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THE HAUSDORFF MEAN OF A FOURIER-STIELTJES TRANSFORM

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ABSTRACT. It is shown that the integral Hausdorff mean $T\hat{\mu}$ of the Fourier-Stieltjes transform of a measure on the real line is the Fourier transform of an L^1 function if and only if $T\hat{\mu}$ vanishes at infinity or the kernel of T has mean value zero. Also a sufficient condition on the kernel of T and a necessary and sufficient condition on the measure is established in order for $-i\,\mathrm{sign}(x)T\hat{\mu}(x)$ to be the Fourier transform of an L^1-function. These results yield an improvement of Fejer's and Wiener's formulas for the inversion of Fourier-Stieltjes transforms, the uniqueness property of certain generalized Fourier transforms, and a generalization of the mean ergodic theorem for unitary operators.

Let $M(R)$ be the space of complex bounded regular Borel measures on the real line R and $\hat{\mu}(x)$ be the Fourier-Stieltjes transform of a measure μ in $M(R)$,

$$\hat{\mu}(x) = \int_R e^{-ixt}\,d\mu(t), \qquad x \in R.$$

We consider the integral Hausdorff mean $T\hat{\mu}$ generated by a Borel measurable kernel ψ in $L^1(R)$, which is defined for $x \in R$ by

$$T\hat{\mu}(x) = \int_R \hat{\mu}(xs)\psi(s)\,ds = \frac{1}{|x|}\int_R \psi\left(\frac{y}{x}\right)\hat{\mu}(y)\,dy,$$

for $x \neq 0$, and $T\hat{\mu}(0) = \hat{\mu}(0)\int_R \psi(s)\,ds$. When the kernel ψ is the characteristic function of $[0, 1]$, $T\hat{\mu}$ reduces to the integral arithmetic average of $\hat{\mu}$ over $[0, x]$.

The summability properties of T are well known [8, pp. 275–278]. The continuity properties and the spectrum of T as a bounded operator on $L^p(R)$ have been studied by Schur [14], Hardy, Littlewood, and Polya [9], Rhoades [12], Fabes, Jodeit, and Lewis [3], and Leibowitz [11]. Goldberg in [5, 6] studied the properties of the transformation T on the Fourier transform for the Lebesgue class $L^p(R)$ for $1 < p \leq 2$.

The objective here is to determine the Fourier analytic properties of T on the class of Fourier-Stieltjes transforms. Theorems 1 and 3 are the main results,

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which include the analogue of the results of Goes [7] and Georgakis [4], concerning the arithmetic means of Fourier-Stieltjes coefficients of a measure on the circle for the integral arithmetic average of the Fourier-Stieltjes transform of a measure on the real line. As a consequence, we obtain: (a) certain improved variants of the formulas of Fejer and Wiener for the inversion and quadratic variation of Fourier-Stieltjes transforms (Corollary 1); (b) a strengthened generalization of the mean ergodic theorem for a one-parameter group of unitary operators. Theorem 2 establishes the uniqueness of the generalized Fourier transforms introduced by Sz.-Nagy [14] and studied extensively in [1].

We show that in general $T\hat{\mu}$ is a Fourier-Stieltjes transform and, in particular, $T\hat{\mu}$ is the Fourier Transform of a function in $L^1(R)$ if and only if the kernel ψ has mean value zero over R, or $T\hat{\mu}$ tends to zero at infinity. Moreover, $-i\,\mathrm{sign}(x)T\hat{\mu}(x)$ is the Fourier transform of a function in $L^1(R)$ if and only if $T\hat{\mu}$ tends to zero at infinity and $x^{-1}\mu((-x, x))$ belongs to $L^1(R)$, assuming that the kernel $0 \not\equiv \psi \in L^1(R)$ is nonnegative, nonincreasing, bounded, and has support in $[0, \infty)$; the conditions imposed on the kernel ψ for the latter are satisfied by the usual summability kernels including e^{-s}, e^{-s^2}, $1\backslash(1+s^2)$, and $\alpha(1-s)^{\alpha-1}$ for $\alpha \geq 1$ and $0 \leq s \leq 1$.

