PROJECTIONS IN KAC-MOODY LIE ALGEBRAS

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(Communicated by Maurice Auslander)

Abstract. Let g be a Kac-Moody Lie algebra and \( \mathcal{B} \) its set of positive Borel subalgebras. If \( b \in \mathcal{B} \) and \( p \) is a parabolic subalgebra, let \( \text{proj}_p(b) = p \cap b + r_n(p) \) where \( r_n(p) \) denotes the nilradical of \( p \). In this paper we consider the idempotent maps \( E_{p, p^-} = \text{proj}_p \circ \text{proj}_{p^-} : \mathcal{B} \to \mathcal{B} \), where \( p \) and \( p^- \) are opposite parabolic subalgebras with \( p \) being of positive type. We consider the semigroup \( M = M(g) \) generated (with respect to composition) by the maps \( E_{p, p^-} \). In particular we show that the maximal subgroups of \( M \) are closely related to proper Levi subgroups of the Kac-Moody group associated with \( g \).

Introduction

Projections in algebraic groups were introduced and used by Tits [6] in his theory of buildings. If \( G \) is a simple algebraic group, \( B \) a Borel subgroup of \( G \), and \( P \) a parabolic subgroup of \( G \), then \( \text{proj}_P(B) = (P \cap B)R_u(P) \) where \( R_u(P) \) denotes the unipotent radical of \( P \). In this way, we have an idempotent map \( \text{proj}_P : G/B \to G/B \). One of the authors [5] studied the monoid \( M \) (with respect to composition) generated by these projections. In particular, it was shown in [5] that the maximal subgroups of \( M \) are closely related to proper Levi subgroups of \( G \).

In this paper we study projections in a Kac-Moody Lie algebra. Let \( g \) be a Kac-Moody Lie algebra associated to an indecomposable symmetrizable generalized Cartan matrix and let \( G \) be the corresponding Kac-Moody group [2, 3, 4]. Let \( b_+ \) and \( b_- \) denote the standard positive and negative Borel subalgebras of \( g \). It was proved in [4] that any Borel subalgebra of \( g \) is \( \text{Ad}(G) \)-conjugate to \( b_+ \) or \( b_- \). We say a Borel subalgebra \( b \) of \( g \) is of positive type if it is conjugate to \( b_+ \) and that a parabolic subalgebra \( p \) of \( g \) is of positive type if it contains a Borel subalgebra of positive type. Let \( \mathcal{B} \) (resp. \( \mathcal{P} \)) denote the set of all Borel (resp. parabolic) subalgebras of \( g \) of positive type. The parabolic subalgebra \( p^- \) is said to be an opposite of the parabolic subalgebra \( p \) if \( p \cap p^- \) is a common Levi factor. For \( b \in \mathcal{B} \) and \( p \) any parabolic subalgebra, we define \( \text{proj}_p(b) = p \cap b + r_n(p) \), where \( r_n(p) \) denotes the nilradical of \( p \). In general \( \text{proj}_p(b) \) does not belong to \( \mathcal{B} \) (unless \( p \in \mathcal{P} \)). We consider the idempotent maps \( E_{p, p^-} : \mathcal{B} \to \mathcal{B} \), where \( E_{p, p^-} = \text{proj}_p \circ \text{proj}_{p^-} \) and \( p \in \mathcal{P} \), \( p^- \)
an opposite of \( p \). Let \( M = M(g) \) be the monoid generated by \( E_{p, p^-} \), \( p \in \mathcal{P} \), with respect to composition. Let \( 1 = p \cap p^- \), \( St_G(l) = \{ x \in G \mid \text{Ad} x(l) = l \} \), \( H = \{ x \in St_G(l) \mid \text{Ad} x(b) = b \text{ for all } b \in \mathcal{P}(l) \}) \), and \( L = St_G(l)/H \). In this paper it is shown (Theorem 2.8) that the maximal subgroup of \( M \) with identity element \( E_{p, p^-} \) is isomorphic to \( L \). Since the Dynkin diagram of any Kac-Moody Lie algebra can be extended by a single node, we see that a group closely related to any Kac-Moody group arises as a maximal subgroup of some \( M \).

