PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 1, September 1992

SOME APPLICATIONS OF CONVOLUTION OF OPERATORS ON BANACH SPACES

PATRICK N. DOWLING

(Communicated by Paul S. Muhly)

ABSTRACT. Using convolution between functions and cone absolutely summing operators, we obtain characterizations of Banach spaces with the Radon-Nikodym property, the analytic Radon-Nikodym property and Banach spaces not containing a copy of c_0.

1. INTRODUCTION

In [B2] Blasco gave characterizations of the Radon-Nikodym property and the analytic Radon-Nikodym property using convolution between functions and cone absolutely summing operators. The aim of this note is to improve Blasco's characterization of the analytic Radon-Nikodym property and also to extend Blasco's results to give a new characterization of Banach spaces not containing a copy of c_0. In fact, we will prove a general result that give the characterizations of the Radon-Nikodym property, the analytic Radon-Nikodym property, and Banach spaces not containing c_0 as easy corollaries.

2. PRELIMINARIES AND DEFINITIONS

Throughout this note G will denote a compact abelian metrizable group, $\mathscr{B}(G)$ denotes the σ-algebra of Borel subsets of G, and λ is normalized Haar measure on G. The dual group of G will be denoted by Γ and is a countable discrete abelian group.

Definition 2.1. Let $1 < p < \infty$. Given a bounded linear operator $T: L^p(G) \to X$ and a function $g \in L^1(G)$, we define an operator $g * T: L^p(G) \longrightarrow X$ by $(g * T)(f) = T(f * g)$ for all $f \in L^p(G)$. Clearly, $\|g * T\| \le \|g\|_1 \|T\|$.

Definition 2.2. Let $1 \le r < \infty$, $1 < p < \infty$, and $1/p + 1/q = 1$. A bounded linear operator $T: L^p(G) \longrightarrow X$ is called a positive r-summing operator if there exists a constant C such that for every $\{f_i\}_{i=1}^n$ in $L^p(G)$ with $f_i \ge 0$ for $i = 1, 2, \ldots, n$ the following inequality holds,

$$\left(\sum_{i=1}^n \|T(f_i)\|^r\right)^{1/r} \le C \sup\left\{\left(\sum_{i=1}^n \left|\int_G f_i(t)g(t)\,d\lambda(t)\right|^r\right)^{1/r} : \|g\|_q = 1\right\}.$$

Received by the editors December 7, 1990 and, in revised form, February 23, 1991.
1991 *Mathematics Subject Classification*. Primary 46B22.

The positive r-summing norm of T is given by the infinum of all such constants C and is denoted by $\|T\|_{p,r}$. The space of positive r-summing operates from $L^P(G)$ to X is denoted by $\Lambda_r(L^p(G), X)$. The space of operators we will be most interested in is $\Lambda_1(L^p(G), X)$, which is often referred to as the space of cone absolutely summing operators [B1, S].

Definition 2.3. Let $1 \leq p < \infty$ and $1/p + 1/q = 1$. A bounded linear operator $T\colon L^p(G) \longrightarrow X$ is called representable if there exists a function $f \in L^q(G, X)$ such that $T(h) = \int_G f(t)h(t)\,d\lambda(t)$ for all $h \in L^p(G)$. In this case we will write $T = T_f$. It is easily seen that $\|T_f\| \leq \|f\|_q$.

Remark. Schaefer [S] proved that if $f \in L^q(G, X)$ then $T_f \in \Lambda_1(L^p(G), X)$ and $\|T_f\|_{p,1} = \|f\|_q$. Also, it is easily seen that if $g \in L^1(G)$ then $g*T_f = T_{g*f}$ and $\|g * T_f\|_{p,1} \leq \|g\|_1\|T_f\|_{p,1}$. These observations have been extended by Blasco [B2].

Theorem 2.1 (Blasco). *Let $1 \leq r < \infty$, $1 < p < \infty$, $1/p + 1/q = 1$, and let g be a positive element of $L^1(G)$.*

(i) *If $T \in \Lambda_r(L^p(G), X)$ then $g * T \in \Lambda_r(L^p(G), X)$ and $\|g * T\|_{p,r} \leq \|g\|_1\|T\|_{p,r}$.*

(ii) *If $T\colon L^p(G) \longrightarrow X$ is bounded and linear and $g \in L^p(G)$, then there is a continuous function $h\colon G \longrightarrow X$ such that $g * T = T_h$. If, in addition, we also assume that $T \in \Lambda_1(L^p(G), X)$ then $\|h\|_q \leq \|g\|_1\|T\|_{p,1}$.*

Definition 2.4. A sequence of functions $\{g_n\}_{n=1}^{\infty}$ in $L^1(G)$ is called an approximate identity if

(a) $\int_G g_n(t)\,d\lambda(t) = 1$ for all $n \in \mathbb{N}$,
(b) $\sup_n \int_G |g_n(t)|\,d\lambda(t) < \infty$, and
(c) $\lim_{n\to\infty} \int_U g_n(t)\,d\lambda(t) = 1$ for all neighborhoods U of 1 in G.

