mathcal{B}_{\max}) = (M_n(\mathcal{B}))_{\max}$, hence, by Lemmas 2.4 and 2.6, $M_n(\mathcal{B})$ is a finite maximal subdiagonal subalgebra of $M_n(\mathcal{N})$. Therefore, by Lemma 2.8, $s = ua$, where u is unitary and $A \in M_n(\mathcal{B}) \cap M_n(\mathcal{B})^{-1}$. Then $aea^{-1} = u^*ses^{-1}u$ is a selfadjoint projection in $M_n(\mathcal{E})$, hence e is similar to an idempotent in $M_n(\mathcal{E})$. This proves that i_* is onto. \square

3. MAXIMAL TRIANGULAR ALGEBRAS

In this section \mathcal{M} will denote an arbitrary finite von Neumann algebra and \mathcal{F} will denote a σ -weakly closed triangular subalgebra of \mathcal{M} , i.e., the diagonal

$\mathcal{D} = \mathcal{T} \cap \mathcal{T}^*$ is a masa in \mathcal{M} . The object of this section is to show that if \mathcal{D} is a Cartan subalgebra of \mathcal{M} and \mathcal{T} is a finite maximal triangular subalgebra of \mathcal{M} , then $K_0(\mathcal{T}) \cong K_0(\mathcal{D})$. This is the content of Theorem 3.3.

We will need the following result from [5].

Lemma 3.2 [5, 3.6]. *Suppose that \mathcal{T} is a maximal triangular subalgebra of \mathcal{M} , and assume that the diagonal \mathcal{D} of \mathcal{T} is a Cartan of \mathcal{M} with respect to a faithful normal expectation Φ from \mathcal{M} onto \mathcal{D} . Then \mathcal{T} is a maximal subdiagonal subalgebra of \mathcal{M} with respect to Φ .*

Theorem 3.3. *If \mathcal{T} is a finite maximal triangular subalgebra of \mathcal{M} such that the diagonal \mathcal{D} of \mathcal{T} is a Cartan of \mathcal{M} , then $K_0(\mathcal{T}) \cong K_0(\mathcal{D})$.*

Proof. The proof is immediate from Theorem 2.9 and Lemma 3.2. \square

Remark 3.4. Let \mathcal{T} be any triangular subalgebra of an arbitrary von Neumann algebra \mathcal{M} and let \mathcal{D} be the diagonal of \mathcal{T} . Then there exists a compact (second countable) space X and a positive measure ν on X such that \mathcal{D} is $*$ -isomorphic to $L_\infty(X, \nu)$. Define $L_\infty(X, \mathbf{Z}, \nu)$ to be the space of all ν -measurable, essentially bounded \mathbf{Z} -valued functions on X . It is well known that $K_0(L_\infty(X, \nu)) \cong L_\infty(X, \mathbf{Z}, \nu)$. Hence if \mathcal{T} is as in Theorem 3.3, then we have $K_0(\mathcal{T}) \cong L_\infty(X, \mathbf{Z}, \nu)$.

4. σ -WEAK CLOSURES OF STANDARD TUHF ALGEBRAS

Let \mathcal{M} be a fixed type II_1 factor. The object of this section—Theorem 4.9—is to compute explicitly K_0 of those triangular subalgebras \mathcal{T} of \mathcal{M} that have the property that there exists a UHF subalgebra of \mathcal{A} of \mathcal{M} and a standard triangular UHF algebra \mathcal{S} in \mathcal{A} such that \mathcal{A} is σ -weakly dense in \mathcal{M} and \mathcal{T} is the σ -weak closure of \mathcal{S} .