1. Preliminaries

In this section we will recall some basic definitions and facts about Kac-Moody algebras and groups associated to a symmetrizable generalized Cartan matrix. For more details the reader is referred to [2, 3, 4].

Let \( A = (a_{ij})_{i,j=1}^n \) be an \( n \times n \) symmetrizable generalized Cartan matrix, i.e., an integral matrix satisfying \( a_{ii} = 2, a_{ij} \leq 0 \) for \( i \neq j \), \( a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \), and \( DA \) is symmetric for some nondegenerated diagonal matrix \( D \). We assume for simplicity that \( A \) is indecomposable. Let \( C \) denote the field of complex numbers and \( C^* \) denote the set of nonzero complex numbers. Consider the Kac-Moody algebra \( g = g(A) \) over \( C \) (see [2]) with generators \( e_i, f_i, h_i, i \in \mathbb{I} = \{1, 2, \ldots, n\} \).

The Lie algebra \( g \) admits a gradation \( g = \bigoplus_{\alpha \in Q} g_{\alpha} \) by the free abelian group \( Q \) on symbols \( \alpha_i, i \in \mathbb{I} \), which is called the root lattice, such that \( h = g_0 = \bigoplus_{i \in \mathbb{I}} C h_i \), \( g_{\alpha_i} = C e_i \), and \( g_{-\alpha_i} = C f_i \). Call \( h \) the standard Cartan subalgebra of \( g \). Let \( \Delta = \{ \alpha \in Q \mid g_\alpha \neq 0, \alpha \neq 0 \} \) be the set of roots of \( g(A) \). Define \( r_i \in \text{Aut}_C(h) \) \( i \in \mathbb{I} \) by \( r_i(h) = h - \alpha_i(h)h_i \) and take \( S = \{ r_i \mid i \in \mathbb{I} \} \). \( S \) generates the Weyl group \( W \), and \( (W, S) \) is a Coxeter system. Let \( \Gamma \) be the Coxeter graph for \( W \). Since \( A \) is indecomposable, \( \Gamma \) is connected. \( W \) preserves the root system \( \mathcal{A} \). A real (resp. imaginary) root is an element of \( \mathcal{A}^\text{re} := \{ w \cdot \alpha_i \mid w \in W, i \in \mathbb{I} \} \) (resp. \( \mathcal{A}^\text{im} := \mathcal{A}\setminus\mathcal{A}^\text{re} \)). Elements of \( \mathcal{A}^+ := Q^+ \cap \mathcal{A} \) are called positive roots and \( \mathcal{A}^- := \mathcal{A}\setminus\mathcal{A}^+ \) are called negative roots. Define \( n^\pm = \bigoplus_{\alpha \in \mathcal{A}^\pm} g_\alpha \). Then \( g = n^- \otimes h \oplus n^+ \) and \( b^\pm = h \oplus n^\pm \) are the standard Borel subalgebras (positive if \( '+' \), negative if \( '-' \)).

For \( t \in C^* \) and \( u \in C \), define the following elements of \( \text{SL}_2(C) \):

\[
h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad y(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.
\]

The following axioms (1.1)–(1.3) determine (uniquely up to isomorphism) a group \( G = G(A) \) and homomorphisms \( \phi_i : \text{SL}_2(C) \to G(A) \) for \( i \in \mathbb{I} \) [3]. Here and further on, \( \phi_i(h(t)) \), \( \phi_i(x(u)) \), and \( \phi_i(y(u)) \) are denoted by \( h_i(t) \), \( x_i(u) \), and \( y_i(u) \), respectively, for short.

1. There exists a faithful \( G \)-module \((V, \pi)\) over \( C \) such that each \( \text{SL}_2(C) \)-module \((V, \pi \circ \phi_i)\) is a direct sum of rational finite-dimensional submodules.

2. \( h_i(t)x_j(u)h_j(t)^{-1} = x_j(t^{a_{ij}}u) \) and \( h_i(t)y_j(u)h_j(t)^{-1} = y_j(t^{-a_{ij}}u) \) for all \( i, j \in \mathbb{I}, t \in C^*, \) and \( u \in C \).

