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LINEAR COMBINATIONS OF PROJECTIONS IN VON NEUMANN ALGEBRAS

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ABSTRACT. Any operator in a von Neumann algebra is a linear combination of a finite number of projections from the algebra with coefficients from the center of the algebra. Those von Neumann algebras that are the complex linear span of their projections are identified.

INTRODUCTION

Let $\mathscr{A}$ be a von Neumann algebra and $\mathscr{Z}$ its center; let further $\mathscr{A}_h$ denote the set of all self-adjoint elements of $\mathscr{A}$ and $\operatorname{Proj}\mathscr{A}$ the lattice of all projections (i.e., self-adjoint idempotents) in $\mathscr{A}$. In this note, we show that the $\mathscr{Z}$-module $\mathscr{A}$ is generated by $\operatorname{Proj}\mathscr{A}$. Moreover, we prove that the $\mathbf{C}$-module $\mathscr{A}$ is generated by $\operatorname{Proj}\mathscr{A}$ if and only if the center of the finite discrete part of $\mathscr{A}$ is finite dimensional.

Let us first consider $\mathscr{Z}$-linear combinations of projections. As shown by Pearcy and Topping [7], every self-adjoint operator in a properly infinite von Neumann algebra can be written as a real linear combination of eight projections. It follows from the results of Fack and de la Harpe [2] and Pearcy and Topping [6] that an algebra of type II_1, treated as a $\mathscr{Z}$-module, is generated by its projections [2, Corollaire 4.2(ii)]. Namely, in such an algebra, any operator with central trace zero is a sum of ten commutators [2], every commutator a sum of ten operators each having square zero, and every such operator a sum of sixteen projections. This results in a $\mathscr{Z}$-linear combination of 1,601 projections, with all but one coefficient from $\mathbf{C}$. It is also clear that any operator in an algebra of type I_n $(n < \infty)$ can be written as a $\mathscr{Z}$-linear combination of n^2 projections. By using the diagonal representation of a self-adjoint operator [1, Corollary 3.3], the number of projections required for such an operator can be reduced to n. The only obstacle in showing that the $\mathscr{Z}$-module $\mathscr{A}$ is generated by projections has been the case of an infinite direct sum of type I_n algebras. We show that any self-adjoint operator in a discrete finite algebra can be written as a $\mathscr{Z}$-linear combination of four projections.

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Let us next consider complex (or real) linear combinations of projections. As noted in [3, 7], some von Neumann algebras are not linearly spanned by projections. The above-mentioned results of Pearcy, Topping, Fack, and de la Harpe show that any properly infinite algebra and any factor of type II_1 is generated by projections. We prove that an arbitrary algebra of type II_1 is a complex linear span of its projections, thus giving an affirmative answer to the question raised by Pearcy and Topping in §3 of [7]. These results are complemented by the known results on type I_n factors (see [4, 5] and the references therein). With all this in mind it is easy to single out the class of algebras that are not generated by projections.

Finally, in von Neumann algebras without a finite discrete summand it is possible to write any self-adjoint operator using only sums and differences of projections.

In many cases considered, we managed to reduce substantially the number of projections required to span an operator. The results are contained in Theorems 1 and 2 and summed up in Theorem 3. Theorem 4 is a slight improvement of a result of Fack and de la Harpe [2] on commutators.

1. Statement of the results

In the sequel, we consider only self-adjoint operators $A \in \mathscr{A}$. We denote by τ the normalized trace on a finite factor $\mathscr{A}$. To facilitate the reading of the list of results, to each item of the list we add the information regarding the number of projections and the number of central coefficients required in the given representation of A (the other coefficients are simply real numbers).

Theorem 1. *Let $\mathscr{A}$ be a factor and let $A \in \mathscr{A}_h$.*

1. *If $\mathscr{A}$ is of type I_n, $n < \infty$, $\alpha = \tau(A)$, and $C = \tau(A)1_{\mathscr{A}}$, then*

(a) [4 *projections*] *for any β, $\gamma \geq 2\|A\|$, there are projections P, Q, R, S in $\mathscr{A}$ such that*

$$A = (\beta + \alpha)P - \beta Q + (\gamma + \alpha)R - \gamma S \tag{1}$$

when $\alpha \geq 0$ and

$$A = \beta P - (\beta - \alpha)Q + \gamma R - (\gamma - \alpha)S \tag{2}$$

when $\alpha \leq 0$.