Definition 4.1. Let $\mathcal{A} = \overline{\bigcup \mathcal{M}_n}^{\|\cdot\|}$ be a UHF algebra of rank (p_n) contained in \mathcal{M} , where (\mathcal{M}_n) is an increasing sequence of C^* -subalgebras of \mathcal{A} such that each \mathcal{M}_n is $*$ -isomorphic to the $p_n \times p_n$ matrices. Suppose that \mathcal{A} is σ -weakly dense in \mathcal{M} . For $m \leq n$, assume that \mathcal{M}_m is embedded in \mathcal{M}_n via standard embedding. For $n \leq 1$, let $\{e_{ij}^{(n)} \mid 1 \leq i, j \leq p_n\}$ be the standard system of matrix units in \mathcal{M}_n . Define \mathcal{D}_n to be the diagonal of \mathcal{M}_n , and define $\mathcal{E} = \overline{\bigcup \mathcal{D}_n}^{\|\cdot\|}$. Let \mathcal{T}_n be the full algebra of triangular matrices in \mathcal{M}_n . The standard triangular UHF algebra \mathcal{S} in \mathcal{A} is defined to be the Banach algebra $\mathcal{S} = \overline{\bigcup \mathcal{T}_n}^{\|\cdot\|}$. The algebra \mathcal{T} is defined to be the σ -weak closure of \mathcal{S} in \mathcal{M} . Finally, define \mathcal{D} to be the σ -weak closure of \mathcal{E} .

According to Remark 3.5, we may write $K_0(\mathcal{D}) \cong L_\infty(X, \mathbf{Z}, \nu)$, where X is some compact space and ν is some positive measure on X ; in fact, we may take X to be the spectrum of the C^* -algebra \mathcal{E} (see [3, I.7.1.1, I.7.2.4]). This allows us to use Theorem 2.9 to show that $K_0(\mathcal{T}) \cong L_\infty(X, \mathbf{Z}, \nu)$, which gives an explicit expression for $K_0(\mathcal{T})$. In order to apply Theorem 2.9, we need to construct an expectation Φ onto \mathcal{D} such that \mathcal{T} is a finite maximal subdiagonal subalgebra of \mathcal{M} with respect to Φ . To define Φ we make use of the following result.

Lemma 4.2 [2, 2.6]. *Let $1 \leq n \leq k$ be positive integers. Let g_1, \dots, g_p be the minimal projections in $\mathcal{M}_n^c \cap \mathcal{D}_k$, where \mathcal{M}_n^c is the commutant of \mathcal{M}_n in \mathcal{M} .*

Define the map $\varphi_{nk}: \mathcal{M} \rightarrow \mathcal{M}$ by $\varphi_{nk}(x) = \sum_{i=1}^p g_i x g_i$, $x \in \mathcal{M}$. Let $\|\cdot\|_2$ denote the trace norm on \mathcal{M} . Then the following conditions hold:

- (a) For all $x \in \mathcal{M}$, $\|\varphi_{nk}(x)\|_2 \leq \|x\|_2$ and $\|\varphi_{nk}(x)\| \leq \|x\|$.
- (b) For all $x \in \mathcal{M}$, the limit $\varphi_n(x) = \|\cdot\|_2\text{-}\lim_{k \geq n} \varphi_{nk}(x)$ exists and may be written as

$$\varphi_n(x) = \sum_{i,j=1}^{p_n} e_{ij}^{(n)} d_{ij}, \quad d_{ij} \in \mathcal{D}^{(n)}, \quad \mathcal{D}^{(n)} = \overline{\bigcup_m \mathcal{M}_n^c \cap \mathcal{D}_m^w}.$$

- (c) For all $x \in \mathcal{M}$, $\|\varphi_n(x)\| \leq \|x\|$ and $\varphi_n(x) = s\text{-}\lim_{k \geq n} \varphi_{nk}(x)$.
- (d) For all $x \in \mathcal{M}$, $x = s\text{-}\lim_n \varphi_n(x)$.

Definition 4.3. For $1 \leq k$ a positive integer, define the maps $\Theta_k, \Phi: \mathcal{M} \rightarrow \mathcal{M}$ by $\Theta_k = \varphi_{1k}$, $\Phi = \varphi_1$.

Without loss of generality, we may assume that $p_1 = 1$. Then the range of Φ is \mathcal{D} .

