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D'ALEMBERT FUNCTIONAL EQUATIONS IN DISTRIBUTIONS

E. Y. DEEBA AND E. L. KOH

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ABSTRACT. In this paper we shall develop a method to define and solve the D'Alembert functional equation in distributions. We shall also show that for regular distributions (i.e., locally integrable functions) the distributional solution reduces to the classical one.

1. INTRODUCTION

In recent years there has been a good deal of interest in studying functional equations in distributions (e.g., see [5–9, 12, 14, 16]). This interplay between functional equations and distributions may become essential if one recognizes that, although the theory of differential equations provides ample tools to handle functional equations, the assumed differentiability of the latter equation is not typically natural. To overcome this deficiency, it was pointed out (see [1, 7]) that one may appeal to some regularity theorems of the type "continuity implies differentiability" (see [1]) or "measurability implies continuity" (see [1, 4, 10, 13]). The theory of distributions [15], with its elegant justification for such objects as Hadamard's finite part of a divergent integral and Dirac's delta function and with the greater 'reservoir' it has given for solutions of partial differential equation, provides another avenue to overcome such deficiencies and shortcomings.

It is the purpose of this paper to reformulate and solve the D'Alembert functional equation in the domain of distributions.

The D'Alembert functional equation is given by

$$f(x+y)+f(x-y)=2f(x)f(y) \tag{1.1}$$

whose nontrivial solutions are $f(x) = \cos cx$ and $f(x) = \cosh cx$ (see [2] for method of solutions). We shall consider a slightly more general form of equation (1.1). Namely, we shall consider the functional equation (see [2])

$$f(x+y)+g(x-y)=2f(x)g(y) \tag{1.2}$$

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whose solutions satisfy a second order linear differential equations with constant coefficients. We note that the left-hand side of (1.2) is the general form of solution of the wave equation $u_{xx} = u_{yy}$.

We shall develop a method to formulate and solve (1.2) in distributions. Moreover, we shall show that for regular distributions (locally integrable functions) the distributional solution reduces to the classical one. In [6] Baker obtained a similar formulation using translation operators but restricting the right side of (1.2) to a product of a distribution with an infinitely differentiable function. In this case, a tensor product operator is not necessary.

In the next section we summarize the background material needed in the sequel.

Preliminaries

2.1. **Some notation.** Let $I = (0, \infty) \subset R$ and $I^2 = I \times I \subset R^2$. We denote by $\mathscr{D}(I)$ and $\mathscr{D}(I^2)$ the spaces of infinitely differentiable functions on I and I^2 with compact support, respectively. Likewise, $\mathscr{E}(I)$ and $\mathscr{E}(I^2)$ are the spaces of infinitely differentiable functions on I and I^2, respectively. The duals of these spaces will be denoted by a prime, e.g., $\mathscr{D}'(I)$, and we note that $\mathscr{D}(I) \subset \mathscr{E}(I) \subset \mathscr{E}'(I) \subset \mathscr{D}'(I)$ (see [15]). The second inclusion will be interpreted by identifying the smooth function in $\mathscr{E}(I)$ with the regular distribution it generates in $\mathscr{E}'(I)$. The topologies for these spaces will be the usual convergence concepts for $\mathscr{D}(I)$ and $\mathscr{E}(I)$ as given in [15] and the weak topologies for their duals. $L_{\mathrm{Loc}}(I)$ and $L_{\mathrm{Loc}}(I^2)$ denote the spaces of equivalence classes of locally integrable functions on I and I^2, respectively. Two functions are equivalent if they are equal almost everywhere. We shall denote the distribution corresponding to a locally integrable function f by λ_f. If $f \in L_{\mathrm{Loc}}(I)$ then

$$\langle \lambda_f, \phi \rangle = \int_I f(x)\phi(x)\,dx$$

for any $\phi \in \mathscr{D}(I)$.

D denotes the differentiation operator on $\mathscr{D}'(I)$ whereas D_1 and D_2 are the partial differentiation operators on $\mathscr{D}'(I^2)$ with respect to the first and second variable from I^2, respectively. These symbols will also denote the differentiation operators on subspaces of $\mathscr{D}'$, e.g., on $\mathscr{D}$.