(b) [5 *projections*] *for any β, $\gamma \geq \|A\|$, there are projections P, Q, R, S in $\mathscr{A}$ such that*

$$A = C + \beta(P - Q) + \gamma(R - S). \tag{3}$$

2. *If $\mathscr{A}$ is of type II_1 then*

(a) [12 *projections*] *for any $\alpha \geq \|A\|$ and any $\beta_i \geq 2\alpha$, $i = 1, \dots, 5$, there exist projections R_i, S_i, T, T' in $\mathscr{A}$ such that*

$$A = \sum_{i=1}^{5} \beta_i(R_i - S_i) + 2\alpha(T - T'). \tag{4}$$

3. *If $\mathscr{A}$ is properly infinite then*

(a) [4 *projections*] *for any β, $\gamma > 2\|A\|$, there are projections P, Q, R, S in $\mathscr{A}$ and $|\alpha| \leq \|A\|$ such that* (1) *holds when $\alpha \geq 0$ and* (2) *holds when $\alpha \leq 0$.*

(b) [5 *projections*] *for any* $\alpha \geq \|A\|$ *and any* $\beta, \gamma > \frac{3}{2}\alpha$, *there are projections* P, Q, R, S, T *in* $\mathscr{A}$ *such that*

$$A = \beta(P - Q) + \gamma(R - S) + \alpha T \tag{5}$$

or

$$A = \beta(P - Q) + \gamma(R - S) - \alpha T. \tag{6}$$

It should be noted that α is uniquely determined by A in case 1, depends on the spectral properties of A in case 3, and is arbitrary ($\geq \|A\|$) in case 2.

Theorem 2. *Let* $\mathscr{A}$ *be an arbitrary von Neumann algebra and let* $A \in \mathscr{A}_h$.

1. *If* $\mathscr{A}$ *is finite and discrete* (*and* C *is the center-valued trace of* A), *then*

(a) [4 *projections,* 4 *central coefficients*] *there are* $P_i \in \operatorname{Proj}\mathscr{A}$ *and* $C_i \in \mathscr{Z}$, $i = 1, \dots, 4$, *such that* $A = \sum_{i=1}^{4} C_i P_i$. *The operators* C_i *can be chosen so as to satisfy* $\|C_i\| \leq 3\|A\|$.

(b) [5 *projections,* 1 *central coefficient*] *for any* $\beta, \gamma \geq \|A\|$, *there are projections* P, Q, R, S *in* $\mathscr{A}$ *such that* (3) *holds.*

2. *If* $\mathscr{A}$ *is of type* II_1 *then*

(a) [12 *projections*] *the result is identical with* 2(a) *of Theorem* 1.

3. *If* $\mathscr{A}$ *is properly infinite then*

(a) [4 *projections,* 4 *central coefficients*] *there are* $P_i \in \operatorname{Proj}\mathscr{A}$ *and* $C_i \in \mathscr{Z}$, $i = 1, \dots, 4$ *such that* $A = \sum_{i=1}^{4} C_i P_i$. *For a fixed* $\varepsilon > 0$, *the operators* C_i *can be chosen so as to satisfy* $\|C_i\| \leq 3\|A\| + \varepsilon$.

(b) [5 *projections,* 1 *central coefficient*] *for any* $\beta, \gamma > \|A\|$, *there are projections* P, Q, R, S, T *in* $\mathscr{A}$ *and* $C \in \mathscr{Z}$ *such that*

$$A = \beta(P - Q) + \gamma(R - S) + CT, \tag{7}$$

and the operator C *can be chosen so as to satisfy* $\|C\| \leq \|A\|$.

(c) [6 *projections*] *for any* $\alpha \geq \|A\|$ *and any* $\beta, \gamma > \frac{3}{2}\alpha$, *there are projections* P, Q, R, S, T, T' *in* $\mathscr{A}$ *such that*

$$A = \beta(P - Q) + \gamma(R - S) + \alpha(T - T'). \tag{8}$$

Theorem 3. *Let* $\mathscr{A}$ *be a von Neumann algebra.*

1. *Any self-adjoint operator in* $\mathscr{A}$ *can be written as a linear combination of* 12 *projections, with* 4 *central and* 8 *real coefficients.*

2. $\mathscr{A}$ *is the complex linear span of its projections if and only if the center of the finite discrete part of* $\mathscr{A}$ *is finite dimensional. In particular, if it is m-dimensional* ($0 \leq m < \infty$), *then any self-adjoint operator is a real linear combination of at most* $m + 12$ *projections.*