The properties of the transformation T depend on those of the transformation S on $M(R)$, which is defined as

$$S\mu(x) = \begin{cases} \displaystyle\int_{R-\{0\}} \frac{1}{|y|}\psi\left(\frac{x}{y}\right) d\mu(y) & \text{for } 0 \neq x \in R, \\ 0 & \text{for } x = 0. \end{cases}$$

It should be noted that $S\mu$ is an even or odd function if the corresponding measure μ is even or odd, that is, $\mu(-E) = \mu(E)$ or $\mu(-E) = -\mu(E)$ for each Borel subset E of R, respectively. Furthermore, if $\mu \geq 0$ and the kernel $\psi \geq 0$ is nonincreasing with support on $[0, \infty)$, then $S\mu$ is nonincreasing for $x > 0$.

Theorem 1. *Let $T\hat{\mu}$ be the Hausdorff mean of the Fourier-Stieltjes transform of a measure $\mu \in M(R)$ whose kernel $\psi \in L^1(R)$. Then*

(a) $S\mu \in L^1(R)$;

(b) *$T\hat{\mu}$ is a Fourier-Stieltjes transform, and* $T\hat{\mu}(x) = \mu(\{0\}) \int_R \psi(s)\,ds + (S\mu)\hat{}\,(x)$, $x \in R$;

(c) *$T\hat{\mu}$ is the Fourier transform of a function in $L^1(R)$ if and only if either ψ has mean value zero or $T\hat{\mu}$ tends to zero at infinity, which is equivalent to* $\mu(\{0\}) = 0$.

Proof. Clearly, (c) follows from (a) and (b), since $(S\mu)\hat{}(x) \to (x)$ as $|x| \to \infty$ by the Riemann-Lebesgue theorem. For (a) and (b), we may assume without loss of generality that μ is nonnegative. Then Fubini's theorem yields

$$\begin{aligned} \int_R S\mu(x)\,dx &= \int_{R-\{0\}} d\mu(y) \int_R \psi\left(\frac{x}{y}\right)\frac{dx}{|x|} \\ &= -\mu(\{0\}) \int_R \psi(s)\,ds + \int_R \psi(s)\,ds \int_R d\mu(t). \end{aligned}$$

This shows that $S\mu \in L^1(R)$ and $T\hat{\mu}(0) = \mu(\{0\}) \int_R \psi(s)\,ds + (S\mu)\hat{}(0)$ and establishes (b) for $x = 0$. Similarly, for $x \neq 0$ and with the change of variable

$y = (x/t)\, s$, $dy = (|x|/|t|) ds$, we get

$$T\hat{\mu}(x) - \mu(\{0\}) \int_R \psi(s)\, ds = \int_{R-\{0\}} e^{-ixt}\, d\mu(t) \int_R \psi\left(\frac{y}{x}\right) \frac{dy}{|x|}$$
$$= \int_{R-\{0\}} d\mu(t) \int_R e^{-ixs} \psi\left(\frac{s}{t}\right) \frac{ds}{|t|} = (S\mu)\hat{}(x),$$

which is (b) for $x \neq 0$.

Theorem 2. *Let μ be a continuous measure in $M(R)$, $\mu(R) = 0$, and*

$$\dot{\mu}(x) = \int_R \frac{e^{-ixt} - 1}{t}\, d\mu(t) = 0$$

for $0 \neq x \in R$. Then μ is the zero measure.

Proof. Let $T\hat{\mu}$ be the Hausdorff mean whose kernel ψ is the characteristic function of $[0, 1]$. Then $\dot{\mu}(x)/(-ix) = T\hat{\mu}(x) = 0$ for $x \neq 0$. By Theorem 1, $\mu(\{0\}) = 0$ and $\dot{\mu}(x) = -ix(S\mu)\hat{}(x)$ for $x \neq 0$. Since $T\hat{\mu}(0) = \hat{\mu}(0) = \mu(R)$ and $T\hat{\mu}(0) = (S\mu)\hat{}(0)$ by Theorem 1(b), it follows that $(S\mu)\hat{}(x) = 0$ for all $x \in R$. Furthermore, if $E_y = [0, y]$ for $y > 0$ and $E_y = [y, 0]$ for $y < 0$, then it is easy to verify that

