ONE RADIUS THEOREM FOR THE EIGENFUNCTIONS OF THE INVARIANT LAPLACIAN

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(Communicated by Clifford J. Earle, Jr.)

Abstract. Let \( B \) be the open unit ball in \( \mathbb{C}^n \) with its boundary \( S \). Suppose that \( \alpha \geq \frac{1}{2} \) and \( u(z) = (1 - |z|^2)^{\alpha(1 - \alpha)} F(z) \) for some \( F(z) \in C(B) \). If for every \( z \in B \) there corresponds an \( r(z) : 0 < r(z) < 1 \) and an automorphism \( \psi_z \) with \( \psi_z(0) = z \) such that

\[
\tilde{A}u(z) = \frac{1}{g_\alpha(r(z))} \int_S u \circ \psi_z(r(z)\zeta) d\sigma(\zeta),
\]

then \( \tilde{A}u(z) = -4\pi^2 \alpha(1 - \alpha)u(z) \), \( z \in B \). Here \( \tilde{A} \) is the invariant Laplacian and \( g_\alpha(r) \) is the hypergeometric function \( F(n - n\alpha, n + n\alpha; n; r^2) \).

1. Introduction

Let \( \psi_z \) be an automorphism, that is, a one-to-one holomorphic map onto itself, of the open unit ball \( B \) of \( \mathbb{C}^n \) satisfying \( \psi_z(0) = z \). Let \( \phi_z \) be one of such an automorphism defined by

\[
\phi_z(w) = \frac{z - P_z w - s_z(w - P_z w)}{1 - \langle w, z \rangle}, \quad w \in B,
\]

where \( P_z \) is the orthogonal projection of \( \mathbb{C}^n \) onto the subspace generated by \( z \), \( \langle w, z \rangle = \sum_{j=1}^n w_j \bar{z}_j \), \( s_z = \sqrt{1 - |z|^2} \), and \( |z|^2 = \langle z, z \rangle \). For \( u \in C^2(B) \), its invariant Laplacian \( \tilde{A} \) is well defined by

\[
(\tilde{A}u)(z) = \Delta(u \circ \psi_z)(0), \quad z \in B,
\]

where \( \Delta \) is the ordinary Laplacian. Our starting point is the following mean value theorem [4, Theorem 4.2.4].

Theorem A. Given a complex number \( \lambda \), any function \( u(z) \in C^2(B) \) satisfying

(1) \( \tilde{A}u(z) = \lambda u(z) \), \( z \in B \),

has mean value property

(2) \( g_\alpha(r)u(z) = \int_S u \circ \psi_z(r\zeta) d\sigma(\zeta), \quad z \in B, \ 0 < r < 1 \).

Received by the editors July 25, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 31B35.

Key words and phrases. Invariant Laplacian, one radius property.

The research was partially supported by Yonam Foundation.

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0002-9939/92 $1.00 + .25 per page

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Here and throughout \( \lambda \) and \( \alpha \) are related to be
\[
\lambda = \lambda_\alpha = -4n^2n(1 - \alpha),
\]
\( d\sigma \) is the normalized rotation invariant positive Borel measure on the boundary \( S \) of \( B \), and
\[
g_\alpha(r) = \int_S \left( \frac{1 - r^2}{|1 - r_1^2|^2} \right)^{n\alpha} d\sigma(\zeta).
\]
The space of all \( u(z) \) satisfying (1) is denoted by \( X_\lambda \). This space is invariant under automorphism. See [4, Chapter 4] for the terminology and related properties.

When \( \lambda = 0 \), Theorem A has a strong converse known as the one radius theorem [4, Theorem 4.3.4]. See [2] also for the same vein. Motivated by these, our goal here is in the description of the complex numbers \( \alpha \) and the smoothness of functions \( u(z) \) that makes the following property valid.

**One-radius property for \( \alpha \) and \( u \).** If for every \( z \in B \) there corresponds an \( r(z) : 0 < r(z) < 1 \) such that
\[
ga(r(z))u(z) = \int u \circ \phi_z(r(z)\zeta) d\sigma(\zeta),
\]
then \( u(z) \in X_\lambda \).