3. $\mathscr{A}$ *is the linear span, with integer coefficients, of its projections if and only if it has no direct summand of finite discrete type. If this is the case, any self-adjoint operator of norm* ≤ 1 *can be represented in the form*

$$P_1 + \cdots + P_{12} - P_{13} - \cdots - P_{24} \quad \text{with } P_i \in \operatorname{Proj}\mathscr{A}. \tag{9}$$

Theorem 4. *Any self-adjoint operator of central trace zero in a von Neumann algebra of type* II_1 *is a sum of* 4 *commutators, and also a sum of* 12 *operators each having square zero.*

2. Proofs of the results

Lemma 5 (cf. [5]). *Let $E, F \in \operatorname{Proj}\mathscr{A}$, $EF = 0$, and let $U \in \mathscr{A}$ be a partial isometry such that $E = U^*U$, $F = UU^*$. For any $\beta \geq 0$ and any $D \in \mathscr{A}_h$ satisfying $-\beta E \leq D \leq \beta E$, there are projections $P, Q \leq E + F$ in $\mathscr{A}$ such that $D - UDU^* = \beta(P - Q)$. Moreover, for any $D \in \mathscr{A}$ there are operators $S, T \in \mathscr{A}$ satisfying $S^2 = T^2 = 0$ such that $D - UDU^* = S + T$.*

Proof. In the 2×2 matrix representation over $E\mathscr{A}E$,

$$\begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix} = \frac{\beta}{2}\begin{bmatrix} E + \frac{1}{\beta}D & (E - \frac{1}{\beta^2}D^2)^{1/2} \\ (E - \frac{1}{\beta^2}D^2)^{1/2} & E - \frac{1}{\beta}D \end{bmatrix} - \frac{\beta}{2}\begin{bmatrix} E - \frac{1}{\beta}D & (E - \frac{1}{\beta^2}D^2)^{1/2} \\ (E - \frac{1}{\beta^2}D^2)^{1/2} & E + \frac{1}{\beta}D \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} D & D \\ -D & -D \end{bmatrix} + \frac{1}{2}\begin{bmatrix} D & -D \\ D & -D \end{bmatrix}.$$

Lemma 6 (cf. [4, 5]). *Let $\mathscr{A}$ be an infinite factor and let $A \in \mathscr{A}_h$. There exists $a \in \mathbf{R}$ such that for any $\varepsilon > 0$, there are a sequence (ε_i), $i \in \mathbf{Z}\backslash\{0\}$ of positive numbers satisfying $\sum_{i\neq 0} \varepsilon_i \leq \varepsilon$ and a sequence (E_i), $i \in \mathbf{Z}$ of projections from $\mathscr{A}$ such that $E_i \sim 1$ and $E_iA = AE_i$ for all i, $\sum_{i\in\mathbf{Z}} E_i = 1$, and*

$$(a - \varepsilon_i)E_i \leq AE_i \leq (a + \varepsilon_i)E_i \quad \text{for } i \in \mathbf{Z}\backslash\{0\}.$$

Proof. There exists $a \in [-\|A\|, \|A\|]$ such that $e_A((a - \varepsilon, a + \varepsilon)) \sim 1$ for any $\varepsilon > 0$ (here e_A stands for the spectral measure of A). Indeed, it is enough to use the compactness of the interval $[-\|A\|, \|A\|]$. Now consider two cases:

(i) $e_A(\{a\}) \sim 1$. Choose E_i, $i \in \mathbf{Z}\backslash\{0\}$, E_0' so that $E_0' \sim E_i \sim 1$ and $\sum_{i\neq 0} E_i + E_0' = e_A(\{a\})$. Put $E_0 = E_0' + e_A(\mathbf{R}\backslash\{a\})$ and $\varepsilon_i = 0$ for all $i \neq 0$.

(ii) $e_A(\{a\}) \nsim 1$. First observe that we can replace the set of indices $\mathbf{Z}$ by any other countable set. Now define inductively the sequences $\varepsilon_1, \varepsilon_2, \dots$ and $E_0, E_1, \dots$ as follows. Put $\varepsilon_1 = \varepsilon/2$. Assuming $\varepsilon_1, \dots, \varepsilon_n$ already defined, choose $\varepsilon_{n+1} \leq 2^{-(n+1)}\varepsilon$ so that

$$E_n \overset{\text{def}}{=} e_A(]a - \varepsilon_n, a - \varepsilon_{n+1}] \cup [a + \varepsilon_{n+1}, a + \varepsilon_n[) \sim 1.$$

Finally, put $E_0 = 1_{\mathscr{A}} - \sum_{i=1}^{\infty} E_i$.