2.2. **Some operators on $\mathscr{D}'$.** Let E_1 and E_2 be integration operators from $\mathscr{D}(I^2)$ into $\mathscr{D}(I)$ given, respectively, for any $\phi \in \mathscr{D}(I^2)$, by

$$E_1[\phi](x) = \int_I \phi(x, y)\,dy \tag{2.1}$$

and

$$E_2[\phi](y) = \int_I \phi(x, y)\,dx. \tag{2.2}$$

It is easy to see that these are continuous linear operators, and we shall denote this by membership in $L[\mathscr{D}(I^2); \mathscr{D}(I)]$. Their adjoints E_i^* $(i = 1, 2)$ again are continuous linear operators from $\mathscr{D}'(I)$ onto $\mathscr{D}'(I^2)$ and are given by

$$\langle E_1^*[T], \phi \rangle = \langle T, E_1[\phi] \rangle = \langle T(x), \int_I \phi(x, y)\,dy \rangle$$

and

$$\langle E_2^*[T], \phi\rangle = \langle T, E_2[\phi]\rangle = \langle T(y), \int_I \phi(x, y)\,dx\rangle$$

for any $T \in \mathscr{D}'(I)$ and $\phi \in \mathscr{D}'(I^2)$.

In the next proposition we shall summarize some properties of the above operators. The proof of these properties is analogous to [12] and thus we shall omit it.

Proposition 2.1. (a) *If* $f \in L_{\mathrm{Loc}}(I)$ *and* $\phi(x, y) = f(x)$ *for* $(x, y) \in I^2$, *then* $\phi \in L_{\mathrm{Loc}}(I^2)$ *and* $E_1^*(\lambda_f) = \phi$. *Likewise,*

(b) *If* $\alpha \in \mathscr{E}(I)$, *then* $E_i^*[\alpha](i = 1, 2)$ *belong to* $\mathscr{E}(I^2)$.

(c) *If* $\alpha \in \mathscr{E}(I)$ *and* $T \in \mathscr{D}'(I)$, *then, for* $i = 1, 2$, $E_i^*[\alpha T] = E_i^*[\alpha]E_i^*[T]$.

(d) *If* $T \in \mathscr{D}'(I)$, *then*

$$D_1E_1^*[T] = E_1^*[DT] \quad \textit{and} \quad D_1E_2^*[T] = 0,$$
$$D_2E_2^*[T] = E_2^*[DT] \quad \textit{and} \quad D_2E_1^*[T] = 0.$$

In order to define equation (1.2) in distributions one needs to define a tensor product operator on $\mathscr{D}'(I) \times \mathscr{D}'(I)$ to replace the product of the functions $f(x)g(y)$ (see [11]). Namely, we shall define the linear operator $P: \mathscr{D}'(I) \times \mathscr{D}'(I) \to \mathscr{D}'(I^2)$ by

$$\langle P[S, T], \phi\rangle = \langle S(x), \langle T(y), \phi(x, y)\rangle\rangle = \langle T(y), \langle S(x), \phi(x, y)\rangle\rangle$$

for any $S, T \in \mathscr{D}'(I)$ and any $\phi \in \mathscr{D}'(I^2)$. It can be easily shown that, for any $S, T \in \mathscr{D}'(I)$,

$$D_1P[S; T] = P[DS; T] \quad \text{and} \quad D_2P[S; T] = P[S; DT]. \tag{2.3}$$

The relationship between the operator P and the integration operators is given, without proof, in the following proposition.

Proposition 2.2. (a) *If* $T \in \mathscr{D}'(I)$, *then*

$$E_1^*[T] = P[T; 1] \quad \text{and} \quad E_2^*[T] = P[1; T].$$

(b) *If* $\alpha, \beta \in \mathscr{E}(I)$ *and* $S, T, U \in \mathscr{D}'(I)$, *then*

$$E_1^*[\alpha]P[S; U] + E_1^*[\beta]P[T; U] = P[\alpha S + \beta T; U],$$
$$E_2^*[\alpha]P[S; T] + E_2^*[\beta]P[S; U] = P[S; \alpha T + \beta U].$$

Another useful property of the operator P that is needed in solving the distributional analog of equation (1.2) is

Proposition 2.3. *Suppose* S, T, U, *and* V *are nonzero distributions in* $\mathscr{D}'(I)$. *Then* $P[S; T] = P[U; V]$ *if and only if there exist nonzero real numbers* c_1 *and* c_2 *such that* $S = c_1U$ *and* $T = c_2V$.

