ON THE CARTESIAN PRODUCTS OF LINDELÖF SPACES WITH ONE FACTOR HEREDITARILY LINDELÖF

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Abstract. E. Michael asked the following question: Is there a space $X$ such that $Y \times X$ is Lindelöf for every hereditarily Lindelöf space $Y$ but $X^2$ is not. The aim of this paper is to present a construction that provides such an example.

In this paper we construct for every $n \in \mathbb{N}$ a space $X_n$ such that for every hereditarily Lindelöf space $Y$, $Y \times X_n$ is Lindelöf and $X_n^{n+1}$ is not. Moreover, we also obtain, using the same technique, a space $X_\omega$ such that for every $n \in \mathbb{N}$ and every hereditarily Lindelöf space $Y$, $Y \times X_\omega^n$ is Lindelöf but $X_\omega^\omega$ is not.

Let us recall that another example of $X_\omega$ was presented in [A2, A3]. The present construction is much simpler than the previous one.

For related results and constructions see [A5, T, A6, A4].

Our topological terminology follows [E]. In particular, if $M$ is a subspace of a topological space $X$ then the symbol $X_M$ stands for the set $X$ with a new topology generated by $\mathcal{T} = \{U \subset X : U$ is open in $X\} \cup \{\{x\} : x \in X \setminus M\}$. In the sequel $Q$ and $N$ stand for the set of all rational and natural numbers respectively. Let us denote by $\mathcal{B}$ a countable base for $Q$ consisting of open intervals. If $x \in Q$ then put $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. The symbols $\omega$ and $\omega_1$ denote the first infinite ordinal number and the first uncountable ordinal number, respectively.

The projection of $Q^{\omega_1}$ onto $Q^T$ where $T \subset \omega_1$ will be denoted by $P_T$. If $\alpha$ is an ordinal number then we shall identify $\alpha$ with the set of its predecessors. If $\alpha < \beta$, and $x \in Q^\beta$ then let us denote $P_\alpha(x)$ by $x|\alpha$.

Example. For all $n \in \mathbb{N}$ there is $X_n$ such that for every hereditarily Lindelöf space $Y$, $Y \times (X_n)^n$ is Lindelöf but $(X_n)^{n+1}$ is not. Moreover, there is also $X_\omega$ such that for all $n \in \mathbb{N}$, $Y \times (X_\omega)^n$ is Lindelöf but $(X_\omega)^{\omega}$ is not.

We now consider the proof of the example. There exists a family $\{A_\alpha : 1 \leq \alpha < \omega_1\}$ such that $A_\alpha$ is a countable set consisting of strictly increasing sequences of rational numbers of length $\alpha$ for $1 \leq \alpha < \omega_1$.
(2) if $\alpha < \beta < \omega_1$ then $P_\alpha(A_\beta) = A_\alpha$,
(3) if $a = (a(\lambda))_{\lambda < \alpha} \in A_\alpha$ then $a$ is a continuous function from $\alpha$ into $Q$ and $\sup\{a(\lambda): \lambda < \alpha\}$ is a rational number (see [J, p. 91], the construction of the Aronszajn tree).

Let us attach to $a \in A_\alpha$, for $1 \leq \alpha < \omega_1$, $x_\alpha \in Q^{\omega_1}$ such that

$$x_\alpha(\beta) = \begin{cases} a(\beta) & \text{if } \beta < \alpha, \\ \sup\{a(\lambda): \lambda < \alpha\} & \text{if } \beta \geq \alpha. \end{cases}$$

Put $Z_\alpha = \{x \in Q^{\omega_1}: \text{there is a } a \in A_\alpha \text{ such that } x = x_\alpha\}$. If $x \in \bigcup\{Z_\alpha: 1 \leq \alpha < \omega_1\}$ then $a(x)$ is such an ordinal number that $x \in Z_{a(x)}$.

Let $\{S_i: i \leq n+1\}$ be a cover of $\omega_1$ consisting of pairwise disjoint stationary sets and put $K_i = \bigcup\{S_j: j \neq i\}$ for all $i \leq n+1$.