$$\mu(E_y) = \int_0^y S\mu(x)\, dx - y\, \mathrm{sign}(y) S\mu(y). \tag{1}$$

Therefore, if μ is a continuous measure, then $S\mu(y)$ is continuous for $y \neq 0$ and its Fourier integral is Cesaro summable to $S\mu(y)$ for $y \neq 0$. Since $(S\mu)\hat{}(x) = 0$, it follows that $S\mu(y) = 0$ for $y \neq 0$. Hence, $\mu(E_y) = 0$ for $y \in R$ because of equation (1) and the fact that $\mu(\{0\}) = 0$. Therefore, μ is the zero measure.

Theorem 3. *Let $T\hat{\mu}$ be the Hausdorff mean of the Fourier-Stieltjes transform of measure $\mu \in M(R)$, whose kernel $0 \not\equiv \psi \in L^1(R)$ is nonnegative, nonincreasing, bounded, and has support in $[0, \infty)$. Then $-i\, \mathrm{sign}(x) T\hat{\mu}(x)$ is the Fourier transform of a function in $L^1(R)$ if and only if $T\hat{\mu}$ tends to zero at infinity and $x^{-1}\mu((-x, x)) \in L^1(0, \infty)$.*

Proof. First we show that $-i\, \mathrm{sign}(x) T\hat{\mu}(x)$ is the Fourier transform of a function in $L^1(R)$ if and only if $\mu(\{0\}) = 0$ and the Hilbert transform of $S\mu$ belongs to $L^1(R)$. Suppose $-i\, \mathrm{sign}(x) T\hat{\mu}(x) = \hat{g}(x)$ for $g \in L^1(R)$. Then both $T\hat{\mu}(x)$ and $(S\mu)\hat{}(x)$ tend to zero as $|x| \to \infty$ by the Riemann-Lebesgue theorem and $\mu(\{0\}) = 0$ by Theorem 1(b) and the fact that $\psi \geq 0$. Moreover, $-i\, \mathrm{sign}(x) T\hat{\mu}(x) = -i\, \mathrm{sign}(x) (S\mu)\hat{}(x)$. Thus, the Fourier integral of $-i\, \mathrm{sign}(x) T\hat{\mu}(x)$ is Abel summable to $g(x)$ a.e. [2, p.192], and at the same time it is Abel summable to the Hilbert transform $\widetilde{S}\mu(x)$ of the function $S\mu(x)$ a.e. [2, p.327]. Therefore, $(\widetilde{S}\mu)(x) = g(x)$ a.e. and $S\mu \in L^1(R)$. Conversely, if $T\hat{\mu}$ tends to zero at infinity and $\widetilde{S}\mu \in L^1(R)$, then $T\hat{\mu} = (S\mu)\hat{}$ by Theorem 1(b) and $(S\mu)\hat{}(x) = -i\, \mathrm{sign}(x) (S\mu)\hat{}(x)$ [2, p.324]. In order to complete the proof of the theorem, it remains to show that $\widetilde{S}\mu \in L^1(R)$ if and only if $x^{-1}\mu((-x, x)) \in L^1(0, \infty)$, assuming that $\mu(\{0\}) = 0$.

(a) Let $\rho \in M(R)$ be an odd measure in $M(R)$. We will show that $\widetilde{S}\rho \in L^1(R)$. $S\rho(x)$ is an odd function in $L^1(R)$ and nonincreasing for $x > 0$. Its Hilbert transform is even and for $x > 0$

$$-\pi\widetilde{S}\rho(x) = \int_{-\infty}^{\infty} \frac{S\rho(t)}{t-x}\,dt = \left(\int_{3x/2}^{\infty} + \int_{x/2}^{3x/2} + \int_0^{x/2}\right)\left(\frac{1}{t-x} + \frac{1}{t+x}\right)S\rho(t)\,dt$$
$$= g_1(x) + g_2(x) + g_3(x).$$