The proof of this proposition can be carried out exactly as in [8].

The product operator P will be used, therefore, to represent the product $f(x)g(y)$ of equation (1.2) when the equation is interpreted in distributions. In order to represent the left-hand side of equation (1.2), one needs to define two additional operators.This will be discussed in the next section.

3. The operators $Q_\pm^*$

Let Q_+ and Q_- be operators from $\mathscr{D}(I^2)$ into $\mathscr{D}(I)$ given by

$$Q_+[\phi](x) = \int_{\mathbb{R}} \phi(x - y, y)\,dy = \int_{\mathbb{R}} \phi(y, x - y)\,dy$$

and

$$Q_-[\phi](x) = \int_{\mathbb{R}} \phi(x+y, y)\, dy = \int_{\mathbb{R}} \phi(y, y-x)\, dy.$$

These operators are well defined since $\phi(x, y)$ has compact support. We note that $Q_\pm$ belong to $L[\mathscr{D}(I^2); \mathscr{D}(I)]$. Moreover, we remark that the operator Q_+ is similar to the operator Q defined and studied in [8, 12]. The adjoints of these operators are Q_+^* and Q_-^* and are defined from $\mathscr{D}'(I)$ into $\mathscr{D}'(I^2)$ by

$$\langle Q_\pm^*[T], \phi\rangle = \langle T, Q_\pm[\phi]\rangle = \langle T(x), Q_\pm[\phi](x)\rangle$$

for any $\phi \in \mathscr{D}(I^2)$ and $T \in \mathscr{D}'(I)$.

The following proposition is essential to our work.

Proposition 3.1. *Suppose* $f \in L_{\mathrm{Loc}}(I)$. *Then*

(i) $Q_+^*[\lambda_f] \in L_{\mathrm{Loc}}(I^2)$ *and* $Q_+^*[\lambda_f] = f(x+y)$;
(ii) $Q_-^*[\lambda_f] \in L_{\mathrm{Loc}}(I^2)$ *and* $Q_-^*[\lambda_f] = f(x-y)$.

Proof. (i) For any $\phi \in \mathscr{D}(I^2)$,

$$\begin{aligned}\langle Q_+^*[\lambda_f], \phi\rangle &= \langle \lambda_f, Q_+[\phi]\rangle = \left\langle \lambda_f, \int_{\mathbb{R}} \phi(x-y, y)\, dy \right\rangle \\ &= \int_I f(x) \int_{\mathbb{R}} \phi(x-y, y)\, dy\, dx = \int_{I^2} f(x+y)\phi(x, y)\, dy\, dx \\ &= \langle f(x+y), \phi(x, y)\rangle.\end{aligned}$$

That is, $Q_+^*[\lambda_f] = f(x+y)$.

(ii) For any $\phi \in \mathscr{D}(I^2)$,

$$\begin{aligned}\langle Q_-^*[\lambda_f], \phi\rangle &= \langle \lambda_f, Q_-[\phi]\rangle = \left\langle \lambda_f, \int_{\mathbb{R}} \phi(x+y, y)\, dy \right\rangle \\ &= \int_I f(x) \int_{\mathbb{R}} \phi(x+y, y)\, dy\, dx = \int_{I^2} f(x-y)\phi(x, y)\, dy\, dx \\ &= \langle f(x-y), \phi(x, y)\rangle.\end{aligned}$$

That is, $Q_-^*[\lambda_f] = f(x-y)$.

The proof of the next proposition is immediate.

Proposition 3.2. (a) *If* $\alpha \in \mathscr{E}(I)$, *then* $Q_\pm^*[\alpha]$ *belong to* $\mathscr{E}(I^2)$.
(b) *If* $\alpha \in \mathscr{E}(I)$ *and* $T \in \mathscr{D}'(I)$, *then* $Q_\pm^*[\alpha T] = Q_\pm^*[\alpha] Q_\pm^*[T]$.