Lemma 4.4. The map Φ is preserved by the canonical trace on \mathcal{M} , hence Φ is a faithful normal expectation from \mathcal{M} onto \mathcal{D} .

Proof. That Φ is an expectation from \mathcal{M} onto \mathcal{D} follows from Lemma 4.2 and standard results concerning type II_1 von Neumann algebras [5, 1.6.I, 1.3.2]. Let τ be the canonical trace on \mathcal{M} and let $x \in \mathcal{M}$. Because \mathcal{A} is σ -weakly dense in \mathcal{M} , we may write $x = s\text{-}\lim_k x_k$, where $x_k \in \mathcal{M}_k$ and $\sup_k \|x_k\| < \infty$. We have $\tau(\Phi(x)) = \tau(s\text{-}\lim_k \Theta_k(x_k)) = \lim_k \tau(\Phi_k(x_k)) = \lim_k \tau(x_k) = \tau(x)$. Therefore $\tau \circ \Phi = \tau$. By Lemma 2.2, Φ is faithful and normal. \square

The proof that Φ is multiplicative on \mathcal{I} depends on the next lemma.

Lemma 4.5. Let $n \geq 1$ and suppose that $d \in \mathcal{D}^{(n)}$. If $i > j$ and $e_{ij}^{(n)} d \in \mathcal{I}$, then $e_{ij}^{(n)} d = 0$.

Proof. Write $e_{ij}^{(n)} d = \sigma\text{-}w\text{-}\lim_\alpha x_\alpha$, where $x_\alpha \in \bigcup \mathcal{I}_m$. By Lemma 4.4, Φ is σ -weakly continuous, consequently, $e_{jj}^{(n)} d = \Phi(e_{jj}^{(n)} d) = \Phi(\sigma\text{-}w\text{-}\lim_\alpha e_{ji}^{(n)} x_\alpha) = \sigma\text{-}w\text{-}\lim_\alpha \Phi(e_{ji}^{(n)} x_\alpha) = 0$, i.e., $e_{ij}^{(n)} d = e_{ij}^{(n)} e_{jj}^{(n)} d = 0$. \square

Lemma 4.6. For all $x, y \in \mathcal{I}$, we have $\Phi(xy) = \Phi(x)\Phi(y)$.

Proof. Let $n \geq 1$ and let $x, y \in \mathcal{I}$. By Lemmas 4.2 and 4.5, we may write

$$\varphi_n(x) = \sum_{i \leq j}^{p_n} e_{ij}^{(n)} d_{ij}, \quad \varphi_n(y) = \sum_{i \leq j}^{p_n} e_{ij}^{(n)} h_{ij}, \quad d_{ij}, h_{ij} \in \mathcal{D}^{(n)}.$$

Now, for $k \geq n$, a computation shows that $\Theta_k(\varphi_n(x)\varphi_n(y)) = \Theta_k(\varphi_n(x)) \times \Theta_k(\varphi_n(y))$. Because $\Phi = s\text{-}\lim_k \Theta_k$, we see that $\Phi(\varphi_n(x)\varphi_n(y)) = \Phi(\varphi_n(x)) \times \Phi(\varphi_n(y))$. Lemmas 4.2 and 4.4 imply that Φ is strongly continuous, hence $\Phi(xy) = \Phi(x)\Phi(y)$. \square

Lemma 4.7. The algebra \mathcal{I} is a triangular subalgebra of \mathcal{M} with diagonal \mathcal{D} . Moreover, \mathcal{I} is a subdiagonal subalgebra of \mathcal{M} with respect to Φ .

