ATTRACTORS OF ITERATED FUNCTION SYSTEMS

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Abstract. In this paper, the question of which compact metric spaces can be attractors of hyperbolic iterated function systems on Euclidean space is studied. It is shown that given any finite-dimensional compact metric $X$, there is a Cantor set $C$ such that the disjoint union of $C$ and $X$ is an attractor. In the process, it is proved that every such $X$ is the Lipschitz image of a Cantor set.

1. Introduction

If $X$ is a complete metric space and $F = \{f_1, f_2, \ldots, f_k\}$ is a collection of contraction mappings of $X$ to itself, then $F$ is said to be a (hyperbolic) Iterated Function System. It is well known [4] that for such an $F$ there is a unique compact set $A$ such that $A = \bigcup_{i=1}^{k} f_i(A)$. $A$ is called the attractor for $F$. In this paper, we are concerned with the question of which compact sets in Euclidean space can be realized as the attractors of some IFS. Williams [6] initially investigated the topological structure of attractors of IFS’s. Hata [3] generalized Williams’ work and showed that not every compactum can be realized as the attractor of an IFS, since, for example, a connected attractor must be locally connected. Hata posed the question whether every finite-dimensional locally connected continuum is the attractor of some IFS. Barnsley [1] showed to what extent every compactum can be approximated by an attractor of an IFS. As a special case, we note that it is not difficult to show that every compact subpolyhedron in $\mathbb{R}^d$ is an attractor.

We show (Theorem 4.1) that for each $n > 0$ there is a Cantor set $C^n \subset \mathbb{R}^{2n}$ with the property that if $X$ is any compact set in $\mathbb{R}^n$, the set

$$C^n \times \{0\} \cup \{0\} \times X \subset \mathbb{R}^{3n}$$

is the attractor of an IFS. It follows that every compact, finite dimensional metric space that contains a closed and open Cantor subset can be embedded in some Euclidean space as an attractor.
Most of our notation is standard, although we shall use $\| \cdot \|$ to denote the sum norm on $\mathbb{R}^n$. That is, if $x = (x_1, x_2, \ldots, x_n)$ then $\|x\| = |x_1| + |x_2| + \cdots + |x_n|$.

2. Lipschitz mappings of Cantor sets onto compacta

A classical theorem of topology states that every compact metric space is the continuous image of the Cantor set. In this section, we construct, for each $n > 0$, a Cantor set $V^n$ with the property that for every compactum $X \subset \mathbb{R}^n$ there is a Lipschitz mapping of $V^n$ onto $X$.

To begin, fix an $n \geq 1$ and let $B$ be the unit cube in $\mathbb{R}^n$ spanned by the standard basis vectors. Subdivide $B$ into cubes $B_i$, $i \in \Gamma$, where $\Gamma = \{0, 1, \ldots, 2^n - 1\}$ and $B_i = \prod_{j=0}^{n} [e_j, 1/2 + e_j]$ with $e_j$ being one half the $j$th binary digit of $i$. Subdivide the $B_i$'s in a similar manner and continue the process so that at the $k$th stage, $B$ is the union of $2^{kn}$ cubes $B_{i_1 i_2 \cdots i_k}$. For each $k \geq 1$, give the space of $k$-tuples of integers the discrete topology, let

$$V_k = \bigcup_{i_1, i_2, \ldots, i_k \in \Gamma} B_{i_1 i_2 \cdots i_k} \times (i_1, i_2, \ldots, i_k)$$

and define $p_k : V_{k+1} \to V_k$ by $p_k(x, i_1, i_2, \ldots, i_{k+1}) = (x, i_1, i_2, \ldots, i_k)$. If $\bar{i} = \{i_k\}$ is a sequence of integers with $i_k \in \Gamma$, let

$$x(\bar{i}) = \bigcap_{k=1}^{\infty} B_{i_1 i_2 \cdots i_k}.$$

In the next section, we will need the following observation, whose proof is left to the reader.

**Proposition 2.1.** If $i_k = x_k + x_k 2 + \cdots + x_k 2^{n-1}$, with $x_k \in \{0, 1\}$, then the $m$th component of $x(\bar{i})$ has the base 2 expansion $x_{1m} x_{2m} x_{3m} \cdots$.