Thus, it suffices to show that $\widetilde{S}\rho \in L^1(0, \infty)$, assuming without loss of generality that $\mu \geq 0$ on $[0, \infty)$. Let $\varphi(x) = S(S\rho)(x)$ for $x > 0$ when the kernel of S is the characteristic function of $[0, 1]$. Then $\varphi \in L^1(0, \infty)$ by Theorem 1(a). Now for $0 < 3x/2 \leq t < \infty$, $1/(t-x) \leq 3/t$, $1/(t+x) \leq 1/t$, $|g_1(x)| \leq (18/5)\varphi(3x/2)$, and $g_1 \in L^1(0, \infty)$. Next

$$g_2(x) = \int_{x/2}^{3x/2} \frac{S\rho(t)}{t-x}\,dt + \int_{x/2}^{3x/2} \frac{S\rho(t)}{t+x}\,dt = g_4(x) + g_5(x),$$

where $|g_5(x)| \leq \phi(x/2)$ and $g_5 \in L^1(0, \infty)$. Since $S\rho(t) \geq 0$ and is nonincreasing for $t > 0$, we have

$$|g_4(x)| = \int_0^{x/2} [S\rho(x-u) - S\rho(x+u)]\frac{du}{u}$$
$$= \int_0^{x/2} \frac{du}{u} \int_{[x-u,x)} \psi\left(\frac{x}{t}\right)\frac{d\rho(t)}{t} + \int_0^{x/2} \frac{du}{u} \int_{[x,x+u)} \psi\left(\frac{x}{t}\right)\frac{d\rho(t)}{t}$$
$$\leq \|\psi\|_\infty \int_{[x/2,x)} \log\frac{x}{2(x-t)}\frac{d_\rho(t)}{t} + \|\psi\|_\infty \int_{[x,3x/2)} \log\frac{x}{2(t-x)}\frac{d\rho(t)}{t}$$
$$= \|\psi\|_\infty \rho((0, \infty))\left[\int_1^2 \log\frac{s}{2(1-s)}\,ds + \int_{1/3}^1 \log\frac{s}{2(1-s)}\,ds\right] < \infty.$$

Thus, $g_4 \in L^1(0, \infty)$, and $g_2 \in L^1(0, \infty)$. Finally,

$$g_3(x) = \int_0^{x/2}\left(\frac{1}{x} + \frac{1}{t-x}\right)S\rho(t)\,dt + \int_0^{x/2}\left(\frac{1}{t+x} - \frac{1}{x}\right)S\rho(t)\,dt$$
$$= g_6(x) + g_7(x)$$

Then $g_6 \in L^1(0, \infty)$ and $g_7 \in L^1(0, \infty)$ because

$$|g_6(x)| \leq \frac{2}{x^2}\int_0^{x/2} tS\rho(t)\,dt = \omega(x), \qquad |g_7(x)| \leq \frac{1}{2}\omega(x),$$

$$\int_0^\infty \omega(x)\,dx = \int_0^\infty tS\rho(t)\,dt \int_{2t}^\infty \frac{2}{x^2}dx = \int_0^\infty S\rho(t)\,dt < \infty.$$

This implies that $g_3 \in L^1(0, \infty)$. Hence $\widetilde{S}\rho \in L^1(R)$.

(b) Let σ be an even measure in $M(R)$. We show that $\widetilde{S}\sigma \in L^1(R)$ if and only if $x^{-1}\sigma((0, x)) \in L^1(0, \infty)$. We may assume $\sigma \geq 0$ on $[0, \infty)$. $S\sigma(x)$ is an even function in $L^1(R)$ and nonincreasing for $x > 0$; its Hilbert transform is odd and for $x > 0$,

$$-\pi\widetilde{S}\sigma(x) = \left(\int_{3x/2}^{\infty}\int_{x/2}^{3x/2} + \int_0^{x/2}\right)\left(\frac{1}{t-x} - \frac{1}{t+x}\right)S\sigma(t)\,dt$$
$$= f_1(x) + f_2(x) + f_3(x).$$

The argument that showed $g_1 \in L^1(0, \infty)$ in part (a) also shows that $f_1 \in L^1(0, \infty)$. Put

$$f_2(x) = \int_{x/2}^{3x/2} \frac{S\sigma(t)}{t-x}\,dt - \int_{x/2}^{3x/2} \frac{S\sigma(t)}{t+x}\,dt = f_4(x) - f_5(x).$$