Since the action of the partial differential operators D_1 and D_2 on Q_+ and Q_- differs slightly from results obtained in [8, 12], we shall include the proof of the following proposition, which is also essential in the solution of the distributional analog of equation (1.2).

Proposition 3.3. *If* $T \in \mathscr{D}'(I)$, *then*

(a) $D_1(Q_+^*[T]) = D_2(Q_+^*[T]) = Q_+^*[DT]$;
(b) $D_1(Q_-^*[T]) = Q_-^*[DT]$;
(c) $D_2(Q_-^*[T] = -Q_-^*[DT]$.

Proof. (a) Let $\phi(x, y) \in \mathscr{D}(I^2)$ and $\psi(x, y) = -D_1\phi(x, y)$. Then for any $T \in \mathscr{D}'(I)$,

$$\begin{aligned}\langle D_1(Q_+^*[T]), \phi(x, y)\rangle &= \langle Q_+^*[T], \psi(x, y)\rangle = \langle T, Q_+[\psi]\rangle \\ &= \left\langle T(u), \int_{\mathbb{R}} \psi(u-v, v)\,dv \right\rangle \\ &= \left\langle T(u), \int_{\mathbb{R}} -D_1\phi(u-v, v)\,dv \right\rangle \\ &= \left\langle T(u), -\frac{\partial}{\partial u}\int_{\mathbb{R}} \phi(u-v, v)\,dv \right\rangle \\ &= \langle DT, Q_+(\phi)\rangle = \langle Q_+^*[DT], \phi(x, y)\rangle.\end{aligned}$$

Therefore, $D_1(Q_+^*[T]) = Q_+^*[DT]$. Since $\int_{\mathbb{R}} \phi(u-v, v)\,dv = \int_{\mathbb{R}} \phi(v, u-v)\,dv$, the second part of (a) is proved analogously.

(b) Let $\phi(x, y) \in \mathscr{D}(I^2)$ and $\psi(x, y) = -D_1\phi(x, y)$. Then, for any $T \in \mathscr{D}'(I)$,

$$\begin{aligned}\langle D_1(Q_-^*[T]), \phi(x, y)\rangle &= \langle Q_-^*[T], -D_1\phi(x, y)\rangle = \langle Q_-^*[T], \psi(x, y)\rangle \\ &= \langle T, Q_-^*[\psi]\rangle = \left\langle T(u), \int_{\mathbb{R}} \psi(u+v, v)\,dv \right\rangle \\ &= \left\langle T(u), \int_{\mathbb{R}} -D_1\phi(u+v, v)\,dv \right\rangle \\ &= \left\langle T(u), -\frac{\partial}{\partial u}\int_{\mathbb{R}} \phi(u+v, v)\,dv \right\rangle \\ &= \langle DT, Q_-[\phi]\rangle = \langle Q_-^*[DT], \phi\rangle,\end{aligned}$$

which yields $D_1(Q_-^*[T]) = Q_-^*[DT]$.

(c) Let $\phi(x, y) \in \mathscr{D}(I^2)$ and $\psi(x, y) = -D_2\phi(x, y)$. Then for an $T \in \mathscr{D}'(I)$,

$$\begin{aligned}\langle D_2(Q_-^*[T]), \phi(x, y)\rangle &= \langle Q_-^*[T], -D_2\phi(x, y)\rangle = \langle Q_-^*[T], \psi(x, y)\rangle \\ &= \langle T, Q_-[\psi]\rangle - \left\langle T(u), \int_{\mathbb{R}} \psi(u+v, v)\,dv \right\rangle \\ &= \left\langle T(u), \int_{\mathbb{R}} \psi(v, v-u)\,dv \right\rangle = \left\langle T(u), \int_{\mathbb{R}} -D_2\phi(v, v-u)\,dv \right\rangle \\ &= \left\langle T(u), \int_{\mathbb{R}} \frac{\partial}{\partial u}\phi(v, v-u)\,dv \right\rangle = \left\langle T(u), \frac{\partial}{\partial u}\int_{\mathbb{R}} \phi(v, v-u)\,dv \right\rangle \\ &= \left\langle -DT(u), \int_{\mathbb{R}} \phi(v, v-u)\,dv \right\rangle = \left\langle -DT(u), \int_{\mathbb{R}} \phi(v+u, v)\,dv \right\rangle \\ &= \langle -DT, Q_-[\phi]\rangle = \langle -Q_-^*[DT], \phi\rangle,\end{aligned}$$

which implies that $D_2(Q_-^*[T]) = -Q_-^*[DT]$.