We denote this property by \( 1RP(\alpha ; u) \) in short. Rudin's one radius theorem says that \( 1RP(0 ; u) = 1RP(1 ; u) \) is true if \( u(z) \in C(B) \). Izuchi [1] proved that \( 1RP(1 ; u) \) fails for a bounded real analytic \( u(z) \). In this paper, we first confine ourselves to \( \text{Re}\alpha > \frac{1}{2} \). Our results say that under our growth condition on \( u \), \( 1RP(\alpha ; u) \) is true if and only if \( \alpha \) is real (Theorem 3), and the growth condition on \( u \) cannot be improvable (Theorem 2). Our last section deals with the case \( \alpha = \frac{1}{2} \). All unexplained notations and properties will be referred to [4].

2. **One radius theorem for \( X_\lambda \)**

**Lemma 1.** Let \( \alpha \in \mathbb{C} \). Then
\[
g_\alpha(r) = g_{1-\alpha}(r) = (1 - r^2)^{n(1-\alpha)} \mathcal{F}(\alpha : r),
\]
where
\[
\mathcal{F}(\alpha : r) = \int_S \left( \frac{1 - r^2}{|1 - r_1^2|^2n} \right)^{n\alpha} d\sigma(\zeta).
\]
If \( \alpha \) is real then \( \mathcal{F}(\alpha : r) \) is an increasing function of \( r \). If \( \alpha > \frac{1}{2} \) then \( \mathcal{F}(\alpha : r) \) tends to a finite limit \( \mathcal{F}(\alpha) = \lim_{r \to 1} \mathcal{F}(\alpha : r) \). Also
\[
\mathcal{F} \left( \frac{1}{2} : r \right) = \frac{1}{r^2} \log \frac{1}{1 - r^2} \mathcal{F} \left( \frac{1}{2} : r \right)^{-1}
\]
is a continuous function of \( r \) that tends to a finite limit \( \mathcal{F} \left( \frac{1}{2} \right) = \lim_{r \to 1} \mathcal{F} \left( \frac{1}{2} : r \right) \).

**Theorem 1.** Suppose \( \alpha > \frac{1}{2} \) and \( u(z) = (1 - |z|^2)^{n(1-\alpha)}F(z) \) for some \( F(z) \in C(B) \). If for every \( z \in B \) there corresponds an \( r(z) : 0 < r(z) < 1 \) and an
automorphism $\psi_z$ such that

$$g_\alpha(r(z))u(z) = \int_S u \circ \psi_z(r(z)\zeta) \, d\sigma(\zeta),$$

then

$$u(z) = \frac{1}{\mathcal{F}(\alpha)} \int_S P^\alpha(z, \zeta)F(\zeta) \, d\sigma(\zeta), \quad z \in B,$$

where $P(z, w)$ denotes the invariant Poisson kernel. In particular, $1R^\alpha(\alpha; u)$ is true.

Proof of Lemma 1. By [4, Remark, p. 44],

$$\mathcal{F}(\alpha : r) = \int_S \frac{(1 - r^2)^{2n\alpha - n}}{|1 - \langle r\zeta, \phi_{r\zeta}(\zeta)\rangle|^{2n\alpha}} P(\phi_{r\zeta}^{-1}(0), \zeta) \, d\sigma(\zeta)$$

$$= \int_S \frac{(1 - r^2)^{2n\alpha - n}}{|1 - \langle \phi_{r\zeta}(0), \phi_{r\zeta}(\zeta)\rangle|^{2n\alpha}} \frac{(1 - r^2)^n}{|1 - \langle r\eta, \zeta\rangle|^{2n}} \, d\sigma(\zeta),$$

which turns out to be

$$\int_S \frac{d\sigma(\zeta)}{|1 - \langle r\eta, \zeta\rangle|^{2n(1-\alpha)}}$$

once we use the identity [4, Theorem 2.2.5]

$$1 - \langle \psi_a(z), \psi_a(w) \rangle = \frac{(1 - |a|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)},$$

$a \in B$, $z, w \in \overline{B}$. Hence $(1 - r^2)^{n(1-\alpha)}\mathcal{F}(\alpha : r) = g_{1-\alpha}(r)$. That $g_\alpha(r) = g_{1-\alpha}(r)$ is noted in [4, Corollary, p. 51]. The second part follows from the subharmonicity of the integrand and [4, Proposition 1.4.10].