3. If a group \( G' \) and homomorphisms \( \phi'_i : \text{SL}_2(C) \to G' \), \( i \in \mathbb{I} \), satisfy (1.2) and (1.3), then there exists a unique homomorphism \( \psi : G \to G' \) such that \( \phi'_i = \psi \circ \phi_i \) for all \( i \in \mathbb{I} \).

Put \( G_i = \phi_i(\text{SL}_2(C)) \), \( i \in \mathbb{I} \). It follows from the axioms that the subgroups
$G_i$, $i \in I$, generate the group $G$. Put $H_i = \{h_i(t)|t \in \mathbb{C}^*\}$, and let $H$ be the subgroup of $G$ generated by the subgroups $H_i$. Then $H$ is an abelian subgroup of $G$. Let $N_i$ be the normalizer of $H_i$ in $G_i$ and let $N$ be the subgroup of $G$ generated by the $N_i$'s. Then $H$ is an abelian normal subgroup of $N$ and the Weyl group $W$ is isomorphic to the quotient group $N/H$ [3, 4].

To the integrable $g$-module $(g, \text{ad})$ associate the homomorphism $\text{Ad}: G \to \text{Aut}_c(g)$ satisfying $\text{Ad}(\exp x) = \exp(\text{ad} x)$ for all $x \in g_\alpha$, $\alpha \in \Delta^e$. Then the kernel of $\text{Ad}$ is the center $C(G)$ of $G$. For $i \in I$ consider the one-parameter subgroups $U_{\alpha_i} = \{x_i(u)|u \in \mathbb{C}\}$, $U_{-\alpha_i} = \{y_i(u)|u \in \mathbb{C}\}$ of $G$. For a real root $\alpha = w\alpha_i$, take $n \in N$ such that $w = nh$ and put $U_{\pm \alpha} = nU_{\pm \alpha}n^{-1}$. Then $U_\alpha = \exp g_\alpha$ for all $\alpha \in \Delta^e$ [3]. Let $U_+$ (resp. $U_-$) be the subgroup of $G$ generated by the subgroups $U_{\alpha}$ (resp. $U_{-\alpha}$), $\alpha \in \Delta^e_+$. Define the standard Borel subgroups $B_\pm = HU_{\pm}$. Then we have [3, 4]

\begin{align}
(1.4) & \quad G = \bigcup_{w \in W} B_+wB_+ \quad \text{(Bruhat decomposition)}, \\
(1.5) & \quad G = \bigcup_{w \in W} B_-wB_+ \quad \text{(Birkhoff decomposition)}.
\end{align}

The following theorem is known [4, Theorems 2, 3].

**Theorem 1.1** [4]. (i) Every Cartan subalgebra of $g$ is $\text{Ad}(G)$-conjugate to $h$.

(ii) Every Borel subalgebra of $g$ is $\text{Ad}(G)$-conjugate to $b_+$ or to $b_-$.

(iii) Every Cartan subalgebra of $b_+$ is $\text{Ad}(U_+)$-conjugate to $h$.

The following in an immediate corollary of Theorem 1.1(ii) and equations (1.4), (1.5).

**Corollary 1.2.** If $p_1$, $p_2$ are any two parabolic subalgebras of $g$, then $p_1 \cap p_2$ contains a Cartan subalgebra of $g$. In particular, if $b_1$ and $b_2$ are any two Borel subalgebras of $g$, then $b_1 \cap b_2$ contains a Cartan subalgebra of $g$.