Lemma 7 (cf. [2]). *Let $\mathscr{A}$ be a von Neumann algebra acting in a separable Hibert space H, let $\mathscr{Z}$ be its center, and let $H = \int_X^{\oplus} H(x)\, d\mu(x)$ and $\mathscr{A} = \int_X^{\oplus} \mathscr{A}(x)\, d\mu(x)$ be the central direct integral decompositions of H and $\mathscr{A}$ over (X, μ), where X is Polish, locally compact and σ-compact and μ is Radon.*

(a) *Let $I_1, \dots, I_m$ be some compact subsets of the real line. Assume that for any $x \in X\backslash N$, where N is μ-negligible, and any $A \in \mathscr{A}(x)_h$, there are $P_i \in \operatorname{Proj}\mathscr{A}(x)$, $\alpha_i \in I_i$, $i = 1, \dots, m$, such that $A = \sum_{i=1}^m \alpha_i P_i$. It follows that for any $A \in \mathscr{A}_h$, there are $P_i \in \operatorname{Proj}\mathscr{A}$ and $C_i \in \mathscr{Z}_h$, $i = 1, \dots, m$, such that $A = \sum_{i=1}^m C_i P_i$. Moreover, $\|C_i\| \leq \max\{|\alpha| : \alpha \in I_i\}$, and if for some $1 \leq j_1 < \cdots < j_k \leq m$, $I_{j_i} = \{\gamma_{j_i}\}$ for $i = 1, \dots, k$, then (C_i) can be chosen so that $C_{j_k} = \gamma_{j_k} 1_{\mathscr{A}}$.*

(b) *Let $\mathscr{A}$ be a finite von Neumann algebra. If, for $x \in X\backslash N_0$ with μ-negligible N_0, any Hermitian operator from $\mathscr{A}(x)$ with trace zero and norm ≤ 1 is a sum of m operators from $\mathscr{A}(x)$ each having square zero and norm*

≤ 3, *then any Hermitian operator from* $\mathscr{A}$ *with center-valued trace zero is a sum of* m *operators from* $\mathscr{A}$ *each having square zero.*

Proof. (a) We may assume $x \mapsto H(x)$ to be a constant field of Hilbert spaces, so that all the algebras $\mathscr{A}(x)$ act in the same Hilbert space H_0. Let (B_j) be a sequence of self-adjoint operators from $\mathscr{A}$ of norm ≤ 1 such that for μ-almost every x, the set $\{B_j(x)\}$ generates $\mathscr{A}(x)'$. Let X_0 be a Borel subset of X with a μ-negligible complement containing N, such that all the fields $x \mapsto A(x)$ and $x \mapsto B_j(x)$ are Borel when restricted to X_0. Put $Y = X_0 \times I_1 \times \mathscr{P} \times \cdots \times I_m \times \mathscr{P}$ where $\mathscr{P}$ is the set of operators from $\mathbf{B}(H_0)_h$ with norm ≤ 1, equipped with its strong-operator topology. It is evident that Y is Polish. Denote by $\mathscr{S}$ the set of those $(2m+1)$-tuples $(x, \alpha_1, P_1, \dots, \alpha_m, P_m)$ from Y that satisfy the following conditions:

(i) $P_i = P_i^2$;
(ii) $P_i B_j(x) = B_j(x) P_i$ for all $i = 1, \dots, m$ and all j;
(iii) $A(x) = \sum_{i=1}^m \alpha_i P_i$.

It is clear that $\mathscr{S}$ is Borel and that $\mathrm{pr}_1(\mathscr{S}) = X_0$. By the measurable selection principle, there exists a μ-measurable mapping

$$x \mapsto (\alpha_1(x), P_1(x), \dots, \alpha_m(x), P_m(x))$$

defined on X_0 such that

$$(x, \alpha_1(x), P_1(x), \dots, \alpha_m(x), P_m(x)) \in \mathscr{S} \quad \text{for } x \in X_0.$$

Put $\alpha_i(x) = 0$ and $P_i(x) = 0$ for $x \in X \backslash X_0$. Then $P_i = \int_X^\oplus P_i(x)\, d\mu(x)$ are projections from $\mathscr{A}$ and $C_i = \int_X^\oplus \alpha_i(x) 1_{\mathscr{A}(x)}\, d\mu(x)$ are diagonalizable so that $C_i \in \mathscr{Z}_h$. Moreover, $A = \sum_{i=1}^m C_i P_i$, which concludes the proof of the first assertion of the lemma.

