SAEKI'S IMPROVEMENT
OF THE VITALI-HAHN-SAKS-NIKODYM THEOREM
 HOLDS PRECISELY FOR BANACH SPACES HAVING COTYPE

PAUL ABRAHAM

(Communicated by Andrew M. Bruckner)

Abstract. We prove that a Banach space $X$ has nontrivial cotype if and only if given any $\sigma$-field $\Sigma$ and any sequence $\mu_n: \Sigma \to X$ of strongly additive vector measures such that for some $\gamma \geq 1$, $\limsup_{n\to\infty} \|\mu_n(E)\| \leq \gamma \liminf_{n\to\infty} \|\mu_n(E)\| < \infty$ for each $E \in \Sigma$ then $\{\mu_n: n \in \mathbb{N}\}$ is uniformly strongly additive.

In a recent note [S] Saeki introduced the notion of a measuroid and, based on his work in measuroids, was able to substantially improve the classical Vitali-Hahn-Saks-Nikodym Theorem [DU, p. 23]—but a price must be paid. The price: the Banach space must satisfy the following “fatness” condition: for each constant $C > 0$ there exists a positive integer $m$ such that given $x_1, \ldots, x_m \in X$, $\|x_i\| \geq 1$ for each $i = 1, \ldots, m$, there exists $F \subseteq \{1, \ldots, m\}$ such that $\| \sum_{i \in F} x_i \| \geq C$.

The payoff: given a sequence $(\mu_n)$ of strongly additive $X$-valued vector measures defined on a $\sigma$-field $\Sigma$ such that there exists a constant $\gamma \geq 1$ so that for each $E \in \Sigma \limsup_{n\to\infty} \|\mu_n(E)\| \leq \gamma \liminf_{n\to\infty} \|\mu_n(E)\| < \infty$ then $\{\mu_n\}$ is uniformly strongly additive.

In this note we relate Saeki’s fatness condition precisely with the geometry of the Banach space. First, a couple of definitions.

Definition 1. We say a Banach space $X$ has cotype $q$ ($\geq 2$) if there is a constant $K_q > 0$ such that for each $n \geq 1$, $x_1, \ldots, x_n \in X$, we have

$$\left( \sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leq K_q \int_{0}^{1} \left\| \sum_{k=1}^{n} r_k(t)x_k \right\| \, dt$$

where $(r_n)$ denotes the Rademacher sequence on $[0, 1]$.
Definition 2. We say a Banach space $X$ contains the $l^n_\infty$'s uniformly if there is a constant $\lambda > 1$ such that for each $n \geq 1$ there is an $n$-dimensional subspace $E_n$ of $X$ and isomorphism $\varphi_n: l^n_\infty \to E_n$ such that $\|\varphi_n\| \|\varphi_n^{-1}\| < \lambda$.

Maurey and Pisier [MP] have shown that for any Banach space $X$, $X$ has some cotype if and only if $X$ does not contain the $l^n_\infty$'s uniformly.

Proposition 3. Let $X$ be any Banach space. The following are equivalent:

(a) $X$ has cotype.

(b) $X$ satisfies the fatness condition.

(c) If $(\mu_n)$ is a sequence of strongly additive (respectively, countably additive) $X$-valued vector measures on a $\sigma$-algebra $\Sigma$ such that there exists $\gamma \geq 1$ such that for each $A \in \Sigma$ we have $\limsup_{n \to \infty} \|\mu_n(A)\| \leq \gamma \liminf_{n \to \infty} \|\mu_n(A)\| < \infty$, then $(\mu_n)$ is uniformly strongly additive (respectively, uniformly countably additive).

(d) $X$ does not contain the $l^n_\infty$'s uniformly.