4. D'Alembert's equation in distributions

In this section we shall employ the operators $Q_\pm^*$ and the product operator P to write equation (1.2) in the domain of distributions and utilize the properties discussed in §§2 and 3 to solve the distributional equation. We shall indicate

that for regular distribution the results reduce to the "classical" solutions of the D'Alembert equation (1.2).

Let T and U be in $\mathscr{D}'(I)$. We shall call the following equation the D'Alembert equation in distributions T and U in $D'(I)$:

$$Q_+^*[T] + Q_-^*[U] = 2P[T\,;\,U]. \tag{4.1}$$

The next proposition shows that (4.1) reduces to (1.2) in case T and U are regular distributions.

Proposition 4.1. *If T and U are regular distributions, that is, there are functions f and g in $L_{\mathrm{Loc}}(I)$ such that $\lambda_f = T$ and $\lambda_g = U$, then equation* (4.1) *reduces to equation* (1.2).

Proof. In light of Proposition 3.1, we have

$$Q_+^*[\lambda_f] = f(x+y) \quad \text{and} \quad Q_-^*[\lambda_g] = g(x-y).$$

On the other hand,

$$\begin{aligned}\langle P[\lambda_f\,;\,\lambda_g]\,,\,\phi\rangle &= \langle \lambda_g\,,\,\langle \lambda_g\,,\,\phi(x\,,\,y)\rangle\rangle \\ &= \int_{I^2} f(x)g(y)\phi(x\,,\,y)\,dy\,dx = \langle f(x)g(y)\,,\,\phi(x\,,\,y)\rangle.\end{aligned}$$

Thus if equation (4.1) holds for the regular distributions T and U, that is,

$$Q_+^*[\lambda_f] + Q_-^*[\lambda_g] = 2P[\lambda_f\,;\,\lambda_g],$$

then the above calculation implies that

$$\int_{I^2} \{f(x+y) + g(x-y) - 2f(x)g(y)\}\phi(x\,,\,y)\,dy\,dx = 0$$

for all $\phi \in \mathscr{D}(I^2)$. Hence we conclude that $f(x+y) + g(x-y) = 2f(x)g(y)$ for almost every $(x\,,\,y) \in I^2$.

We shall now proceed to solve equation (4.1). Apply D_1^2 and D_2^2 to equation (4.1) and use the properties developed in Propositions 3.3, 2.2(a), and 2.1(d). We obtain the system

$$\begin{aligned}Q_+^*(D^2T) + Q_-^*(D^2U) &= 2P[D^2T\,;\,U],\\ Q_+^*(D^2T) + Q_-^*(D^2U) &= 2P[T\,;\,D^2U].\end{aligned}$$

This implies that $P[D^2T\,;\,U] = P[T\!:\!D^2U]$. By virtue of Proposition 2.3, we deduce the existence of real constants c_1 and c_2 such that $D^2T = c_1T$ and $D^2U = c_2U$. The second order homogeneous linear differential system with constant coefficients yields

$$T = k_1e^{\sqrt{c_1}x} + k_2e^{-\sqrt{c_1}x} \quad \text{and} \quad U = l_1e^{\sqrt{c_2}x} + l_2e^{-\sqrt{c_2}x}$$

where l_1, l_2, k_1, and k_2 are to be determined.

In the case T and U are regular distributions, that is, there are locally integrable functions f and g such that $T = \lambda_f$ and $U = \lambda_g$, then f and g satisfy the differential equations $f'' = c_1f$ and $g'' = c_2g$ whose solutions represent the solution of equation (1.2).