Proof of Theorem 1. Let

$$v(z) = \frac{1}{\mathcal{F}(\alpha)} \int_S P^\alpha(z, \zeta)F(\zeta) \, d\sigma(\zeta), \quad z \in B.$$
Now the function $H(z)$ defined by
$$H(z) = \begin{cases} 
(1 - |z|^2)^{n(\alpha - 1)}h(z), & z \in B, \\
0, & z \in S,
\end{cases}$$
is of $C(\overline{B})$. We will show $H \equiv 0$ in $B$. Suppose $|H(z)| > 0$ for some $z \in B$. Then the set $E$ on which $|H(z)|$ takes the maximum is compact. We can take $z_0 \in E$ such that dist$(z_0, S) = \text{dist}(E, S)$. Since both $u(z)$ and $v(z)$ satisfy (4), we have (4) with $h(z)$ in place of $u(z)$, so that
$$H(z)g_\alpha(r(z)) = \int_S (1 - |z|^2)^{n(\alpha - 1)}h(z)\psi_z(r(z)\zeta)d\sigma(\zeta), \quad z \in B.$$But the right side of (6) is
$$\int_S P^{1-\alpha}(r(z)\zeta, z)H \circ \psi_z(r(z)\zeta)d\sigma(\zeta)$$once we use the identity (5). If we apply (6) and (7) to $z_0$ with its corresponding $r(z_0) = r_0$, we obtain, by (3),
$$|H(z_0)| = \frac{1}{\mathcal{F}(\alpha : r_0)}\left|\int_S \frac{H \circ \psi_{z_0}(r_0\zeta)}{|1 - \langle r_0z_0, \zeta \rangle|^{2n(1-\alpha)}}d\sigma(\zeta)\right|,$$which is, by (3) once more, strictly less than
$$\frac{\mathcal{F}(\alpha : r_0)|z_0|}{\mathcal{F}(\alpha : r_0)}|H(z_0)| \leq |H(z_0)|.$$
This contradiction proves our assertion.

3. On smoothness

Because of the symmetry, $g_\alpha = g_{1-\alpha}$ and $X_{\lambda_\alpha} = X_{\lambda_{1-\alpha}}$, we can have the proper substitute for the case $\alpha < \frac{1}{2}$, that is, Theorem 1 with all $\alpha$ replaced by $1 - \alpha$. On the other hand, if $\alpha < \frac{1}{4}$ and $(1 - |z|^2)^{n(\alpha - 1)}u(z)$ can be of $C(\overline{B})$, then we can easily check from Theorem 1 that $1\text{RP}(\alpha; u) = 1\text{RP}(1 - \alpha; u)$ is true. But if $\alpha \leq \frac{1}{4}$, there is no $u \in \mathcal{X}_z$, $u \neq 0$ with $(1 - |z|^2)^{n(\alpha - 1)}u(z)$ bounded. This can be checked by comparing the order of $g_\alpha(r)$ as $r \to 1$ after integrating $(1 - |\psi_z(r\zeta)|^2)^{n(\alpha - 1)}u \circ \psi_z(r\zeta)$ with respect to $d\sigma(\zeta)$ and using (2) in Theorem A and (5). This fact and Theorem 1 gives the following solution to the boundary value problem.

**Corollary.** Let $\alpha$ be real and let $f \in C(S)$, $f \neq 0$, be given. Then there is a solution $u(z)$ such that
(i) $\tilde{\Delta}u(z) = \lambda u(z)$ in $B$,
(ii) with boundary value $f(\zeta)$, $(1 - |z|^2)^{n(\alpha - 1)}u(z)$ is of $C(\overline{B})$
if and only if $\alpha > \frac{1}{2}$. In fact,
$$u(z) = \frac{1}{\mathcal{F}(\alpha)}\int_S P^\alpha(z, \zeta)f(\zeta)d\sigma(\zeta)$$is the unique solution when $\alpha > \frac{1}{2}$.