Proof. (a) $\Rightarrow$ (b). Suppose $X$ has cotype $q > 2$. Observe that if $x_1, \ldots, x_n \in X, \|x_i\| \geq 1$ for each $i = 1, \ldots, n$ then we have

$$n^{1/q} \leq \left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leq K_q \int_{0}^{1} \left(\sum_{k=1}^{n} r_k(t)x_k\right) dt \leq K_q 2^{-n} \sum_{\varepsilon_1 = \pm 1, \ldots, \varepsilon_n = \pm 1} \|\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n\|,$$

where $K_q > 0$ is the cotype $q$ constant. Since the right-hand sum has $2^n$ terms, for some choice say $\varepsilon'_1 = \pm 1, \ldots, \varepsilon'_n = \pm 1$ we have

$$n^{1/q} K_q^{-1} \leq \|\varepsilon'_1 x_1 + \varepsilon'_2 x_2 + \cdots + \varepsilon'_n x_n\|.$$

Let $P = \{i|\varepsilon'_i = 1\}$ and $N = \{i|\varepsilon'_i = -1\}$. From the triangle inequality and (*) we deduce $\|\sum_{i \in P} x_i\| \geq 2^{-1} n^{1/q} K_q^{-1}$ or $\|\sum_{i \in N} x_i\| \geq 2^{-1} n^{1/q} K_q^{-1}$.

Hence, given $C > 0$, choose $n$ so that $2^{-1} n^{1/q} K_q^{-1} \geq C$ in order to fulfill the fatness condition.

(b) $\Rightarrow$ (c). [S, Corollary 8].

(c) $\Rightarrow$ (d). Suppose $X$ contains the $l^n_\infty$'s uniformly. Then, we have a constant $\lambda > 1$ such that for each $n \geq 1$ there is an $n$-dimensional subspace $E_n$ of $X$ and an isomorphism $\varphi_n: l^n_\infty \to E_n$ such that $\|\varphi_n\| = 1$ and $\|\varphi_n^{-1}\| < \lambda$.

Therefore, for each $n \geq 1$ and each $F \subseteq \{1, \ldots, n\}, F \neq \emptyset$,

$$\left(\sum_{i \in F} \varphi_n(e^{(n)}_i)\right), \left(\sum_{i \in F} \varphi_n(e^{(n)}_i)\right) \geq \lambda^{-1},$$

where $e^{(n)}_1, \ldots, e^{(n)}_n$ denotes the unit vector basis elements of $l^n_\infty$.

Now, for each $n \geq 1$, define $\mu_n: P(\mathbb{N}) \to X$ by $\mu_n(\Delta) = \sum_{i \in \Delta \cap \{1, \ldots, n\}} \varphi_n(e^{(n)}_i)$. Clearly, each $\mu_n$ is finitely additive. In fact, each $\mu_n$ is countably additive since given $(B_i) \subseteq P(\mathbb{N}), B_i \cap B_j = \emptyset$ for each $i \neq j$, $B_i \cap \{1, \ldots, n\} = \emptyset$ for all $i$ sufficiently large. From (**) and (***) it follows that for each $\Delta \in P(\mathbb{N})$,
\[ \limsup_{n \to \infty} \|\mu_n(\Delta)\| \leq \lambda \liminf_{n \to \infty} \|\mu_n(\Delta)\| < \infty. \]
However, \((\mu_n)\) is not even uniformly strongly additive since \( \|\mu_n(\{n\})\| \geq \lambda^{-1} \) for each \( n \geq 1 \).

(d) \Rightarrow (a). One direction of the theorem of Maurey and Pisier already noted.

**Remark 4.** It is not difficult to see that in Proposition 3 we can add the following equivalent statement:

\((c')\) If \((\mu_n)\) is a sequence of \(X\)-valued vector measures on a \(\sigma\)-field \(\Sigma\) and \(m\) is a countably additive nonnegative measure such that for each \(n\), then \(\mu_n\) is \(m\)-continuous, and if there exists a constant \(\gamma \geq 1\) such that for each \(A \in \Sigma\) we have \(\limsup_{n \to \infty} \|\mu_n(A)\| \leq \gamma \liminf_{n \to \infty} \|\mu_n(A)\| < \infty\) then \(\{\mu_n\}\) is uniformly \(m\)-continuous.

Hence, we also have an improvement of the classical Vitali-Hahn-Saks Theorem [DU, p. 24] that holds precisely when the Banach space has cotype.

**References**


**Department of Mathematics and Computer Science, Kent State University, Kent, Ohio 44242**

*E-mail address:* pabraham@mcs.kent.edu