(b) It τ is the center-valued trace on $\mathscr{A}$ and $\tau = \int_X^\oplus \tau_x\, d\mu(x)$ is its central direct integral decomposition, then for μ-almost every x, τ_x is a (not necessarily normalized) trace on $\mathscr{A}(x)$. If $\tau(A) = 0$, then for $x \in X \backslash N_1$ with some μ-negligible N_1, $\tau_x(A(x)) = 0$. The rest of the proof can be obtained as in (a), mutatis mutandis. In particular, one should start with a Hermitian operator of norm ≤ 1 and replace N by $N_0 \cup N_1$.

Proof of Theorem 1. 1(a) This is a result of Paszkiewicz [5]. It should only be noted that the proof given there for the case of even n works equally well if n is odd and > 1. If $n = 1$, put $P = Q = 1$, $R = S = 0$.

1(b) (cf. [5, 2.5]). Assume that $\|A\| \leq 1$, $\tau(A) = 0$, and $n > 1$ (if $n = 1$, put $P = Q = R = S = 0$). Take any $\beta, \gamma \geq 1$. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ in H consisting of eigenvectors of A, so that $A = \sum_{i=1}^n \alpha_j \hat{e}_j$ where $\hat{e}_j = \langle \cdot, e_j \rangle e_j$. The terms in the spectral representation of A can be ordered so that for each j, $j = 1, \dots, n$, $\beta_j \overset{\text{def}}{=} \alpha_1 + \cdots + \alpha_j \in [-1, 1]$. Indeed, let α_1 be an arbitrary eigenvector of A. Note that $\beta_n = 0$, which makes it possible to order the terms inductively as follows: we choose α_{j+1} positive when $\beta_j < 0$, negative when $\beta_j > 0$, and arbitrary when $\beta_j = 0$ $(j < n)$. Assume this particular ordering. By Lemma 5, there are projections P_j, Q_j in $\mathscr{A}$ such that

$$\beta_j(e_j - e_{j+1}) = \begin{cases} \beta(P_j - Q_j) & \text{for even } j, \\ \gamma(P_j - Q_j) & \text{for odd } j. \end{cases}$$

Since $\beta_n = 0$, we have

$$A = \beta_1(\hat{e}_1 - \hat{e}_2) + \cdots + \beta_{n-1}(\hat{e}_{n-1} - \hat{e}_n)$$
$$= \beta\left(\sum_{j \text{ even}} P_j - \sum_{j \text{ even}} Q_j\right) + \gamma\left(\sum_{j \text{ odd}} P_j - \sum_{j \text{ odd}} Q_j\right).$$

2(a) (cf. [2]). We start with taking any $\alpha \geq \|A\|$ and replacing A by A/α. As in [2], denote by $V(Z)$ the diameter of the spectrum of Z. Let P, Q be two complementary projections from $\mathscr{A}$, commuting with A and such that $\tau(P) = \tau(Q) = \frac{1}{2}$ (see [2, Lemma 1.4]). Choose any orthogonal sequence (P_n) of projections from $\mathscr{A}$ such that $P_1 = P$, P_2 commute with AQ, and $\tau(P_n) = 2^{-n}$. Put $Z_1 = A_1 = AP_1$ so that $V(A_1) \leq 2$, and define (A_n) (and (Z_n)) inductively as follows. Suppose $A_1, \dots, A_n$ are already defined, $\|A_n\| \leq 2$, and $V(A_n) \leq 2$. According to Lemma 2.2 of [2] (where we do not assume $\tau(Z) = 0$ and do not insist on having $\tau(S) = 0$), there exist in $\mathscr{A}$ projections P_n', P_n'' that commute with A_n and have sum P_n and partial isometries W_n', W_n'' with $W_n'^* W_n' = P_n'$, $W_n''^* W_n'' = P_n''$, and $W_n' W_n'^* = P_{n+1} = W_n'' W_n''^*$ such that $V(Z_{n+1}) \leq V(A_n)$ where $Z_{n+1} = W_n' A_n W_n'^* + W_n'' A_n W_n''^*$. Put $A_2 = Z_2$ and (noting that $\|Z_{n+1}\| \leq 4$) define A_{n+1} for $n \geq 2$ to be $Z_{n+1} - \xi_n P_{n+1}$ where $\xi_n \in \{-2, 0, 2\}$ is chosen so that $\|A_{n+1}\| \leq 2$. Obviously, $V(A_{n+1}) \leq 2$, and it is not difficult to see that $A_1 = (A_1 - A_2) + (A_2 - A_3) + \cdots$ in the strong-operator topology. By Lemma 5, for any $\beta_i \geq 2$, $i = 1, \dots, 4$, and any n, there are projections R_n', S_n', R_n'', S_n'' in $\mathscr{A}$ such that

