PRIMARY SUMMAND FUNCTIONS
ON THREE-DIMENSIONAL COMPACT SOLVMANIFOLDS

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Abstract. Leonard Richardson has shown that for a certain class of three-
dimensional compact solvmanifolds, projections onto \( \pi \)-primary summands of 
\( L^2(M) \) do not preserve the continuity of functions on \( M \). It is shown here 
that if the \( \pi \)-primary projection of a continuous function is \( L^\infty \) then it is 
actually continuous. From this it follows that there are continuous functions on 
\( M \) whose \( \pi \)-primary projections are essentially unbounded.

Introduction

Let \( G \) be a solvable, connected, and simply connected Lie group, with Lie 
algebra \( g \) and cocompact discrete subgroup \( \Gamma \). By a representation \( \pi \) of \( G \) we 
shall mean a strongly continuous, unitary representation of \( G \) in some separable 
Hilbert space \( H_\pi \); \( \pi \) will be called irreducible if the space \( H_\pi \) contains no 
proper closed nontrivial subspace invariant under \( \pi \).

Let \( M \) be the space of right \( \Gamma \) cosets in \( G \) and give \( M \) the quotient topol-
ygy. There is a unique probability measure on \( M \) that is invariant under the 
natural action of \( G \) by right translation. Forming \( L^2(M) \) with respect to this 
measure, we have that the action of \( G \) on \( M \) induces a unitary action of \( G \) 
on \( L^2(M) \): \( R(g)f(m) = f(mg) \) for \( f \in L^2(M) \), \( m \in M \), and \( g \in G \). \( R \) is 
called the quasi-regular representation of \( G \) on \( L^2(M) \).

It is well known that \( L^2(M) \) decomposes into the direct sum \( \bigoplus H_\pi \), where 
the spaces \( H_\pi \) are mutually orthogonal \( R(G) \)-invariant subspaces, and that \( R \) 
on the space \( H_\pi \) is a finite multiple of the irreducible representation \( \pi \). [GGP, 
§1.2]. We let \( (\Gamma \backslash G)^\Lambda \) denote the set of irreducible representations appearing 
in the quasi-regular representation \( R \) of \( G \) on \( L^2(M) \). Then the orthogonal 
projection \( P_\pi \) of \( L^2(M) \) onto \( H_\pi \) is \( L^2 \)-continuous and preserves \( C^\infty(M) \) 
[AB, Theorem 5].

Now let \( N \) be a nilpotent Lie group, connected and simply connected, with 
Lie algebra \( n \) and cocompact discrete subgroup \( \Gamma \).

If the coadjoint orbits of the action of \( N \) on the dual \( n^* \) are linear vari-
eties, then \( \Gamma \backslash N \) possesses the property that the orthogonal projections \( P_\pi \) of 
\( L^2(\Gamma \backslash N) \) onto \( H_\pi \) preserve continuity [Ri1, B]. These flat-orbit nilmanifolds

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share this property with compact quotients of the three-dimensional solvable group \( S_R \) by discrete subgroups. Here \( S_R \) denotes the semidirect product \( \mathbb{R} \ltimes \mathbb{R}^2 \), where \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) via a one-parameter subgroup of rotations \([Ril]\).

This work was motivated by the fact that orthogonal projections onto \( \pi \)-primary summands preserves continuity of functions in \( L^2 \) of both compact quotients of flat-orbit nilmanifolds and compact quotients of the group \( S_R \).

In \([Ri2]\) Richardson proved a Fejer theorem for flat-orbit nilmanifolds. It can be shown that a similar Fejer theorem holds for compact quotients of \( S_R \) by discrete subgroups. Let \( S_H \) be the semidirect product \( \mathbb{R} \ltimes \mathbb{R}^2 \), where \( \mathbb{R} \) acts upon \( \mathbb{R}^2 \) via the one-parameter subgroup \( t \mapsto [x^\lambda, x^{-1}] \) in \( SL_2(\mathbb{R}) \), where \( \lambda + \lambda^{-1} = k + 1 \) for some integer \( k \geq 2 \). Let \( \Gamma \) be a cocompact discrete subgroup of \( S_H \). It is known that orthogonal projections \( P_\pi \) of \( L^2(\Gamma \backslash S_H) \) onto \( H_\pi \) do not preserve continuity \([Ril]\). It can be shown that no standard Fejer theorem exists for solvmanifolds of this type. However, the series determining the \( L^2 \) equivalence class of a projected function bears a resemblance to a standard lacunary Fourier series on \( \mathbb{R} \backslash \mathbb{Z} \) (see \([Z]\)). An adaptation of Sidon’s theorem on convergence of lacunary Fourier series \([Z, \text{Theorem VI.6.1}]\) is used to demonstrate that if the orthogonal projection \( P_\pi f \) of a continuous function \( f \) on \( \Gamma \backslash S_H \) is an \( L^\infty \) function, then \( P_\pi f \) is actually continuous. Together with Theorem 3.13 in \([Ri1]\), this implies that for each \( H_\pi \) there is continuous \( f \in L^2(\Gamma \backslash S_H) \) such that \( P_\pi f \) is discontinuous and essentially unbounded.