Let $V^n$ be the inverse limit of the sequence

$$V_1 \xrightarrow{p_1} V_2 \xrightarrow{p_2} \cdots.$$

Note that we may view a point in $V^n$ as a pair $(x, \bar{i})$, where $\bar{i}$ is a sequence of integers from $\Gamma$ and $x = x(\bar{i})$. Define a metric $\rho$ on $V^n$ by

$$\rho((x, \bar{i}), (y, \bar{j})) = \|x - y\| + 1/2^k,$$

where $\| \cdot \|$ is the sum norm described in the introduction and $k + 1$ is the first index in which $\bar{i}$ and $\bar{j}$ disagree (note that if $\bar{i} = \bar{j}$ then $x = y$).

**Proposition 2.2.** $V^n$ is a Cantor set and $\rho$ is a metric consistent with the inverse limit topology on $V^n$. 

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Proof. Recall that if \( \{X_k\} \) is a sequence of metric spaces with bounded metrics \( d_k \), then

\[
d(\{x_k\}, \{y_k\}) = \sum_{k=1}^{\infty} \frac{d_k(x_k, y_k)}{2^k}
\]

is a metric on the product of the \( X_k \)'s that induces the product topology. Let \( d_k \) be the metric on \( V_k \) given by

\[
d_k((x, i_1, i_2, \ldots, i_k), (y, j_1, j_2, \ldots, j_k)) = \|x - y\| + \varepsilon;
\]

where \( \varepsilon = 0 \) if \( (i_1, i_2, \ldots, i_k) = (j_1, j_2, \ldots, j_k) \), 1 otherwise. Then, if \( (x, \vec{i}) \) and \( (y, \vec{j}) \) are in \( V^n \) and \( k + 1 \) is the first index for which \( \vec{i} \) and \( \vec{j} \) disagree, the construction of \( d \) above gives

\[
d((x, \vec{i}), (y, \vec{j})) = \sum_{m=1}^{\infty} \frac{\|x - y\|}{2^m} + \sum_{m=k+1}^{\infty} \frac{1}{2^m} = \|x - y\| + \frac{1}{2^k} = \rho((x, \vec{i}), (y, \vec{j})).
\]

Thus \( \rho \) agrees with the product metric. To see that \( V^n \) is a Cantor set, note that the map \( (x, \vec{i}) \to \vec{i} \) is a homeomorphism of \( V^n \) onto the space of sequences of integers from \( \Gamma \). \( \square \)

Now let \( X \) be a compact subset of \( B \), and define a mapping \( \phi : V^n \to X \) as follows. For each \( k \)-tuple \( (i_1, i_2, \ldots, i_k) \) such that \( B_{i_1, i_2, \ldots, i_k} \) intersects \( X \), choose a point \( x_{i_1, i_2, \ldots, i_k} \in B_{i_1, i_2, \ldots, i_k} \cap X \). Given \( (x, \vec{i}) \in V^n \), define

\[
\phi(x, \vec{i}) = \begin{cases} x_{i_1, i_2, \ldots, i_k} & \text{if } x \notin X \text{ and } k \text{ is the last index with } X \cap B_{i_1, i_2, \ldots, i_k} \neq \emptyset, \\ x & \text{if } x \in X. \end{cases}
\]

**Theorem 2.1.** \( \phi \) is a Lipschitz map of \( V^n \) onto \( X \).

Proof. Given \( a = (x, \vec{i}) \), \( b = (y, \vec{j}) \in V^n \), there are three cases to consider.

Case 1. Both \( x \) and \( y \) are in \( X \). In this case,

\[
\|\phi(a) - \phi(b)\| = \|x - y\| \leq \rho(a, b).
\]

Case 2. \( x \in X \), but \( y \notin X \). Let \( k + 1 \) be the first index in which \( \vec{i} \) and \( \vec{j} \) disagree, and let \( \phi(b) = x_{j_1, j_2, \ldots, j_l} \). Then \( k \leq l \) and

\[
\|y - x_{j_1, j_2, \ldots, j_l}\| \leq n/2^l \leq n/2^k
\]

since both \( y \) and \( x_{j_1, j_2, \ldots, j_l} \) are in \( B_{j_1, j_2, \ldots, j_l} \). We have

\[
\|\phi(a) - \phi(b)\| = \|x - x_{i_1, i_2, \ldots, i_l}\| \leq \|x - y\| + \|y - x_{j_1, j_2, \ldots, j_l}\| \
\leq \|x - y\| + n/2^k \leq \rho(a, b).
\]