Remark. Since G is a compact abelian metrizable group approximate identities exist. In fact, approximate identities of positive functions exist [R, p. 23].

3. Radon-Nikodym properties and applications

As in the previous section, G will denote a compact abelian metrizable group and Γ will denote the dual group of G. Let M be a subset of Γ and let X be a Banach space. Let us recall that for an X-valued measure, μ, on $\mathscr{B}(G)$ we can define $\mathbb{E}(\mu|\pi)$ by

$$\mathbb{E}(\mu|\pi) = \sum_{E\in\pi} \frac{\mu(E)}{\lambda(E)}\chi_E,$$

where π is a finite measurable partition of G, along with the convention $\frac{0}{0} = 0$. We say that μ is of bounded average range (respectively, of bounded variation) if $\sup_\pi \|\mathbb{E}(\mu|\pi)\|_{L^\infty(G;X)} < \infty$ (respectively, $\sup_\pi \|\mathbb{E}(\mu|\pi)\|_{L^1(G;X)} < \infty$), where the supremum is taken over all finite measurable partitions of G.

Definition 3.1. (a) X is said to have type I-M-Radon-Nikodym property (type I-M-RNP) if every X-valued measure, μ, on $\mathscr{B}(G)$ that is of bounded average range and such that $\hat{\mu}(\gamma) = 0$ for all a $\gamma \notin M$ has Radon-Nikodym derivative with respect to λ.

(b) X is said to have type II-M-Radon-Nikodym property (type II-M-RNP) if every X-valued measure, μ, on $\mathscr{B}(G)$ that is of bounded variation, which is absolutely continuous with respect to λ, and such that $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin M$ has a Radon-Nikodym derivative with respect to λ.

Remarks. (i) Type I-M-RNP was introduced by Edgar [E]. More information on types I and II-M-RNP can be found in [D2].

(ii) If $G = \mathbb{T}$ then $\Gamma = \mathbb{Z}$. In this case, type I-$\mathbb{Z}$-RNP is equivalent to type II-$\mathbb{Z}$-RNP which, in turn, is equivalent to the Radon-Nikodym property. Type I-$\mathbb{N}$-RNP is equivalent to type II-$\mathbb{N}$-RNP and these properties are equivalent to the analytic Radon-Nikodym property [BD].

(iii) In general, it is unknown if type I-M-RNP and type II-M-RNP are equivalent properties for each subset M of Γ. However, if M is a Sidon set then this is true. Recall that M is a Sidon subset of Γ if $C_M(G)$ is isomorphic to $l^1(M)$, where $C_M(G)$ is the space of continuous functions on G whose Fourier transform is supported on M, with the supremum norm. It is shown in [D1] that if M is a Sidon subset of Γ then a Banach space X has type I-M-RNP if and only if it has type II-M-RNP if and only if it does not contain a copy of c_0.

The result that allows us to apply the ideas in §2 is the following result of Edgar [E]:

Proposition 3.1. *A Banach space X has type* I-M-RNP *if and only if for every bounded linear operator $T\colon L^1(G)/L^1_{M'}(G) \longrightarrow X$ the operator TQ is representable, where $M' = \{\gamma \in \Gamma\colon \overline{\gamma} \notin M\}$ and $Q\colon L^1(G) \longrightarrow L^1(G)/L^1_{M'}(G)$ is the natural quotient map.*

Theorem 3.1. *Let $1 < p < \infty$, let $\{g_n\}_{n=1}^{\infty}$ be an approximate identity of positive functions in $L^p(G)$, and let M be a subset of Γ such that type* I-M-RNP *and type* II-M-RNP *are equivalent properties. Then the following statements are equivalent for a Banach space X:*

(a) *X has type* I-M-RNP.
(b) *For every operator $T \in \Lambda_1(L^p(G), X)$ with $T(\overline{\gamma}) = 0$ for all $\gamma \notin M$, the convolution $g_n * T$ converges to T in $\Lambda_1(L^p(G), X)$.*

Proof. Suppose X has type I-M-RNP, and let $T \in \Lambda_1(L^p(G), X)$ with $T(\overline{\gamma}) = 0$ for all $\gamma \notin M$. Define a vector measure $\mu\colon \mathscr{B}(G) \longrightarrow X$ by

$$\mu(E) = T(\chi_E) \quad \text{for all } E \in \mathscr{B}(G).$$

From this definition, it is easily seen that

$$\int_G \varphi \, d\mu = T(\varphi) \quad \text{for all } \varphi \in C(G).$$

In particular,

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma} \, d\mu = T(\overline{\gamma}) = 0 \quad \text{for all } \gamma \notin M.$$

By the same method of proof of Blasco's Theorem 2.1 [B2] we see that μ is of bounded variation and is absolutely continuous with respect to Haar measure

λ. Now X has type II-M-RNP, since type I-M-RNP and type II-M-RNP are equivalent, so there exists a function $f \in L^1(G, X)$ such that

$$\mu(E) = \int_E f(t)\, d\lambda(t) \quad \text{for all } E \in \mathscr{B}(G).$$

As noted by Blasco [B2], a standard argument shows that $f \in L^q(G, X)$, where $1/p + 1/q = 1$, and T is represented by f (see also [DU, p. 62]). Thus $g_n * T$ is represented by $g_n * f$. However, $g_n * f$ converges to f in $L^q(G, X)$-norm so therefore $g_n * T$ converges to T in $\Lambda_1(L^p(G), X)$ by the remark preceding Theorem 2.1.