The proof that $f_4, f_5 \in L^1(0, \infty)$ is identical with that given for $g_4, g_5 \in L^1(0, \infty)$ in (a). Finally, let

$$\begin{aligned} f_3(x) &= \int_0^{x/2} \left(\frac{1}{x} + \frac{1}{(t-x)}\right) S\sigma(t)\,dt - \int_0^{x/2} \left(\frac{1}{t+x} - \frac{1}{x}\right) - S\sigma(t)\,dt \\ &\quad - \frac{2}{x}\int_0^{x/2} S\sigma(t)\,dt \\ &= f_6(x) - f_7(x) - \theta\left(\frac{x}{2}\right). \end{aligned}$$

Here, $f_6, f_7 \in L^1(0, \infty)$ for the same reason that $g_6, g_7 \in L^1(0, \infty)$ in (a). Thus, f_3 differs from an integrable function on $(0, \infty)$ by the function $-\theta(x)$. But

$$\begin{aligned} \theta(x) &\geq \frac{1}{x}\int_0^x du \int_u^\infty \psi\left(\frac{u}{t}\right)\frac{d\sigma(t)}{t} = x^{-1}\sigma((0, x))\int_0^1 \psi(s)\,ds, \\ \theta(x) &= \frac{1}{x}\int_0^x du \int_0^x \psi\left(\frac{u}{t}\right)\frac{d\sigma(t)}{t} + \frac{1}{x}\int_0^x du \int_x^\infty \psi\left(\frac{u}{t}\right)\frac{d\sigma(t)}{t} \\ &\leq x^{-1}\sigma((0, x))\|\psi\|_1 + \|\psi\|_\infty S\sigma(x), \end{aligned}$$

by Fubini's theorem. From the preceding inequalities it follows that $\theta \in L^1(0,\infty)$ if and only if $x^{-1}\sigma((0, x)) \in L^1(0, \infty)$.

Finally, let $\mu \in M(R)$ and $2\rho(E) = \mu(E) + \mu(-E)$ and $2\sigma(E) = \mu(E) + \mu(-E)$. Then ρ is an odd measure, σ is an even measure, and $\widetilde{S}\mu = \widetilde{S}\rho + \widetilde{S}\sigma$. By (a) $\widetilde{S}\rho \in L^1(R)$, and by (b) $\widetilde{S}\sigma \in L^1(R)$ if and only if $x^{-1}\sigma((0, x)) \in L^1(0,\infty)$, or equivalently $x^{-1}\mu((-x, x)) \in L^1(0, \infty)$, and thus completes the proof of Theorem 3.

Corollary 1. *For $\psi \in L^1(R)$, $\mu \in M(R)$, and $p \in R$, let*

$$\begin{aligned} r(x) &= T(\hat{\mu}\cdot\hat{\delta}_p)(x) - \mu(\{p\})\int_R \psi(s)\,ds, \\ R(x) &= T|\hat{\mu}|^2(x) - \sum_{p\in R}|\mu(\{p\})|^2\int_R \psi(s)\,ds. \end{aligned}$$

Then

(a) *$r(x)$ and $R(x)$ are Fourier transforms of functions in $L^1(R)$.*

(b) *$-i\,\mathrm{sign}(x)r(x)$ is the Fourier transform of a function in $L^1(R)$ if and only if $T(\hat{\mu}, \hat{\delta}_p)$ tends to zero at infinity and $x^{-1}\mu((-p-x, -p+x)) \in L^1(R)$, provided $0 \not\equiv \psi \geq 0$, is nonincreasing, bounded, and has support in $[0, \infty)$.*

(c) *$-i\,\mathrm{sign}(x)R(x)$ is the Fourier transform of a function in $L^1(R)$ if and only if $T|\hat{\mu}|^2$ tends to zero at infinity and $x^{-1}\mu*\mu^*((-x, x)) \in L^1(R)$, provided ψ satisfies the conditions for (b).*

Corollary 1 follows from Theorems 1 and 3 when applied to the convolution $\mu * \delta_p$, where δ_p is the unit mass at the point p, and to $\mu * \mu^*$, where $\mu^*(E) = \overline{\mu(-E)}$. When ψ is the characteristic function of $[0, 1]$ and μ is replaced by the even part of μ, given by $2\sigma(E) = \mu(E) + \mu(-E)$, the Hausdorff mean reduces to the integral arithmetic average and Corollary 1 provides a considerable improvement of the Theorems of Fejer and Wiener [10, pp. 102–104].