Case 3. Neither \( x \) nor \( y \) are in \( X \). Let \( \phi(a) = x_{i_1, i_2, \ldots, i_m} \), \( \phi(b) = x_{j_1, j_2, \ldots, j_m} \), and let \( k + 1 \) be the first index in which \( \vec{i} \) and \( \vec{j} \) disagree. We may assume that \( l \leq m \), and we would like to have \( k \leq l \). If \( m \leq k \), then \( \phi(a) = \phi(b) \).
and there is nothing to prove. By the definition of $\phi$, the case $l < k < m$ leads to a contradiction, so we assume $k \leq l$. Then

$$\| \phi(a) - \phi(b) \| = \| x_i x_j \cdots x_m - x_i x_j \cdots x_m \|$$

$$\leq \| x - y \| + \| x - x_i x_j \cdots x_m \| + \| y - x_i x_j \cdots x_m \|$$

$$\leq \| x - y \| + 2n/2^k \leq 2n \rho(a, b).$$

Thus $\phi$ is a Lipschitz map with Lipschitz constant at most $2n$. $\square$

Since every compact set in $\mathbb{R}^n$ can be moved into $B$ by similarity transformations, we have

**Corollary 2.1.** If $X$ is any compact set in $\mathbb{R}^n$, there is a Lipschitz map of $V^n$ onto $X$.

### 3. Standard Cantor sets

Let $C^1$ be the set of points in $\mathbb{R}^2$ of the form $(x, y)$, where $x, y \in [0, 1]$ have base 2 expansions of the form $x = .x_1 x_2 x_3 \cdots$ and $y = .y_1 y_2 y_3 \cdots$, with $x_i = 0$ for $i$ even and $y_i = 0$ for $i$ odd. Define contractions $h_0$ and $h_1$ on $\mathbb{R}^2$ by

$$h_0(x, y) = \left(\frac{y}{2}, \frac{x}{2}\right) \quad \text{and} \quad h_1(x, y) = \left(\frac{y}{2} + \frac{1}{2}, \frac{x}{2}\right).$$

**Proposition 3.1.** $C^1$ is a Cantor set and is the attractor for the IFS $F^1 = \{h_0, h_1\}$.

**Proof.** It is easy to check that $h_0(C^1)$ and $h_1(C^1)$ are disjoint and that their union is $C^1$. Thus $C^1$ is the attractor and the disjointness implies that it is a Cantor set [3, Theorem 4.4]. $\square$

If $F = \{f_1, f_2, \ldots, f_k\}$ and $G = \{g_1, g_2, \ldots, g_l\}$ are IFSs on the complete metric spaces $X$ and $Y$ with attractors $A$ and $B$, respectively, define the IFS $F \times G$ on $X \times Y$ to be the set of all products $f_i \times g_j$.

The reader can easily verify the following proposition.

**Proposition 3.2.** The attractor of $F \times G$ is $X \times Y$.

It follows that for each $n > 1$ the $n$-fold product $C^n = C^1 \times C^1 \times \cdots \times C^1$ is a Cantor set in $\mathbb{R}^{2n}$ that is the attractor of the IFS $F^n = F^1 \times F^1 \times \cdots \times F^1$.

We note in passing that since $F^n$ consists of $2^n$ similitudes with similarity constant $\frac{1}{2}$, the Hausdorff dimension of $C^n$ is $n$ [2].

In the previous section, we showed that each compactum in a finite-dimensional Euclidean space is the Lipschitz image of an abstractly defined Cantor set. The purpose of the present construction is to be able to replace $V^n$ by the more geometrically appealing $C^n$. Again, we need some preliminary notation. If $u \in C^n$, we can represent $u$ as a $2n$-tuple $u = (x_1, x_2, x_3, \ldots, x_n, y_n)$ where each $x^m$ has the binary expansion $x^m = .x_1^m x_2^m \cdots x^m_n$ and $x_k^m = 0$ if $k$ is even and the $y^m$ have similar expansions with $y_k^m = 0$ for $k$ odd. Then we
can define a mapping \( \psi: C^n \to V^n \) by \( \psi(u) = (x(\bar{i}), \bar{v}) \), where

\[
i_k = \sum_{m=1}^{n} \frac{x_k^m + y_k^m}{2^{m-1}}.
\]