**Preliminaries**

Let \( G \) be a connected, simply connected Lie group with Lie algebra \( \mathfrak{g} \), and let \( \mathfrak{g}^* \) be the vector space of linear functionals on \( \mathfrak{g} \). We define a sequence of ideals of the Lie algebra \( \mathfrak{g} \) by \( \mathfrak{g}^{(0)} = \mathfrak{g} \), \( \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \); this is called the derived series of \( \mathfrak{g} \), and \( \mathfrak{g} \) is said to be solvable if \( \mathfrak{g}^{(n)} = 0 \) for some \( n \in \mathbb{N} \). We define another sequence of ideals of the Lie algebra \( \mathfrak{g} \) by \( \mathfrak{g}^{(0)} = \mathfrak{g} \), \( \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}] \); this is called the lower central series of \( \mathfrak{g} \), and \( \mathfrak{g} \) is said to be nilpotent if \( \mathfrak{g}^{(n)} = 0 \) for some \( n \in \mathbb{N} \) (see \([H, \S 3]\)). Throughout this paper, the term “nilmanifold” ("solvmanifold") will refer to compact spaces \( \Gamma \backslash G \), where \( G \) is nilpotent (solvable) and \( \Gamma \) is discrete and cocompact.

The coadjoint representation of \( G \) is of central importance in the representation theory of nilpotent and solvable Lie groups. The set of equivalence classes of irreducible representations of a nilpotent Lie group \( G \) is naturally parametrized by the orbit space \( \mathfrak{g}^*/\text{Ad}^*G \); this is also true for the (completely) solvable Lie groups examined in this work. This parametrization, due to A. A. Kirillov, is freely drawn upon in this work; for details, see \([CG, \text{Chapter II}]\).

There are two three-dimensional, solvable, nonnilpotent Lie groups with cocompact discrete subgroups, the groups \( S_H \) and \( S_R \). Their Lie algebras are three-dimensional vector spaces spanned by the vectors \( T, X, \) and \( Y \), where \( \exp sT = (s, 0, 0) \), \( \exp sX = (0, s, 0) \), and \( \exp sY = (0, 0, s) \).

There are five distinct compact quotients of \( S_R \) and infinitely many distinct compact quotients of \( S_H \).

We have the following compact quotients of \( S_H \), with convenient coordinatizations.

Suppose \( k \in \mathbb{Z}, \ k \geq 2 \). Define \( S_{H,k} = \mathbb{R} \ltimes \mathbb{R}^2 \), where \( \mathbb{R} \) acts on \( \mathbb{R}^2 \) via
the one-parameter subgroup \( \sigma_k(t) \) in \( SL_2(\mathbb{R}) \) with \( \sigma_k(1) = \begin{pmatrix} 1 & 1 \\ k-1 & k \end{pmatrix} \). (We will henceforth refer to the normal subgroup \( \mathbb{R}^2 \) as \( N \).) Then \( S_{H,k} \cong S_H \) for each \( k \). Let \( \Gamma_{H,k} = \{(p, m, n) \in S_{H,k} ; p, m, n \in \mathbb{Z}\} \); then each \( \Gamma_{H,k} \backslash S_{H,k} = M_{H,k} \) is a distinct compact quotient of \( S_H \).