Proof. We first show that \mathcal{D} is a masa in \mathcal{M} . Let $\mathcal{D} \subseteq \mathcal{E}$, where \mathcal{E} is an abelian von Neumann subalgebra of \mathcal{M} . Let $n \geq 1$ and $x \in \mathcal{E}$. Then Lemma

4.2 implies that $\varphi_n(x) \in \mathcal{E}$, hence $\varphi_n(x)$ commutes with $e_{ii}^{(n)}$, for $1 \leq i \leq p_n$. It is then a consequence of Lemma 4.2(b) that for $i \neq j$, $e_{ij}^{(n)}d_{ij} = 0$. Therefore, $\varphi_n(x) \in \mathcal{D}$. It follows from Lemma 4.2 that $x \in \mathcal{D}$. Hence $\mathcal{E} = \mathcal{D}$, and therefore \mathcal{D} is a masa in \mathcal{M} .

To show that \mathcal{T} has diagonal \mathcal{D} , write $\mathcal{E} = \mathcal{T} \cap \mathcal{T}^*$ and let $x \in \mathcal{E}$. Then for $n \geq 1$, we have $\varphi_n(x) = s\text{-}\lim_{k \geq n} \varphi_{kn}(x)$, with $\varphi_{kn}(x) \in \mathcal{E}$. Because \mathcal{E} is a von Neumann subalgebra of \mathcal{M} , we see that $\varphi_n(x) \in \mathcal{E}$. Express $\varphi_n(x)$ as in Lemma 4.2(b). For $1 \leq i \neq j \leq p_n$, we have $e_{ij}^{(n)}d_{ij} \in \mathcal{E}$. Hence Lemma 4.5 then implies that $e_{ij}^{(n)}d_{ij} = 0$. Hence, for all n , $\varphi_n(x) \in \mathcal{D}$. Lemma 4.2 then implies that $\mathcal{E} = \mathcal{D}$.

Because $\mathcal{T} + \mathcal{T}^*$ is clearly σ -weakly dense in \mathcal{M} , Lemma 4.6 gives that \mathcal{T} is a subdiagonal subalgebra of \mathcal{M} with respect to Φ . \square

Theorem 4.8. *The algebra \mathcal{T} is a finite maximal subdiagonal subalgebra of \mathcal{M} with respect to the expectation Φ .*

Proof. Lemma 4.4 implies that \mathcal{T} is a finite subdiagonal subalgebra in \mathcal{M} . By Lemma 2.6, the proof will be complete if we show that $\mathcal{T}_{\max} \subseteq \mathcal{T}$. First, observe that \mathcal{T}_{\max} is strongly closed. Let $n \geq 1$ and let $x \in \mathcal{T}_{\max}$. Because $\mathcal{D} \subseteq \mathcal{T}_{\max}$, Lemma 4.2 implies that $\varphi_n(x) \in \mathcal{T}_{\max}$. Express $\varphi_n(x)$ as in 4.2(b). Let $1 \leq j < i \leq p_n$. Define $y = e_{ij}^{(n)}d_{ij}$, then we have $y^* = e_{ji}^{(n)}d_{ij}^* \in \mathcal{T}$. Therefore $y + y^* \in \mathcal{T}_{\max} \cap \mathcal{T}_{\max}^* = \mathcal{D}_{\max} = \mathcal{D}$ because \mathcal{T} and \mathcal{T}_{\max} have the same diagonal (see [1]). It follows from Lemma 4.5 that $e_{ij}^{(n)}d_{ij} = 0$. Consequently, for all n , $\varphi_n(x) \in \mathcal{T}$. Because \mathcal{T} is strongly closed, and $x = s\text{-}\lim_n \varphi_n(x)$ by Lemma 4.2, we see that $x \in \mathcal{T}$. Therefore, $\mathcal{T} = \mathcal{T}_{\max}$. \square

We are now ready to compute $K_0(\mathcal{T})$.

Theorem 4.9. *We have $K_0(\mathcal{T}) \cong K_0(\mathcal{D}) \cong L_\infty(X, \mathbf{Z}, \nu)$.*

Proof. By Theorems 4.9 and 2.9, we have $K_0(\mathcal{T}) \cong K_0(\mathcal{D})$. Remark 3.4 then implies that $K_0(\mathcal{T}) \cong L_\infty(X, \mathbf{Z}, \nu)$. \square

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