**Theorem 3.1.** The map \( \psi: C^n \to V^n \) is a Lipschitz equivalence.

**Proof.** It follows from the uniqueness of representation of integers and the special properties of the points in \( C^n \) that \( \psi \) is one-to-one and onto. Let \( u \) be as above, let \( v = (w^1, z^1, w^2, z^2, \ldots, w^n, z^n) \) be a second point, and let \( \psi(u) = (a, \bar{x}), \psi(v) = (b, \bar{y}) \). By Proposition 2.1 the \( m \)th coordinate of \( a \) is \( x^m + y^m \) and the \( m \)th coordinate of \( b \) is \( w^m + z^m \), so we have

\[
\|a - b\| = \sum_{m=1}^{n} |x^m + y^m - w^m - z^m| \\
\leq \sum_{m=1}^{n} |x^m - w^m| + |y^m - z^m| = \|u - v\|.
\]

Suppose that \( k + 1 \) is the first index in which \( \bar{x} \) and \( \bar{y} \) disagree. Since \( a \) and \( b \) must have some coordinates that disagree in the \( k + 1 \)st position of their binary expansions, \( \|a - b\| \geq 1/2^{k+1} \). Therefore,

\[
\rho(\psi(u), \psi(v)) = \|a - b\| + \frac{1}{2^k} \leq 3\|a - b\| \leq 3\|u - v\|.
\]

On the other hand,

\[
\|u - v\| = \sum_{m=1}^{n} |x^m - w^m| + |y^m - z^m| \leq \frac{2n}{2^k},
\]

since \( x^m, w^m \) and \( y^m, z^m \) agree in at least their first \( k \) places. Therefore

\[
\|u - v\| \leq 2n\|a - b\| + \frac{2n}{2^k} = 2n\rho(\psi(u), \psi(v)).
\]

**Corollary 3.1.** If \( X \) is a compact set in \( R^n \), there is a Lipschitz map of \( C^n \) onto \( X \).

4. **Constructing IFSs**

Recall the contractions \( h_0 \) and \( h_1 \) on \( R^2 \) constructed in §3. For an \( n > 1 \) we index the members of the IFS \( F^n \) as follows. For an integer \( i \in \Gamma \), let \( i = \sum_{k=0}^{n-1} x_k 2^k \) be its binary representation, and let \( f_i = h_{x_0} \times h_{x_1} \times \cdots \times h_{x_{n-1}} \).

For \( i_1, i_2, \ldots, i_k \in \Gamma \), let \( f_{i_1 i_2 \cdots i_k} \) be the composition \( f_{i_k} f_{i_{k-1}} \cdots f_{i_1} \). It is easy to see that
Proposition 4.1. Each $f_{i_1 i_2 \ldots i_k}$ is a contraction of $\mathbb{R}^{2n}$ with contractivity constant $1/2^k$ and

$$C^n = \bigcup_{i_1, i_2, \ldots, i_k \in \Gamma} f_{i_1 i_2 \ldots i_k} (C^n)$$

for each $k$.

Given a compact set $X \subset \mathbb{R}^n$ with $0 \not\in X$, let $C + X$ denote the union

$$C + X = (C^n \times \{0\}) \cup (\{0\} \times X) \subset \mathbb{R}^{3n}.$$

Theorem 4.1. There is an IFS on $\mathbb{R}^{3n}$ whose attractor is $C + X$.

Proof. Write $\mathbb{R}^{3n}$ as $\mathbb{R}^{2n} \times \mathbb{R}^n$ and let $\lambda : C^n \to X$ be a Lipschitz map with Lipschitz constant $L$. By the “Lipschitz Tietze Theorem” [5], we may assume that $\lambda$ is defined on all of $\mathbb{R}^{2n}$. For each $i \in \Gamma$ let $\hat{f}_i$ be defined by

$$\hat{f}_i(x, y) = (f_i(x), 0).$$

Choose a $k$ such that $L/2^k < 1$ and for each $k$-tuple $i_1, i_2, \ldots, i_k \in \Gamma$, let

$$g_{i_1 i_2 \ldots i_k} (x, y) = (0, \lambda f_{i_1 i_2 \ldots i_k} (x)).$$

Then the collection

$$\{\hat{f}_i | i \in \Gamma\} \cup \{\lambda g_{i_1 i_2 \ldots i_k} | i_1, i_2, \ldots, i_k\}$$

is an IFS whose attractor is $C + X$. □

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