$$Z_n P_n' - W_n' Z_n W_n'^* = \begin{cases} \beta_1(R_n' - S_n') & \text{for even } n, \\ \beta_2(R_n' - S_n') & \text{for odd } n \end{cases}$$

and

$$Z_n P_n'' - W_n'' Z_n W_n''^* = \begin{cases} \beta_3(R_n'' - S_n'') & \text{for even } n, \\ \beta_4(R_n'' - S_n'') & \text{for odd } n. \end{cases}$$

Hence

$$A_1 = \beta_1 \sum_{n \text{ even}} (R_n' - S_n') + \beta_2 \sum_{n \text{ odd}} (R_n' - S_n')$$
$$+ \beta_3 \sum_{n \text{ even}} (R_n'' - S_n'') + \beta_4 \sum_{n \text{ odd}} (R_n'' - S_n'') + \sum_{n=1}^{\infty} \xi_n P_{n+1}.$$

The last term of the above equality can be written as $2(T_1 - T_1')$ where $T_1, T_1' \in \operatorname{Proj}\mathscr{A}$ and $T_1, T_1' \leq Q$.

Now choose an orthogonal sequence (Q_n) of projections from $\mathscr{A}$ such that $Q_1 = Q$, $Q_2 = P_1'$, and $\tau(Q_n) = 2^{-n}$. Put $Z_1^Q = A_1^Q = AQ$ and imitate the procedure from the first paragraph of the proof. To define (A_n^Q) and (Z_n^Q), use Q_n, Q_n', Q_n'' in place of P_n, P_n', P_n'' and U_n', U_n'' in place of W_n', W_n'', guaranteeing additionally that $Q_1' = P_2$ and $U_1' = W_1'^*$ (now, $V(Z_{n+1}^Q) \leq V(A_n^Q)$ for $n \geq 2$). Write A_1^Q in the same form as A_1 with R_n', S_n', R_n'', S_n'' replaced by ${}^Q R_n'$, ${}^Q S_n'$, ${}^Q R_n''$, ${}^Q S_n''$, respectively, with P_{n+1} replaced by Q_{n+1} and β_4 replaced by some $\beta_5 \geq 2$. Note that $R_n' - S_n'$, $R_n'' - S_n'' \leq Q$ and ${}^Q R_n' - {}^Q S_n'$, ${}^Q R_n'' - {}^Q S_n'' \leq P$ for $n \geq 2$, and the last term in the representation of A_1^Q is $\leq Q$. Moreover, by virtue of Lemma 5 with $D = A_1 P_1' - W_1'^* A_1^Q W_1'$,

$$\begin{aligned}\beta_2((R_1' - S_1') + ({}^Q R_1' - {}^Q S_1')) &= (A_1 - A_2) P_1' + (A_1^Q - A_2^Q) Q_1' \\ &= D - W_1' D W_1'^* = \beta_2(R_1 - S_1) \leq P_2 + Q_2\end{aligned}$$

for some R_1, $S_1 \in \operatorname{Proj}\mathscr{A}$. All the terms with $n \geq 3$ are orthogonal to P_2+Q_2, which yields (4).

3(a) The proof is essentially the same as the proof of Theorem 1.B from [5]. One should only replace Lemma 2.2 of [5] with our Lemma 6.