Conversely, suppose (b) holds, and let $S\colon L^1(G)/L^1_{M'}(G) \longrightarrow X$ be a bounded linear operator. Define $T\colon L^1(G) \longrightarrow X$ by $T = SQ$ where $Q\colon L^1(G) \longrightarrow L^1(G)/L^1_{M'}(G)$ is the natural quotient map. Then, clearly, $T(\overline{\gamma}) = 0$ for all $\gamma \notin M$. Again, the method of proof of Blasco's Theorem 2.1 [B2] shows that $T \in \Lambda_1(L^p(G), X)$. Thus, since (b) holds, $g_n * T$ converges to T in $\Lambda_1(L^p(G), X)$. However, by Theorem 2.1(ii), $g_n * T$ is represented by functions f_n in $L^q(G, X)$ and the sequence $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^q(G, X)$ because $g_n * T$ converges to T in $\Lambda_1(L^p(G), X)$. Therefore, $\{f_n\}_{n=1}^{\infty}$ converges in $L^q(G, X)$ to a function f and f represents the operator T. Hence X has type I-M-RNP by Proposition 3.1. □

Corollary 3.1. *Let $1 < p < \infty$ and let $\{g_n\}_{n=1}^{\infty}$ be an approximate identity of positive functions in $L^p(G)$. Then the following are equivalent for a Banach space X:*

(a) *X has the Radon-Nikodym property.*
(b) *For every operator $T \in \Lambda_1(L^p(G), X)$ the convolution $g_n * T$ converges to T in $\Lambda_1(L^p(G), X)$.*

Corollary 3.2. *Let $1 < p < \infty$, let $e_n(t) = e^{-\mathrm{int}}$ for all $n \in \mathbb{Z}$, and let $\{g_n\}_{n=1}^{\infty}$ be an approximate identity of positive functions in $L^p(\mathbb{T})$. Then the following are equivalent for a Banach space X:*

(a) *X has the analytic Radon-Nikodym property.*
(b) *For every $T \in \Lambda_1(L^p(\mathbb{T}), X)$ with $T(e_n) = 0$ for all $n < 0$, the convolution $g_n * T$ converges to T in $\Lambda_1(L^p(\mathbb{T}), X)$.*

Corollary 3.3. *Let $1 < p < \infty$, let M be a Sidon subset of Γ, and let $\{g_n\}_{n=1}^{\infty}$ be an approximate identity of positive functions in $L^p(G)$. Then the following are equivalent for a Banach space X:*

(a) *X does not contain a copy of c_0,*
(b) *For every $T \in \Lambda_1(L^p(G), X)$, with $T(\overline{\gamma}) = 0$ for all $\gamma \notin M$, the convolution $g_n * T$ converges to T in $\Lambda_1(L^p(G), X)$.*

Remarks. (i) Corollary 3.1 is the same as Blasco's Theorem 2.1 [B2] if $G = \mathbb{T}$.

(ii) Corollary 3.2 improves Blasco's Theorem 2.2 [B2] since it works for general approximate identities of positive functions in $L^p(\mathbb{T})$.

REFERENCES

[B1] O. Blasco, *Positive p-summing operators on L^p-spaces*, Proc. Amer. Math. Soc. **100** (1987), 275–280.

[B2] ——, *Convolution of operators and applications*, Math. Z. **199** (1988), 109–114.

[BD] A. V. Bukhvalov and A. A. Danilevich, *Boundary properties of analytic and harmonic functions with values in a Banach space*, Mat. Z. **31** (1982), 203–214; English transl., Math. Notes Acad. Sci. USSR **31** (1982), 104–110.

[DU] J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, RI, 1977.

[D1] P. N. Dowling, *Duality in some vector-valued function spaces*, Rocky Mountain J. Math. **22** (1992).

[D2] ——, *Radon-Nikodym properties associated with subsets of countable discrete abelian groups*, Trans. Amer. Math. Soc. **327** (1991), 879–890.

[E] G. A. Edgar, *Banach spaces with the analytic Radon-Nikodym property and compact abelian groups*, Proceeding of the International Conference on Almost Everywhere Convergence in Probability and Ergodic Theory, Academic Press, Boston, MA, 1989, pp. 195–213.

[R] W. Rudin, *Fourier analysis on groups*, Tracts in Math., No. 12, Interscience, New York, 1962.

[S] H. H. Schaefer, *Banach lattices of positive operators*, Grundlehren Math. Wiss., vol. 215, Springer-Verlag, Berlin and New York, 1974.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MIAMI UNIVERSITY, OXFORD, OHIO 45056