Let $b$ be any Borel subalgebra and $p$ any parabolic subalgebra of $g$. Define $B = \{g \in G|(\text{Ad} g)(b) = b\}$ and $P = \{g \in G|(\text{Ad} g)(p) = p\}$. Then $B$ is a Borel subgroup and $P$ is a parabolic subgroup of $G$. We define $B$ (resp. $P$) to be the Borel (resp. parabolic) subgroup associated with the Borel (resp. parabolic) subalgebra $b$ (resp. $p$) of $g$. A parabolic subgroup $P^-$ is said to be an opposite of the parabolic subgroup $P$ of $G$ if $L = P \cap P^-$ is a reductive group. Then we have a Levi decomposition $P = LU$ where $U = R_u(P)$ is the unipotent radical of $P$. For any parabolic subalgebra $p$ of $g$ and a Cartan subalgebra $c$, the Levi factor $l$ of $p$ is the subalgebra spanned by $c$ and all pairs of opposite root spaces with respect to $c$ occurring in $p$. Then we have the Levi decomposition $p = l \oplus u$ where $u = \text{rad}(p)$ is the nilradical of $p$. A parabolic subalgebra $p^-$ is said to be an opposite of the parabolic subalgebra $p$ of $g$ if $l = p \cap p^-$ is a Levi factor of $p$ (and $p^-$).

2. Main section

As in §1, let $g$ and $G$ denote the Kac-Moody algebra and Kac-Moody group, respectively, associated to a symmetrizable indecomposable Cartan matrix $A$. We say a Borel subalgebra $b$ of $g$ is of positive (resp. negative) type if $b$ is $\text{Ad} G$-conjugate to the standard positive (resp. negative) Borel subalgebra $b_+$.
A parabolic subalgebra \( p \) of \( g \) is of positive (resp. negative) type if \( p \) contains a Borel subalgebra of positive (resp. negative) type. Let \( \mathcal{B} = \mathcal{B}(g) \) denote the set of Borel subalgebras of \( g \) of positive type and \( \mathcal{P} = \mathcal{P}(g) \) denote the set of parabolic subalgebras of \( g \) of positive type. For any subalgebra \( \mathfrak{k} \) of \( g \), whenever defined we will write \( \mathcal{B}(\mathfrak{k}) \) and \( \mathcal{P}(\mathfrak{k}) \) to denote the set of Borel subalgebras and parabolic subalgebras, respectively, of positive type.

For any Borel subalgebra \( b \) and parabolic subalgebra \( p \) of \( g \) define
\[
\text{proj}_p(b) = p \cap b + r_n(p),
\]
where \( r_n(p) \) is the nilradical of \( p \). For \( b \in \mathcal{B} \) and \( p \in \mathcal{P} \) define
\[
E_{p, p^-} = \text{proj}_p \circ \text{proj}_{p^-},
\]
where \( p^- \) is an opposite of \( p \). Then \( E_{p, p^-}(b) \in \mathcal{B} \) for all \( b \in \mathcal{B}, \ E_p, g = 1 \), and \( E_{p, p^-} \circ E_{p, p^-} = E_{p, p^-} \). Let \( \mathcal{E} \) denote the set of idempotent maps \( E_{p, p^-} \), \( p \in \mathcal{P} \). Observe that if \( l = p \cap p^- \) and \( b \in \mathcal{B} \) have a common Cartan subalgebra then \( E_{p, p^-}(b) = \text{proj}_p(b) \). For a subalgebra \( \mathfrak{k} \subseteq g \) and \( x \in G \), we will write \( x^\mathfrak{k} \) to denote \( (\text{Ad}x)(\mathfrak{k}) \) and \( x^{\mathfrak{k}^{-1}} \) to denote \( (\text{Ad}x^{-1})(\mathfrak{k}) \). Then the next lemma follows immediately from the definitions.

**Lemma 2.1.** For \( x \in G \) and \( p \in \mathcal{P} \) we have (i) \( \text{Ad}x \circ E_{p, p^-} = E_{xp, xp^-} \circ \text{Ad}x \), and (ii) \( E_{p, p^-} \circ \text{Ad}x = \text{Ad}x \circ E_{xp, xp^-} \).

For two parabolic subalgebras \( p \) and \( q \) of \( g \), we define \( p \leftrightarrow q \) if and only if \( p \) and \( q \) have a common Levi factor. We say \( E_{p, p^-} \leftrightarrow E_{q, q^-} \) if \( p^- \leftrightarrow q^- \).