The coordinatizations of the solvmanifolds \( S_{H,k} \) just described will be referred to as the integral coordinatizations of \( S_{H,k} \). Let \( A \in GL_2(\mathbb{R}) \) be such that

\[
A \sigma_k(t) A^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}
\]

where \( \lambda + \lambda^{-1} = k + 1 \);

if we recoordinatize \( N \) so that the action of \( A \) on \( N \) is given by this one-parameter subgroup, then \( \Gamma_{H,k} \cap N = A(\mathbb{Z}^2) \); the nondegenerate coadjoint orbits in this case are hyperbolic cylinders of the form \( xy = \lambda \), \( \lambda \in \mathbb{R} \). The 2-torus \( N \cap \Gamma_{H,k} \backslash N \) in this coordinatization will be a nonstandard torus, for all \( k \geq 2 \). This coordinatization of \( S_{H,k} \) will be referred to as the hyperbolic coordinatization. We henceforth fix \( S_{H,k} \) and refer to it as \( S \) and to \( \Gamma_{H,k} \) as \( \Gamma \).

Consider the quasi-regular representation of \( S \) on \( L^2(M) \). Since \( M \) is a compact solvmanifold, \( L^2(M) \) decomposes canonically into the discrete direct sum of subspaces \( H_\pi \), which are invariant with respect to the action of \( S \), and are such that when the action of \( S \) is restricted to the subspace \( H_\pi \), \( S \) acts on \( H_\pi \) as some finite multiple of an irreducible representation \( \pi \) of \( S \). There exists no canonical decomposition of \( H_\pi \) into irreducible subspaces unless \( H_\pi \) is itself irreducible.

\( (\Gamma \backslash S)^\wedge \) will denote the set of unitary irreducible representations of \( S \) appearing in the quasi-regular representation \( R \) of \( S \) in \( L^2(\Gamma \backslash S)^\wedge \).

\( (\Gamma \backslash S)^\wedge _\infty \) will denote the set of those representations \( \pi \in (\Gamma \backslash S)^\wedge \) that are infinite dimensional.

In the integral coordinatization of \( S \), the coadjoint orbits satisfy

\[
(1) \quad (k - 1)x^2 + (k - 1)xy - y^2 = K,
\]

so that the orbits are saturated in the \( T^* \)-direction. We will call \( \lambda \) an integral functional if \( \lambda|_n = \alpha X^* + \beta Y^*, \alpha, \beta \in \mathbb{Z} \), and denote by \( \mathscr{C}_\pi \) the orbit of \( \lambda \) in \( S^* \).

Fix some nonzero integral functional \( \lambda \in \mathscr{C}_\pi \). We define the character \( \chi_\lambda \) on the (abelian) nilradical \( n \) as follows: if \( \lambda|_n = \alpha X^* + \beta Y^* \) then

\[
(2) \quad \chi_\lambda(0, r, s) = e^{2\pi i (ar + \beta s)}.
\]

In the hyperbolic coordinatization of \( S \), integral functionals are elements of a nonstandard lattice obtained by transforming \( \mathbb{Z}^2 \). Let \( \{\lambda_i\}^{mul}_i \) be a set of representatives of distinct \( \Gamma \)-orbits in the integral functionals of \( \mathscr{C}_\pi \), \( \lambda_i \) an integral functional for each \( i \). Then for \( f \) a continuous function on the solvmanifold, we have

\[
(3) \quad P_\pi(f)(\Gamma(t, x, y)) = \sum_{i=1}^{mul} \sum_{\pi \in \mathbb{Z}} \hat{f}(n + t) \gamma_\pi^*(n) \lambda_i(0, x, y)
\]

where for fixed \( t \), \( \hat{f}(n + t) \) is the Fourier coefficient of \( f \) at \( \gamma_k(n) \lambda_i \) and where \( \gamma_k(n) = A \sigma_k(n) A^{-1} \).
1. A Sidon Theorem for Primary Summand Functions in $H_\pi \subset L^2(M)$

Suppose $S = R \ltimes N$ is coordinatized so that $R$ acts on $N$ via the one-parameter subgroup $\sigma_R(t) = [e^{it}, 1]$. Then the coadjoint orbits will be “hyperbolic cylinders,” saturated in the $t$-direction, given by the equations $xy = k$, $k \in R$. Let $\pi \in (\Gamma\backslash S)^\wedge$ be an infinite-dimensional representation. Let $P_\pi: L^2(\Gamma\backslash S) \to L^2(\Gamma\backslash S)$ be the orthogonal projection onto the $\pi$-primary summand of $L^2$; $P_\pi$ does not preserve continuity of functions [Ri1]. Let $(\alpha, \beta)$ be a fixed lattice point in the coadjoint orbit $\mathcal{O}_\pi$, lying in the plane $RX^* + RY^*$, noting that with the chosen coordinatization of $S$, the torus $N \cap \Gamma\backslash N$ will be a nonstandard torus, and so $(\alpha, \beta) \in \mathcal{O}_\pi$ satisfying $\chi_{(\alpha,\beta)}(N \cap \Gamma) = 1$ will not have integer coordinates. We will call the set of $(\alpha', \beta')$ satisfying $\chi_{(\alpha',\beta')}(N \cap \Gamma) = 1$ the lattice $\mathcal{L}^*$. $\mathcal{L}^* \cap \mathcal{O}_\pi$ consists of finitely many $\Gamma$ orbits of integral points.