Our proof of Theorem 1 covers the case $\alpha = 1$. But in (8) equality occurs only when $\alpha = 1$ because $\mathcal{F}(\alpha : r)$ is strictly increasing if $\alpha$ is real and $\alpha \neq 1$. 

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So it seems to be that the compactness of $E$ when $\alpha \neq 1$ is not so much crucial compared to the case $\alpha = 1$. Nonetheless we can explain by use of an example of Izuchi that our growth condition cannot be weakened.

**Theorem 2.** For $\alpha > \frac{1}{2}$, there is a function $u(z)$ such that

(i) $u(z) \notin X_\lambda$, 
(ii) $(1 - |z|^2)^{n(\alpha - 1)}u(z)$ is bounded real analytic in $B$, 
(iii) for every $z \in B$, there is a radius $r(z) : 0 < r(z) < 1$ by which $u(z)$ satisfies (4).

**Proof.** Izuchi [1] constructed a bounded positive radial real analytic function $F(z)$ on $B$ such that

$$\sup_{\zeta \in S} F \circ \phi_z(\delta_j \zeta) < F(z) < \inf_{\zeta \in S} F \circ \phi_z(\sigma_j \zeta)$$

for some $\delta_j$, $\sigma_j$ depend on $z$ (see [1, p. 830]). If we fix $\alpha$ and set $u(z) = g_\alpha(z)F(z)$, then by (9) and (2),

$$\int_S u \circ \phi_z(\delta_j \zeta) d\sigma(\zeta) < F(z) \int_S g_\alpha \circ \phi_z(\delta_j \zeta) d\sigma(\zeta) = u(z)g_\alpha(\delta_j)$$

and

$$u(z)g_\alpha(\sigma_j) = F(z) \int_S g_\alpha \circ \phi_z(\sigma_j \zeta) d\sigma(\zeta) < \int_S u \circ \phi_z(\sigma_j \zeta) d\sigma(\zeta),$$

so that by the continuity on $r$ of

$$\frac{1}{g_\alpha(r)} \int_S u \circ \phi_z(r \zeta) d\sigma(\zeta),$$

(iii) is valid. This $u(z)$ satisfy (i), because of the strict inequality in (10), and (ii) also, because $F(z)$ and $(1 - |z|^2)^{n(\alpha - 1)}g_\alpha(z)$ are bounded real analytic.

4. One-radius property fails for nonreal $\alpha$

**Lemma 2.** (1) If $\frac{1}{2} \leq a_1 < a_2$ then $g_{a_1}(r) < g_{a_2}(r)$, $0 < r < 1$.

(2) For a given nonreal $\beta$ with $\text{Re} \beta > \frac{1}{2}$, there exist $r$ and $a$ such that

(i) $0 < r < 1$, (ii) $\frac{1}{2} < a < \text{Re} \beta$, and (iii) $g_\alpha(r) = g_\beta(r)$.

**Theorem 3.** Let $\text{Re} \alpha > \frac{1}{2}$. Then the following are equivalent:

(1) $1\text{RP}(\alpha ; u)$ is true for all $u(z)$ with $(1 - |z|^2)^{n(\alpha - 1)}u(z) \in C(B)$.

(2) $\alpha$ is real.

**Proof of Lemma 2.** (1) Suppose $g_{a_1}(r) \geq g_{a_2}(r)$ for some $r : 0 < r < 1$. Then since $g_{a_1}(r) < g_{a_2}(r)$ near $r = 1$, there exists $r_0$ such that $g_{a_1}(r_0) = g_{a_2}(r_0)$. This $g_{a_1}$ and $r_0$ satisfy the condition of Theorem 1 with $\alpha = a_2$, $r(z) = r_0$, and $u = g_{a_1}$, but $g_{a_1} \notin X_\lambda_{a_2}$. This is a contradiction.