**Lemma 2.2.** Let \( p, q \in \mathcal{P} \) with respective opposites \( p^- \) and \( q^- \). Suppose \( p \cap p^- \cap q \cap q^- \) contains a Cartan subalgebra of \( g \). Then there exists \( p_1, q_1 \in \mathcal{P} \) with respective opposites \( p_1^- \) and \( q_1^- \) such that \( p_1 \subseteq p \), \( p_1^- \subseteq p^- \), \( q_1 \subseteq q \), \( q_1^- \subseteq q^- \), \( p_1^- \leftrightarrow q_1 \) (so \( E_{p_1, p_1^-} \leftrightarrow E_{q_1, q_1^-} \)), and \( E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \).

**Proof.** Let \( l_1 = p \cap p^- \), \( l_2 = q \cap q^- \), and \( l = l_1 \cap l_2 \). Then we have the Levi decompositions \( p = l_1 + u, \ p^- = l_1 + u^-, \ q = l_2 + v, \ q^- = l_2 + v^- \), where \( u = r_n(p), \ u^- = r_n(p^-), \ v = r_n(q), \ v^- = r_n(q^-) \). Let \( P, P^-, Q, Q^- \) denote the parabolic subgroups associated with the parabolic subalgebras \( p, p^-, q, q^- \), respectively. We have the Levi decomposition of the parabolic subgroups \( P = L_1U, \ P^- = L_1U^-, \ Q = L_2V, \ Q^- = L_2V^- \), where \( U = R_u(P), \ U^- = R_u(P^-), \ V = R_u(Q), \) and \( V^- = R_u(Q^-) \). Consider the parabolic subalgebras
\[
\begin{align*}
p_1 &= (p \cap q) + u = l + (l_1 \cap v) + u, \\
p_1^- &= (p^- \cap q^-) + u^- = l + (l_1 \cap v^-) + u^-, \\
q_1 &= (p \cap q) + v = l + (l_2 \cap u) + v, \\
q_1^- &= (p^- \cap q^-) + v^- = l + (l_2 \cap u^-) + v^-.
\end{align*}
\]
Observe that \( p_1^- \) and \( q_1 \) have \( l \) as common Levi factor. Let \( b \in \mathcal{B} \). By Corollary 1.2, \( b \cap q_1 \) contains a Cartan subalgebra \( \mathfrak{h} \), say. There exists \( x \in L_2 \cap U^- \), \( y \in V^- \) such that \( xyh \subseteq l \). Therefore, \( \mathfrak{h} \subseteq l_2 \). So \( E_{q_1, q_1^-}(b) = (l_2 \cap yb) + v \). Also \( xyh \subseteq l \), \( x \in L_2 \cap U^- \) implies that \( yh \subseteq p^- \). Hence
\[
E_{p, p^-} \circ E_{q, q^-}(b) = (l \cap xyb) + (l_1 \cap xvb) + u = (l \cap xyb) + (l_1 \cap v) + u.
\]
Now \( v = xv \) as \( x \in L_2 \subseteq Q \). But
\[
E_{q_1, q_1^-}(b) = (l \cap xyb) + (l_2 \cap u) + v,
\]
so

\[ E_{p_1, p_1^-} \circ E_{q_1, q_1^-}(b) = (I \cap x^y b) + (I_1 \cap v) + u = E_{p, p^-} \circ E_{q, q^-}(b), \]

which proves the lemma. \( \square \)

**Corollary 2.3.** Let \( p, q \in \mathcal{P} \) with respective opposites \( p^- \) and \( q^- \). Then there exists \( p_1, q_1 \in \mathcal{P} \) with respective opposites \( p_1^- \) and \( q_1^- \) such that \( p_1^- \leftrightarrow q_1 \) (so \( E_{p_1, p_1^-} \leftrightarrow E_{q_1, q_1^-} \)), \( p_1 \subseteq p \), \( p_1^- \subseteq p^- \), \( q_1 \subseteq q \), \( q_1^- \subseteq q^- \), and \( E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \).