If $S$ is a subset of $\omega_1$ then put $X(S) = \bigcup\{Z_\alpha: a \in S\}$.

The desired space $X_n$ is defined by $X_n = \bigoplus_{i \leq n+1}(X(\omega_1)_{X(K_i)})$ where $X(\omega_1)$ and $X(K_i)$ are subspaces of $Q^{\omega_1}$.

In case of $X_\omega$ we have to consider infinite partition $\{S_n: n \in N\}$ of $\omega_1$ consisting of stationary sets and put

$$X_\omega = \bigoplus_{i=1}^{\infty}(X(\omega_1))_{X(K_i)}, \quad \text{where } K_i = \bigcup\{S_j: j \neq i\}.$$ 

Since $D = \{x = (x_i)_{i=1}^{n+1} \in \prod_{i=1}^{n+1}(X(\omega_1)_{X(K_i)}): \text{the coordinates of } x \text{ are equal}\}$ is a discrete closed and uncountable subset of $(X_n)^{n+1}$, we infer that $(X_n)^{n+1}$ is not Lindelöf.

In order to prove that $Y \times (X_n)^n$ is Lindelöf for every hereditarily Lindelöf space $Y$ we shall need

**Lemma 1.** If $S$ is a stationary subset of $\omega_1$ then $Y \times (X(S))^n$ is Lindelöf for every hereditarily Lindelöf space $Y$.

**Proof.** Let $Y$ be a hereditarily Lindelöf space and $\mathcal{U}$ an open cover of $Y \times (X(S))^n$. If $x = x_\alpha \in Z_\alpha$, where $a \in A_\alpha$, $x_\alpha(\alpha) \in B \in \mathcal{B}$, and $\alpha < \lambda$, then put

$$F(x, B, \lambda) = \left(\prod_{\beta < \omega_1} F_\beta \right) \cap X(\omega_1),$$

where

$$F_\beta = \begin{cases} \{x(\beta)\} & \text{if } \beta \leq \alpha, \\ B & \text{if } \beta = \lambda, \\ Q & \text{otherwise.} \end{cases}$$

For $x = (x_1, \ldots, x_n) \in (X(S))^n$ and $\prod_{i=1}^{n} B_i \in (\mathcal{B})^n$, where $x_i \in Z_{a(x_i)}$ and $x_i(\alpha(x_i)) \in B_i$ for all $i \leq n$ put $A(x, \prod_{i=1}^{n} B_i) = \{y \in Y: \exists \text{ open } H_y, \sup\{\alpha(x_i): i \leq n\} < \lambda(y) < \omega_1, \text{ and } U \in \mathcal{U} \text{ such that } y \in H_y \text{ and } H_y \times \prod_{i=1}^{n}(F(x_i, B_i, \lambda(y))) \subset U\}.$

We can assume, without loss of generality, that $\lambda(y)$ for $y \in A(x, \prod_{i=1}^{n} B_i)$ is as small as possible. Since $\{y \in A(x, \prod_{i=1}^{n} B_i): \lambda(y) \leq \beta\}$ is open for all $\beta < \omega_1$ and $A(x, \prod_{i=1}^{n} B_i)$ is Lindelöf as a subspace of $Y$, we infer that if
A(x, \prod_{i=1}^{n} B_i) \neq \emptyset \text{ then } \sup \{\lambda(y) : y \in A(x, \prod_{i=1}^{n} B_i)\} \text{ is less than } \omega_1. \text{ Put }
\lambda \left( x, \prod_{i=1}^{n} B_i \right) = \begin{cases} \sup \{\lambda(y) : y \in A(x, \prod_{i=1}^{n} B_i)\} & \text{if } A(x, \prod_{i=1}^{n} B_i) \neq \emptyset, \\
0 & \text{otherwise.} \end{cases}

Using again the Lindelöf property of A(x, \prod_{i=1}^{n} B_i) we can find \{y_j : j \in N\} \subset A(x, \prod_{i=1}^{n} B_i) \text{ such that } \bigcup\{H_{y_j} : j \in N\} = A(x, \prod_{i=1}^{n} B_i), \text{ where } H_{y_i} \text{ was defined in connection with } A(x, \prod_{i=1}^{n} B_i).