(2) Let $\beta = b + ic$, $c \neq 0$, be fixed. Let $G(r) = g_\beta(r)/g_b(r)$, $0 \leq r < 1$, and let $R = \{(a, r) : \frac{1}{2} < a < b \text{ and } 0 < r < 1\}$. Define

$$G(a ; r) = g_a(r)/g_b(r), \quad (a, r) \in \overline{R}.$$ 

Since $g_b(r) = (1 - r^2)^{n(1-b)}F(b : r)$, $F(b : 0) = 1$, and $F(a : r)$ is increasing, it follows that $g_b(r)$ cannot be zero and that $G(a ; r)$ is well defined continuous.

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in each variable. Note also, by (1), that \( G(a; r) \) is strictly increasing in the first variable. Let \( \Omega \) be the simply connected open region surrounded by
\[ \{G(\frac{1}{2}; r) : 0 \leq r \leq 1\} \cup \{(a, 1) : 0 \leq a \leq 1\} \cup \{(1, r) : 0 \leq r \leq 1\}. \]

Then
\[
(11) \quad \Omega \subset \bigcup_{(a, r) \in \mathbb{R}} (r, G(a; r));
\]
because if \((x, y) \in \Omega \) then \( G(\frac{1}{2}; x) < y < G(b; x) = 1 \), and by use of the intermediate value theorem we can find \( a : \frac{1}{2} < a < b \) such that \( G(a; x) = y \).

Next let \( \mathcal{B} = \lim_{r \to 1} |\mathcal{B}(r)| \). We claim
\[
(12) \quad 0 < \mathcal{B} < 1.
\]
It is obvious that \( \mathcal{B} \leq 1 \). Also, since \( |\mathcal{F}(\beta : r)| \leq \mathcal{F}(b : r) \uparrow \mathcal{F}(b) \), by the dominated convergence theorem and (3),
\[
|\mathcal{B}(1)| = \lim_{r \to 1} \left| \frac{g_{\beta}(r)}{g_{\beta}(r)} \right| = \lim_{r \to 1} \frac{|\mathcal{F}(\beta : r)|}{\mathcal{F}(b : r)} = \frac{|\mathcal{F}(\beta : 1)|}{\mathcal{F}(b)}.
\]
Since
\[
\mathcal{F}(\beta : 1) = \lim_{r \to 1} \mathcal{F}(\beta : r) = \lim_{r \to 1} F(n - n\beta, n - n\beta, n; r^2) = \frac{\Gamma(n)\Gamma(2n\beta - n)}{\Gamma^2(n\beta)},
\]
where \( F(\cdots) \) is the Gaussian hypergeometric function (see [3] or [4, p. 54]), we have \( \mathcal{B} \neq 0 \). Suppose \( |\mathcal{F}(\beta : 1)| = \mathcal{F}(b) \). Then \( \gamma \mathcal{F}(\beta : 1) = \mathcal{F}(b) \) for some \( \gamma \) with \( |\gamma| = 1 \), that is,
\[
\int_{S} \left( \frac{1}{|1 - \zeta|^{2n(1-\beta)}} - \frac{\gamma}{|1 - \zeta|^{2n(1-\beta)}} \right) d\sigma(\zeta) = 0,
\]
from which follows \( 1 = \text{Re}(\gamma|1 - \zeta|^{2n}) \) a.e., \( \zeta \in S \). This is impossible.

Now, since \( \mathcal{F}(\beta : r)/\mathcal{F}(b : r) \) has a nonzero finite limit as \( r \to 1 \),
\[
\mathcal{B}(r) = (1 - r^2)^{-inc} \frac{\mathcal{F}(\beta : r)}{\mathcal{F}(b : r)}
\]
makes a curve that winds the origin infinitely many times as \( r \) approaches to 1. In particular, we can take, by (12), a positive sequence \( \mathcal{B}(r_j) \) such that \( 1 > \mathcal{B}(r_j) \to \mathcal{B} > 0 \) as \( r_j \to 1 \). Recalling (11), we can conclude that there are infinitely many \( r_j \) such that
\[
(r_j, \mathcal{B}(r_j)) \subset \Omega \subset \bigcup_{(a, r) \in \mathbb{R}} (r, G(a; r)).
\]
Therefore there are \( a \) and \( r_j \) such that \( \mathcal{B}(r_j) = G(a; r_j) \), that is, \( g_{\beta}(r_j) = g_{a}(r_j) \).