**Proof.** Let \( l_1 = p \cap p^- \), \( l_2 = q \cap q^- \). By Corollary 1.2, \( p^- \cap q \) contains a Cartan subalgebra \( h \), say. Let \( P, P^- \), \( Q, Q^- \) denote the parabolic subgroups associated with \( p, p^- \), \( q, q^- \), respectively. Let \( U = Ru(P) \), \( U^- = Ru(P^-) \), \( V = Ru(Q) \), and \( V^- = Ru(Q^-) \). Then there exists \( x \in U^- \) and \( y \in V \) such that \( h \subseteq l_1 \) and \( h \subseteq l_2 \). Then using Lemma 2.1,

\[ E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ Ad x \circ Ad y \circ E_{q, q^-} = Ad x \circ E_{p, p^-} \circ E_{q, q^-} \circ Ad y \]

since \( p^- x = p^- \) and \( y q = q \). By Lemma 2.2, since \( h \subseteq \langle p \rangle \cap \langle p^- \rangle \cap \langle q \rangle \cap \langle q^- \rangle \), we can find \( p_1, q_1 \in \mathcal{P} \) with opposites \( p_1^- \), \( q_1^- \), respectively, such that \( p_1 \subseteq \langle p \rangle \), \( p_1^- \subseteq \langle p^- \rangle \), \( q_1 \subseteq \langle q \rangle \), \( q_1^- \subseteq \langle q^- \rangle \), \( p_1^- \leftrightarrow q_1 \), and \( E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \). Hence using Lemma 2.1,

\[ E_{p, p^-} \circ E_{q, q^-} = Ad x \circ E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \circ Ad y = E_{x p_1, p_1^-} \circ Ad x \circ Ad y \circ E_{q_1, q_1^-} \circ E_{p_1, p_1^-} \circ E_{q_1, q_1^-}, \]

since \( x \in U^- \), \( y \in V \), so that \( x p_1^- = p_1^- \) and \( q_1^\gamma = q_1 \). Hence the corollary follows. \( \square \)

**Corollary 2.4.** Let \( p, q \in \mathcal{P} \) with respective opposites \( p^- \) and \( q^- \). Suppose \( p \subseteq q \) and \( p^- \subseteq q^- \). Then \( E_{p, p^-} \circ E_{q, q^-} = E_{p, p^-} \circ E_{q, q^-} \circ E_{p, p^-} \).

**Proof.** By Lemma 2.2, we have

\[ E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \]

where \( p_1 = p \cap q + \tau_n(p) = p \), \( p_1^- = p^- \cap q^- + \tau_n(p^-) = p^- \), \( q_1 = p \cap q + \tau_n(q) = p \), and \( q_1^- = p^- \cap q^- + \tau_n(q^-) = p^- \). Hence \( E_{p, p^-} \circ E_{q, q^-} = E_{p, p^-} \circ E_{q, q^-} \circ E_{p, p^-} \). Similarly, \( E_{q, q^-} \circ E_{p, p^-} = E_{p, p^-} \). \( \square \)

Let \( M \) denote the monoid generated by the idempotent maps \( E_{p, p^-} \in \mathcal{P} \) with respect to composition.

**Proposition 2.5.** Let \( p \in \mathcal{P} \) be a maximal parabolic subalgebra of \( \mathfrak{g} \) with opposite \( p^- \). Let \( P \) and \( P^- \) be the parabolic subgroups of \( G \) associated with \( p \) and \( p^- \), respectively, and \( L = P \cap P^- \). Let \( \mathcal{A}_1 \{ x \in G | E_{p, p^-} \circ Ad x \circ E_{p, p^-} \in M \} \). Then \( L \subseteq \mathcal{A}_1 \).