Let C be a subset of all countable ordinal numbers satisfying the following conditions:

(a) If \( \alpha \in C \) then there is a sequence \((\alpha_n)_{n=1}^{\infty} \) in \( S \) converging to \( \alpha \); we do not require that \( \alpha \in S \).

(b) If \( \alpha \in C \) then for all \((\beta_1, \ldots, \beta_n) \in (S \cap \alpha)^n \), \( x = (x_1, \ldots, x_n) \in \prod_{i=1}^{n} Z_{\beta_i} \) and \( \prod_{i=1}^{n} B_i \in (\mathcal{B})^n \) where \((x_i(\beta_i))_{i=1}^{n} \in \prod_{i=1}^{n} B_i \), \( \lambda(x, \prod_{i=1}^{n} B_i) < \alpha \).

In order to continue the proof of Lemma 1 we need

Claim 1. C is closed and unbounded in \( \omega_1 \).

Proof. Observe that \( C \) is closed. Hence the proof will be finished if we show that \( C \) is unbounded. Fix \( \beta < \omega_1 \). There is \( \beta_0 \in S \setminus \beta \). Put

\[ \beta'_1 = \sup \left\{ \lambda \left( x, \prod_{i=1}^{n} B_i \right) : x = (x_i)_{i=1}^{n} \in \left( \bigcup \{Z_\alpha : \alpha \in S \cap (\beta_0 + 1)\} \right)^n, \prod_{i=1}^{n} B_i \in \prod_{i=1}^{n} \mathcal{B}_{X_i(\alpha(x_i))} \right\} \]

and choose \( \beta_1 \in S \setminus \beta'_1 \). If \( \beta'_j \) is defined then put

\[ \beta'_j+1 = \sup \left\{ \lambda \left( x, \prod_{i=1}^{n} B_i \right) : x = (x_i)_{i=1}^{n} \in \left( \bigcup \{Z_\alpha : \alpha \in S \cap (\beta_j + 1)\} \right)^n, \prod_{i=1}^{n} B_i \in \prod_{i=1}^{n} \mathcal{B}_{X_i(\alpha(x_i))} \right\} \]

and choose \( \beta_{j+1} \in S \setminus \beta'_j+1 \). Then \( \gamma = \sup \{\beta_j : j \in N\} \in C \).

Since \( S \) is a stationary subset in \( \omega_1 \) and \( C \) is closed and unbounded in \( \omega_1 \), there is \( \alpha \in S \cap C \). Put

\[ \mathcal{D} = \left\{ H_{y_j} \times \prod_{i=1}^{n} F(x_j, B_j, \lambda(y_j)) : y_i \in A \left( x, \prod_{j=1}^{n} B_j \right), i \in N, \prod_{j=1}^{n} B_j \in \prod_{j=1}^{n} \mathcal{B}_{X_j(\alpha(x_j))}, \text{ and } x = (x_1, \ldots, x_n) \in \left( \bigcup \{Z_\gamma : \gamma \in S \cap \alpha\} \right)^n. \right\} \]

Observe that \( \mathcal{D} \) is a countable family that refines \( \mathcal{U} \). In order to finish the proof of Lemma 1 it is enough to show that

Claim 2. \( \bigcup \mathcal{D} = Y \times (X(S))^n \).

Proof. Let \((y, x)\) be an arbitrary point of \( Y \times (X(S))^n \). We shall consider two cases:
Case 1. \( x \in \left( \bigcup \{ Z_\gamma : \gamma \in S \cap (\alpha + 1) \} \right)^n \).

Case 2. Case 1 does not hold.