3(b) (cf. [5, 2.8]). Start with replacing A by A/α so that $\|A\| \leq 1$. Take ε such that β, $\gamma > \frac{3}{2} + \varepsilon$. Let a, (ε_n), and (E_n) be as in Lemma 6. Choose a sequence (ξ_n) with $\xi_0 = 0$ and, for $n \geq 1$, $\xi_n \in \{0, 1\}$ or $\xi_n \in \{0, -1\}$ depending on the sign of a, such that for any $n \geq 1$, $\sum_{j=1}^n (a - \xi_j) \in [-\frac{1}{2}, \frac{1}{2}]$. As in [5, 2.6], we denote by h the mapping $\operatorname{ad}(U)$ on $\mathbf{B}(H)$, where $U = \sum_{j\neq 0} U_j$ and U_j are such that $U_j^* U_j = E_j$, $U_j U_j^* = E_{j+1}$. Put $A_j = AE_j - \xi_{|j|} E_j$ and define

$$B_n = \begin{cases} 0 & \text{for } n = 0, \\ A_1 & \text{for } n = 1, \\ -h^{-1} A_0 & \text{for } n = -1, \\ h^{n-1} A_1 + \cdots + A_n & \text{for } n \geq 2, \\ -h^{-1} A_{n+1} - \cdots - h^{-n} A_0 & \text{for } n \leq -2. \end{cases}$$

We easily check that $A_n = B_n - hB_{n-1}$ and that for any n, $-(\frac{3}{2} + \varepsilon) E_n \leq B_n \leq (\frac{3}{2} + \varepsilon) E_n$. Now, apply Lemma 5 with $D = B_n$ to obtain

$$A - \sum_{j\in\mathbf{Z}} \xi_j E_j = \beta(P - Q) + \gamma(R - S)$$

for some P, Q, R, $S \in \operatorname{Proj}\mathscr{A}$.

Proof of Theorem 2. 1(a) If $\mathscr{A}$ is of type I and acts in a separable Hilbert space, we can apply Lemma 7(a) and Theorem 1.1(a) with β, $\gamma = 2$ to obtain the conclusion. If, in turn, $\mathscr{A}$ is σ-finite, then the subalgebra of $\mathscr{A}$ generated by A and a system $\{U_{ij}\}_{i,j=1,\dots,n}$ of matrix units from $\mathscr{A}$ is of type I_n and its predual is separable (i.e., it is isomorphic to an algebra acting in a separable Hilbert space). Finally, an arbitrary discrete and finite von Neumann algebra is a direct sum of σ-finite algebras of type I_n.

1(b) We consider the operator $A - C$ in place of A and proceed (almost) as in 1(a) (cf. proof of Lemma 7(b)).

2(a) For algebras acting in a separable Hilbert space the result follows immediately from Theorem 1.2(a) and Lemma 7(a). If $\mathscr{A}$ is σ-finite, we generate a subalgebra containing A, which is of type II_1 with separable predual. The generators consist of A and a suitable sequence of partial isometries (see the remark following the proof of Lemma 3.1 in [2]). If $\mathscr{A}$ is not σ-finite, it is a direct sum of σ-finite algebras of type II_1.

3(a) If $\mathscr{A}$ has separable predual, the result follows from Theorem 1.3(a) and Lemma 6(a). If $\mathscr{A}$ is arbitrary, we replace $\mathscr{A}$ by a properly infinite subalgebra $\mathscr{B}$ generated by A and a system $\{U_{ij}\}_{i,j\in N}$ of matrix units from $\mathscr{A}$. Let $\mathscr{Z}$ be the center of $\mathscr{B}$, and let $C_k \in \mathscr{Z}$ be such that the algebras $\mathscr{Z} C_k$ are

σ-finite and $\sum_k Z_k = 1_{\mathscr{B}}$. Then $\mathscr{B}$ is a direct sum of the σ-finite algebras $\mathscr{B}C_k$, each generated by the operators AC_k and $U_{ij}C_k$. Hence, $\mathscr{B}$ is a direct sum of algebras with separable predual.

3(b). Formula (7) follows from Theorem 1.3(b) in exactly the same way as described above.

3(c) Note that (5) and (6) can be replaced by one formula only:

$$A = \beta(P - Q) + \gamma(R - S) + \alpha(T - T'),$$

with $T = 0$ or $T' = 0$. The rest is the same as in 3(a) and 3(b).

Proof of Theorem 3. 1. This is an immediate consequence of Theorem 2, points 1(a), 2(a), 3(a)—note that centrally orthogonal projections, as well as mutually orthogonal central coefficients, may be summed up to yield one term in the final representation of the operator.

2. Suppose the center of the finite discrete part of $\mathscr{A}$ is m-dimensional. Put $\alpha = \|A\|$, $\beta_i = \beta = \gamma = 2\|A\|$. Represent $\mathscr{A}$ as a direct sum of a properly infinite algebra, a type II_1 algebra and m type I_n factors (with possibly different n). Use each of the Theorems 2.2(a) and 2.3(c) once, and Theorem 1.1(b) m times, to write the respective part of A as a real combination of projections. A simple grouping of the terms gives the required formula for A.