Theorem 1.1. $\mathcal{L}^* \cap \mathcal{O}_\pi$ is a Sidon set for each $\pi \in (\Gamma\backslash S)^\wedge_\infty$.

Proof. See [Ru, p. 127].

Corollary 1.2. Suppose $f \in P_\pi(L^2(M))$ and $f \in L^\infty(M)$. Then for almost all fixed $t = t_0$, we have

$$f(\Gamma(t_0, x, y)) = \sum_{(\alpha, \beta) \in \mathcal{L}^* \cap \mathcal{O}_\pi} f(t_0, \cdot, \cdot)\chi_{(\alpha,\beta)}(x, y)$$

absolutely and uniformly convergent to $f$.

Proof. Follows from the definition of a Sidon set.

Corollary 1.3. Suppose $\pi \in (\Gamma\backslash S)^\wedge_\infty$ and that $f \in L^2(\Gamma\backslash S)$ is continuous on $\Gamma\backslash S$. If $P_\pi f$ is $L^\infty$ then $P_\pi f$ is continuous.

Proof. Let $R = \{(t, 0, 0) \in S: t \in R\}$. Then $R$ is a subgroup of $S$.

Suppose $f \in L^2(\Gamma\backslash S)$ is continuous. Then for $(\alpha, \beta) \in \mathcal{L}^*$, we have

$$f(t, \cdot, \cdot)\chi_{(\alpha,\beta)}(x, y) = \int_{T^2} f(t, x, y)\chi_{-(\alpha,\beta)}(x, y)\,dx\,dy$$

continuous on $R$.

If $P_\pi f$ is in $L^\infty$ then we have

$$\sum_{(\alpha, \beta) \in \mathcal{L}^* \cap \mathcal{O}_\pi} |f(t, \cdot, \cdot)\chi_{(\alpha,\beta)}| < K\|f\|_\infty$$

where $K$ is a constant not depending on $f$.

Since the inequality is independent of $t$, we have that

$$P_\pi f = \sum_{(\alpha, \beta) \in \mathcal{L}^* \cap \mathcal{O}_\pi} f(t, \cdot, \cdot)\chi_{-(\alpha,\beta)}(x, y)$$

is the uniformly convergent sum of functions continuous on $S$. Since $P_\pi f$ also possesses left $\Gamma$-invariance, $P_\pi f$ is continuous on $\Gamma\backslash S$.

Corollary 1.4. There exists $f$, continuous on $M$, such that $P_\pi f$ is essentially unbounded.
Proof. Example 5.3 in [Ri1] shows that for each orthogonal projection $P_\pi$, there must exist an $f \in C(M)$ such that $P_\pi f$ is not continuous. By Corollary 1.3, however, $P_\pi f$ must then be essentially unbounded.

**Corollary 1.5.** If $f \in P_\pi(L^2(M))$ is $L^\infty$, then for a.e. fixed $t_0$ we have that 

$$f(t_0, \cdot, \cdot): N \cap \Gamma \setminus N \to \mathbb{C}$$

is a continuous function on $N \cap \Gamma \setminus N \cong T^2$.

**Proof.** By Corollary 1.2, we have that for $f \in P_\pi(L^2(\Gamma \setminus S))$ essentially bounded, inequality $(4)$ holds, for a.a. $t_0$.

Thus for a.a. fixed $t_0$, 

$$f(\Gamma(t_0, x, y)) = \sum f(t_0, \cdot, \cdot)\chi(\alpha, \beta)(x, y)$$

is a uniformly convergent sum of functions that are continuous on $N \cap \Gamma \setminus N$ and so is itself continuous on $N \cap \Gamma \setminus N$.

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