Proof of Case 1. There are an open subset \( y \in H \) of \( Y \), \( U \in \mathcal{Z} \), finite subsets \( T_i \) of \( \omega_1 \), and \( B_\gamma \in \mathcal{B} \) for all \( \gamma \in T_i \) and \( i \leq n \) such that
\[
(y, x) \in H \times \prod_{i=1}^{n} \left( P_{T_i}^{-1} \left( \prod_{\gamma \in T_i} B_\gamma \right) \right) \cap (X(S))^n \subset U.
\]

Put \( T_i = T_{i,1} \cup T_{i,2} \), where \( T_{i,1} = T_i \cap \alpha(x_i) \) and \( T_{i,2} = T_i \setminus T_{i,1} \) for all \( i \leq n \). Since the coordinates of \( x_i \) for \( \gamma \geq \alpha(x_i) \) are the same, we can assume, without loss of generality, that \( T_{i,2} \) is not empty and that there is \( B_i \in \mathcal{B} \) such that \( B_\gamma = B_i \) for all \( \gamma \in T_{i,2} \) and \( i \leq n \).

The sequence \( (x_i(\gamma))_{\gamma < \alpha(x_i)} \) converges to \( x_i(\alpha(x_i)) \) so there is \( \sup T_{i,1} < \gamma_i \leq \alpha(x_i) \) such that \( \gamma_i \in S \cap \alpha \) and \( x_i(\gamma_i) \in B_i \) for all \( i \leq n \). If \( \lambda_i = \sup T_{i,2} \), \( x_i = x_{\lambda_i} \), and \( \lambda = \sup \{ \lambda_i : i = 1, 2, \ldots, n \} \) then \( (y, x) \in H \times \prod_{i=1}^{n} F(x_i, B_i, \lambda) \). Since elements of \( X(S) \) are increasing sequences and \( B_i \) are intervals for all \( i \leq n \), we infer that
\[
(4) \quad (y, x) \in H \times \prod_{i=1}^{n} \left( p_{T_i}^{-1} \left( \prod_{\gamma \in T_i} B_\gamma \right) \right) \cap (X(S))^n \subset U.
\]

From the last fact it follows that
\[
(5) \quad y \in A \left( (x_{\lambda_i})_{\gamma_i}, \ldots, (x_{\lambda_i})_{\gamma_n}, \prod_{i=1}^{n} B_i \right).
\]

Observe that since \( \gamma_i < \alpha \) for all \( i \leq n \), we have
\[
(6) \quad (x_{\lambda_i})_{\gamma_i}, \ldots, (x_{\lambda_i})_{\gamma_n} \in \left( \bigcup \{ Z_\gamma : \gamma \in S \cap \alpha \} \right)^n.
\]

From (5) it follows that there is
\[
y_k \in A \left( (x_{\lambda_i})_{\gamma_i}, \ldots, (x_{\lambda_i})_{\gamma_n}, \prod_{i=1}^{n} B_i \right)
\]
such that
\[
(7) \quad (y, (x_{\lambda_i})_{\gamma_i}, \ldots, (x_{\lambda_i})_{\gamma_n}) \in H_{y_k} \times \prod_{i=1}^{n} F(x_{\lambda_i}, B_i, \lambda(y_k)).
\]

Since \( x_i(\gamma_i) = x_{\lambda_i}(\gamma_i) \) and for all \( \gamma_i < \gamma < \omega_1 \), \( x_i(\gamma) \in B_i \) and \( i \leq n \), we infer by (7) that \( (y, x) \in H_{y_k} \times \prod_{i=1}^{n} F(x_{\lambda_i}, B_i, \lambda(y_k)) \). This completes the proof of Case 1.

Proof of Case 2. Let use assume that \( x = (x_1, \ldots, x_n) \) is an arbitrary point of \( (X(S))^n \) such that \( x_i = x_{\alpha_i} \). Then let \( z = (z_1, \ldots, z_n) \) be such that
\[
(z_i) = \begin{cases} 
  x_i & \text{if } x_i \in U \{ Z_\gamma : \gamma \in S \cap (\alpha + 1) \}, \\
  x_{\alpha_i} & \text{otherwise}.
\end{cases}
\]
Then from Case 1 it follows that there is \( D = D_1 \times D_2 \in \mathcal{D} \) containing \((y, z)\) where \( D_1 \subset Y, \ D_2 = \prod_{i=1}^n D_{2,i} \), and \( D_{2,i} \subset X(S) \) for all \( i \leq n \). Since \( \alpha \in C \), we infer that

\[
P^{-1}_\alpha P_\alpha(D_{2,i}) = D_{2,i} \quad \text{for all } i \leq n.
\]

Since \( z_i|\alpha = x_i \) for all \( i \leq n \), we conclude, applying (8), that \((y, x) \in D\). This completes the proof of Lemma 1.