Finally, it is of interest to note that Corollary 1 yields a strengthened generalization of the mean ergodic theorem for the Hausdorff mean of a continuous parameter group of unitary operators. Consider a one-parameter group U_t, $t \in R$, of unitary operators on a Hilbert space X such that $(U_t f, g)$ is continuous for $f, g \in X$. By Stone's theorem

$$U_t = \int_R e^{-it\lambda}\, dE(\lambda),$$

where $E(\lambda)$ a projection-valued spectral measure on R. Moreover, $(E(\lambda)f, g)$ is a regular bounded complex Borel measure on R whose total variation is $\leq \|f\| \cdot \|g\|$ and Fourier-Stieltjes transform is given by $(U_t f, g)$ and $\|E(\lambda)f\|^2$ is a positive regular Borel measure on R with total variation $\|f\|^2$ (see [13, pp. 381–389]). Let ψ be continuous in $L^1(R)$, and consider the Hausdorff mean $M_x^{(a)} f$ of the vector-valued function $U_{a+t} f$, $f \in X$, given by

$$M_x^{(a)} f = \frac{1}{|x|} \int_R \psi\left(\frac{t}{x}\right) U_{a+t} f\, dt, \qquad 0 \neq x \in R,\ a \in R,$$

defined as a limit of Riemann sums, and so that $M_0^{(a)} f = \left(\int_R \psi(s)\, ds\right) U_a f = \overline{\psi} U_a f$. It is easy to see that

$$\hat{\delta}_\lambda(a) T \hat{\delta}_\lambda(x) - \delta_\lambda(\{0\})\overline{\psi} = \hat{\delta}_\lambda(a)[T\hat{\delta}_\lambda(x) - \delta_\lambda(\{0\})\overline{\psi}]$$

for λ, $a \in R$ and that it is convergent to zero as $|x| \to \infty$ by Corollary 1(a). In fact, the convergence is uniform for $p \in R$ and bounded by $\|\psi\|_1 + |\overline{\psi}|$ for $\lambda \in R$. Therefore,

$$M_x^{(a)} - \overline{\psi} E(\{0\}) = \int_R (\hat{\delta}_\lambda(a) T \hat{\delta}_\lambda(x) - \delta(\{0\})\overline{\psi})\, dE(\lambda),$$

$$\|M_x^{(a)} f - \overline{\psi} E(\{0\}) f\|^2 = \int_R |\hat{\delta}_\lambda(a) T \hat{\delta}_\lambda(x) - \delta_\lambda(\{0\})\overline{\psi}|^2 d\|E(\lambda) f\|^2,$$

which tends to zero as $|x| \to \infty$ uniformly for $p \in R$ and yields part of Corollary 2 below. The rest is simply a restatement of Corollary 1(a), because as noted above $(U_t f, g)$ is the Fourier-Stieltjes transform of the measure $(E(\lambda)f, g)$ in $M(R)$ and

$$\left(\frac{1}{|x|} \int_R \psi\left(\frac{t}{x}\right) U_t f\, dt - \overline{\psi} E(\{0\}) f, g\right) = r(x).$$

Note that when ψ is the characteristic function of $[0, 1]$, Corollary 2 reduces to von Neumann's mean ergodic theorem (see [3, p. 389]), and when $\overline{\psi} = 0$, Corollary 2 says that the Hausdorff mean of U_{a+t} converges strongly to zero.

Corollary 2. *Let U_t be a one-parameter continuous group of unitary operators for $t \in R$. Then the Hausdorff mean of U_{a+t} with a continuous L^1-kernel converges strongly to a scalar multiple of a projection, and for $a = 0$ their difference tends weakly to zero at the rate of a Fourier transform of an L^1-function as in Corollary* 1(a).

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