Now suppose that the center of the finite discrete part of $\mathscr{A}$ is infinite dimensional. We can restrict our consideration to the following two cases: 1° $\mathscr{A}$ is of type I_n $(n < \infty)$ and the center of $\mathscr{A}$ is infinite dimensional. In this case we denote by (Z_k) an infinite sequence of mutually orthogonal, nonzero projections from the center of $\mathscr{A}$. 2° $\mathscr{A}$ is an infinite direct sum of type I_{n_k} factors $\mathscr{A}_{n_k}$ $(k = 1, 2, \dots)$ with $n_1 < n_2 < \cdots$. This time we denote by (Z_k) the sequence of (nonzero) minimal projections of $\mathscr{A}$ for which $Z_k \in \mathscr{A}_{n_k}$.

Let (β_k) be a bounded sequence of real numbers satisfying $\beta_{k+1} \notin \beta_1\mathbf{Q} + \cdots + \beta_k\mathbf{Q}$ for $k = 1, 2, \dots$, $\beta_1 \neq 0$ (i.e., β_k are linearly independent over $\mathbf{Q}$). We shall show that, in either case, the operator $A = \sum_{k=1}^{\infty} \beta_k Z_k$ is not a finite complex linear combination of projections. Suppose that, on the contrary, $A = \sum \alpha_i P_i$ for some $\alpha_1, \dots, \alpha_m \in \mathbf{C}$, $P_1, \dots, P_m \in \operatorname{Proj} \mathscr{A}$.

For case 1°, the center-valued trace $\tau(P)$ of any projection P is a finite linear combination of nonzero central projections. Hence, $A = \tau(A)$ would also be such a linear combination. This is clearly impossible, as the sequence (β_k) consists of distinct numbers.

In case 2°, we have $\tau(P_i) = \sum_{k=1}^{\infty} l_k^{(i)} Z_k$ for some $l_k^{(i)} \in \mathbf{Q}$, $i, k = 1, 2, \dots$. Therefore,

$$\begin{gathered} l_1^{(1)}\alpha_1 + \cdots + l_1^{(m)}\alpha_m = \beta_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ l_{m+1}^{(1)}\alpha_1 + \cdots + l_{m+1}^{(m)}\alpha_m = \beta_{m+1}. \end{gathered}$$

The independence of $\beta_1, \dots, \beta_{m+1}$ over $\mathbf{Q}$ implies that the above system of linear equations in $\alpha_1, \dots, \alpha_m$ is contradictory.

3. If $\mathscr{A}$ is of type I_n, the center-valued trace of a projection is a rational linear combination of projections. Hence an operator such as $A = \pi 1$ is not an integral linear combination of projections.

Now let $\mathscr{A}$ be a direct sum of an algebra of type II_1 and a properly infinite algebra, and let $A \in \mathscr{A}_h$, $\|A\| \leq 1$. Put $\alpha = 1$ and $\beta_i = \beta = \gamma = 2$ in (4) and

(8) and sum the representations of the parts of A corresponding to the finite and properly infinite part of the algebra.

This ends the proof of the theorem.

Proof of Theorem 4. We can proceed as in [2] (using Lemma 5 if applicable) and reduce the numbers of terms needed by using the argument from the proof of Theorem 1.2(a).

REFERENCES

1. D. Deckard and C. Pearcy, *On matrices over the ring of continuous complex-valued functions on a Stonian space*, Proc. Amer. Math. Soc. **14** (1963), 322–328.
2. Th. Fack and P. de la Harpe, *Somme de commutateurs dans les algèbres de von Neumann finies continues*, Ann. Inst. Fourier (Grenoble) **30** (1980), 49–73.
3. P. A. Fillmore and D. M. Topping, *Operator algebras generated by projections*, Duke Math. J. **34** (1967), 333–336.
4. A. Paszkiewicz, *Any self-adjoint operator is a finite linear combination of projectors*, Bull. Acad. Polon. Sci. **28** (1980), 337–345.
5. ——, *Self-adjoint operators as a real span of* 4 *projections*, Math. Japon. (to appear).
6. C. Pearcy and D. Topping, *Commutators and certain* II_1*-factors*, J. Funct. Anal. **3** (1969), 69–78.
7. ——, *Sums of small numbers of idempotents*, Michigan Math. J. **14** (1968), 453–465.

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