Now we are in a position to prove

**Lemma 2.** If \( S \) is a stationary subset of \( \omega_1 \) then for every hereditarily Lindelöf space \( Y \) the product \( Y \times (X(\omega_1)_{X(S)})^n \) is Lindelöf.

**Proof.** Suppose that \( Y \) is a hereditarily Lindelöf space. We shall prove Lemma 3 by induction with respect to \( n \). In order to simplify the induction we shall assume that \((X(\omega_1)_{X(S)})^0\) is a one point set. If \( n = 0 \) then there is nothing to prove. Let us suppose that \( Y \times (X(\omega_1)_{X(S)})^n \) is Lindelöf and that \( \mathcal{U} \) is an open cover of \( Y \times (X(\omega_1)_{X(S)})^{n+1} \) consisting of the sets of the form \( H \times \prod_{i=1}^{n+1} P^{-1}_{\gamma_i}(\prod_{\gamma \in \gamma_i} B_\gamma) \cap (X(\omega_1))^{n+1} \) where \( B_\gamma \in \mathcal{B} \) for all \( \gamma \in T_i \) and \( T_i \) is a finite subset of \( \omega_1 \) for \( i \leq n + 1 \). Then by Lemma 1 there is a countable subfamily \( \mathcal{U}' \) of \( \mathcal{U} \) such that \( Y \times (X(S))^{n+1} \subset \bigcup \mathcal{U}' \). Since \( \mathcal{U}' \) is countable, there is \( \alpha < \omega_1 \) such that

\[
\text{if } U = U_1 \times \prod_{i=1}^{n+1} U_{2,i} \in \mathcal{U}', \quad \text{where } U_1 \subset Y \text{ and } U_{2,i} \subset X(\omega_1)
\]

for all \( i \leq n + 1 \), then \( P^{-1}_\alpha P_\alpha(U_{2,i}) = U_{2,i} \).

Observe that from (9) it follows that

\[
\text{if } y \in Y \text{ and } x = (x_1, \ldots, x_{n+1}) \in \left( \bigcup \{Z_\gamma : \gamma \in \omega_1 \setminus \alpha \} \right)^{n+1},
\]

then \((y, x) \in \bigcup \mathcal{U}'\).

Hence

\[
Y \times (X(\omega_1)_{X(S)})^{n+1} \setminus \bigcup \mathcal{U}' \subset \bigcup_{i=1}^{n+1} E_i
\]

where

\[
E_i = \{ (y, (x_1, \ldots, x_{n+1})) \in Y \times (X(\omega_1)_{X(S)})^{n+1} : x_i \in \bigcup \{Z_\gamma : \gamma \leq \alpha \} \}.
\]

The space \( E_i \) is a countable union of subsets that are Lindelöf by the inductive assumption. We conclude that \( \mathcal{U}' \) has a countable subcover. This completes the proof of Lemma 2.

To finish the proof of the fact that \( Y \times (X_n)^n \) is Lindelöf for every hereditarily Lindelöf space \( Y \) it is sufficient to note that \((X_n)^n\) is a finite sum of elements each of which is a continuous image of a space of the form \((X(\omega_1)_{X(S)})^n\), where \( S \) is a stationary subset of \( \omega_1 \) and to apply Lemma 2.

**Remark 1.** R. Pol pointed out to me that one can also apply similar technique to the topology defined by G. Kurepa on an Aronszajn tree (see [T]) in order to obtain another example of the kind described in this paper.

**Remark 2.** Using the pressing-down Lemma one can show that the product \((X_n)^{n+1}\) is not normal for all \( n \in \mathbb{N} \).


[A6] _____, On the class of all spaces of weight not greater than $\omega_1$ whose Cartesian product with every Lindelöf space is Lindelöf, Fund. Math. 129 (1988), 133–140.