We have thus shown that

Theorem 4.1. *If T and U belong to $\mathscr{D}'(I)$ and satisfy equation* (4.1), *then there exist real constants c_1 and c_2 such that $T = \lambda_f$ and $U = \lambda_g$ where f and g satisfy the differential equations $f'' = c_1 f$ and $g'' = c_2 g$.*

D'Alembert's equation usually refers to equation (1.1). In distributions, equation (1.1) would take the form

$$Q_+^*[T] + Q_-^*[T] = 2P[T\,;\,T]. \tag{4.2}$$

As in Proposition 4.1, equation (4.2) reduces to equation (1.1) when T is a regular distribution. As before, the solution of (4.2) is

$$T = k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}. \tag{4.3}$$

To determine k_1 and k_2, we substitute (4.3) into (4.2). For $\phi \in \mathscr{D}(I^2)$,

$$\begin{aligned}
\langle Q_+^*[T] + Q_-^*[T], \phi\rangle &= \langle T, Q_+\phi\rangle + \langle T, Q_-\phi\rangle \\
&= \int_I (k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}) \int_{\mathbb{R}} \phi(x-y, y)\,dy\,dx \\
&\quad + \int_I (k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}) \int_{\mathbb{R}} \phi(x+y, y)\,dy\,dx \\
&= \int_{I^2} (k_1 e^{\sqrt{c}(x+y)} + k_2 e^{-\sqrt{c}(x+y)})\phi(x, y)\,dy\,dx \\
&\quad + \int_{I^2} (k_1 e^{\sqrt{c}(x-y)} + k_2 e^{-\sqrt{c}(x-y)})\phi(x, y)\,dy\,dx \\
&= \langle k_1 e^{\sqrt{c}(x+y)} + k_2 e^{-\sqrt{c}(x+y)} + k_1 e^{\sqrt{c}(x-y)} + k_2 e^{-\sqrt{c}(x-y)}, \phi(x, y)\rangle.
\end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{aligned}
\langle 2P[T\,;\,T], \phi\rangle &= 2\langle k_1 e^{\sqrt{c}x} + k_2 e^{-\sqrt{c}x}, \langle k_1 e^{\sqrt{c}y} + k_2 e^{-\sqrt{c}y}, \phi(x, y)\rangle\rangle \\
&= \langle 2k_1^2 e^{\sqrt{c}(x+y)} + 2k_2^2 e^{-\sqrt{c}(x+y)} + 2k_1 k_2 e^{\sqrt{c}(x-y)} \\
&\qquad + 2k_1 k_2 e^{-\sqrt{c}(x-y)}, \phi(x, y)\rangle.
\end{aligned} \tag{4.5}$$

Equating (4.4) and (4.5), we see that $k_1 = k_2 = \frac{1}{2}$ or 0. Thus the only nontrivial solutions are $T = \cos ax$ and $T = \cosh ax$ depending on whether the real constant c is positive or negative. We, therefore, have

Theorem 4.2. *If $T \in \mathscr{D}'(I)$ satisfy equation* (4.2), *then there exists a constant $a \in \mathbb{R}$ such that $T = \lambda_f$ where $f(x) = 0$, $\cos ax$, or $\cosh ax$ for all $x \in I$.*

Remark. A natural extension of the D'Alembert functional equation is to consider the case for $2n+2$ unknown functions, viz.

$$f(x+y) + g(x-y) = \sum_{k=1}^{n} f_k(x) g_k(y).$$

This equation has arisen in signal processing (see [2]). Following our method here, this equation in distribution will be

$$Q_+^*[T] + Q_-^*[U] = \sum_{k=1}^{n} P[T_k\,;\,U_k]$$

where T, U, T_k, U_k, $k = 1, \ldots, n$, will be distributions in $\mathscr{D}'(I)$. This will be the subject of a later paper.

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DEPARTMENT OF APPLIED MATHEMATICAL SCIENCES, UNIVERSITY OF HOUSTON-DOWNTOWN, HOUSTON, TEXAS 77002

E-mail address: deeba@dt.uh.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF REGINA, REGINA, CANADA S4S 0A2

E-mail address: elkoh@max.cc.uregina.ca