**Proof of Theorem 3.** That \( (2) \Rightarrow (1) \) follows from Theorem 1. For the converse, first fix a nonreal \( \beta \) with \( \text{Re} \beta > \frac{1}{2} \). Take \( a \) and \( r \) as in Lemma 2(2). Then by (ii),
\[
g_{a}(z)(1 - |z|^2)^{n(\beta - 1)} \in C(B),
\]
and by (iii) and (2),
\[
g_{a}(z)g_{\beta}(r) = \int_{S} g_{a}(\phi_z(r\zeta)) d\sigma(\zeta),
\]
but \( g_{a}(z) \in X_{\lambda_a} \) (so that \( g_{a}(z) \notin X_{\lambda_{\beta}} \)). Therefore 1RP(\( \beta; g_{a} \)) is false.
5. The case $\alpha = \frac{1}{2}$

When $\alpha = \frac{1}{2}$, we have a stronger result as expected in the corollary to Theorem 1.

**Theorem 4.** Suppose that

$$u(z) = (1 - |z|^2)^{n/2} \frac{1}{|z|^2} \log \frac{1}{1 - |z|^2} F(z)$$

for some $F \in C(\overline{B})$. If for every $z \in B$ there corresponds an $r(z) : 0 < r(z) < 1$ and an automorphism $\psi_z$ such that (4) holds, then

$$u(z) = \frac{1}{\mathcal{F}(1/2)} \int_S \left( \frac{1}{2} \right)^{(1/2)(z)} F(\zeta) \ d\sigma(\zeta), \quad z \in B.$$ 

In particular, $1RP(\frac{1}{2}; u)$ is true.

**Proof of Theorem 4.** The idea is easier than, but almost the same as, that of the proof of Theorem 1. Let $F'(z) = F(z)/\mathcal{F}(1/2 : |z|)$. Then by Lemma 1 $F' \in C(\overline{B})$ also. Let

$$v(z) = \int_S \left( \frac{1}{2} \right)^{(1/2)(z)} F'(\zeta) \ d\sigma(\zeta),$$

and

$$G(z) = \begin{cases} g_{1/2}(z)^{-1} v(z), & z \in B, \\ F'(z), & z \in S. \end{cases}$$

Then since $(1 - r^2)^{n/2} g_{1/2}(r)^{-1} \rightarrow 0$ as $r \rightarrow 1$, it follows that $G^r \rightarrow G$ uniformly on $\zeta \in S$ as $r \rightarrow 1$. Hence $G \in C(\overline{B})$. Now let $h(z) = u(z) - v(z)$. It suffices to show that the function

$$H(z) = \begin{cases} g_{1/2}(z)^{-1} h(z), & z \in B, \\ 0, & z \in S, \end{cases}$$

is identically zero in $B$. $H \in C(\overline{B})$ by Lemma 1. Suppose $|H(z)| > 0$ for some $z \in B$. Let $E$ be the set on which $|H(z)|$ takes its maximum, and let $z_0 \in E$ be such that $\text{dist}(z_0, S) = \text{dist}(E, S)$. Then by (4)

$$H(z) g_{1/2}(r(z)) = \int_S g_{1/2}(z)^{-1} h \circ \psi_z(r(z)\zeta) \ d\sigma(\zeta)$$

$$= \int_S \frac{g_{1/2}(\psi_z(r\zeta))}{g_{1/2}(z)} H \circ \psi_z(r\zeta) \ d\sigma(\zeta).$$

If we apply $z_0$ with its corresponding $r(z_0) = r_0$, we obtain, by (2) and by the location of $z_0$,

$$|H(z_0)| g_{1/2}(r_0) < |H(z_0)| g_{1/2}(r_0),$$

which is a contradiction.
The growth condition in Theorem 4 is sharp in the sense that there is \( u(z) \) such that \( F(z) \) is bounded on \( B \) and \( 1RP\left(\frac{1}{2}; u\right) \) is false. We can see this by following exactly the same lines of the proof of Theorem 2.

**Acknowledgment**

This work was completed while the author visited the University of Wisconsin-Madison. The author would like to thank Professor Patrick Ahern for helpful discussions.

**References**


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