**Proof.** Without loss of generality, we may assume that \( P = P_J \) for some \( J = I \setminus \{i\} \), \( 1 \leq i \leq n \). Let \( L = L_J \), \( P^- = P_J^- \), \( P = LU \), where \( U = U_J = Ru(P_J) \) and \( P^- = LU^- \), where \( U^- = U_J^- = Ru(P_J^-) \). Let \( u \in U \), \( v \in U^- \). Then by
Lemma 2.1, we have

\[ \begin{align*}
E_{p,p^-} \circ \text{Ad } u \circ \text{Ad } v \circ E_{p,p^-} &= \text{Ad } u \circ E_{p,p^-} \circ \text{Ad } v \circ E_{p,p^-} \quad \text{since } p^u = p, \quad v p^- = p^- \\
&= E_{p,p^-} \circ E_{p,p^-} \circ \text{Ad } v \quad \text{since } u \in U, \quad v \in U^-.
\end{align*} \]

Hence \( U U^- \subseteq \mathfrak{A}_1 \). Again, since \( E_{p,p^-} \circ \text{Ad } v = E_{p,p^-} \) as \( v \in U^- \) and \( \text{Ad } u \circ E_{p,p^-} = E_{p,p^-} \) as \( u \in U \), it follows that \( U^- U U^- U \subseteq \mathfrak{A}_1 \). Let \( \sigma = \alpha_i \) and \( U_\sigma, U_{-\sigma} \) be the two root subgroups corresponding to \( \sigma \) and \( -\sigma \), respectively. Consider the group \( G_\sigma = \langle U_\sigma, U_{-\sigma} \rangle \) that is isomorphic to \( SL(2, \mathbb{C}) \). One then observes by direct computation that \( G_\sigma = U_\sigma U_{-\sigma} U_{-\sigma} U_{-\sigma} \). Since \( \sigma \notin \{\alpha_j | j \in J\} \) (see [1]) we have \( U_\sigma \subseteq U \) and \( U_{-\sigma} \subseteq U^- \). Thus \( G_\sigma \subseteq U^- U U^- U \subseteq \mathfrak{A}_1 \). Hence the maximal torus \( T_\sigma \) of \( G_\sigma \) is contained in \( \mathfrak{A}_1 \).

Now let \( \sigma_1 \) and \( \sigma_2 \) be two simple roots corresponding to two adjacent nodes in the Coxeter graph \( \Gamma \) of \( G \) and suppose that \( G_{\sigma_1} \subseteq \mathfrak{A}_1 \). Then \( T_{\sigma_1} \subseteq L_{\sigma_2} \cap \mathfrak{A}_1 \). Let \( U_{\sigma_1} = C(L_{\sigma_2}) \). But since \( \sigma_1 \) and \( \sigma_2 \) are adjacent, \( T_{\sigma_1} \notin C(L_{\sigma_2}) \). Hence \( G_{\sigma_1} \subseteq L_{\sigma_2} \cap \mathfrak{A}_1 \), so \( G_{\sigma_2} \subseteq \mathfrak{A}_1 \). Clearly, \( C(L) \subseteq \mathfrak{A}_1 \). Since the generalized Cartan matrix \( A = (a_{ij}) \) is indecomposable, \( \Gamma \) is connected. Hence \( C(L) \subseteq \mathfrak{A}_1 \). Since \( \sigma \in \{\alpha_i | i \in I\} \). Since \( L \) is generated by \( C(L) \) and \( G_\sigma, \sigma \in \{\alpha_i | i \in I\} \), it follows that \( L \subseteq \mathfrak{A}_1 \).

**Proposition 2.6.** Let \( p \in \mathcal{P} \), \( p \neq g \), with opposite \( p^- \). Let \( \mathfrak{A} = \{x \in G | E_{p,p^-} \circ \text{Ad } x \in M\} \). Then \( \mathfrak{A} = G \).

**Proof.** It is enough to prove the result for \( p \) maximal. For if \( p_1 \in \mathcal{P} \), \( p_1 \subseteq p \), \( p^- \subseteq p^- \), then by Corollary 2.4,

\[ E_{p_1,p^-} \circ \text{Ad } x = E_{p_1,p^-} \circ (E_{p,p^-} \circ \text{Ad } x) \in M \]

if \( E_{p,p^-} \circ \text{Ad } x \in M \). Let \( P, P^- \) denote the parabolic subgroups associated with \( p \) and \( p^- \), respectively. Let \( L = P \cap P^- \), \( P = LU \), \( P^- = LU^- \), where \( U = R_u(P) \) and \( U^- = R_u(P^-) \). Let \( u \in U \), \( v \in U^- \). Then by Lemma 2.1,

\[ E_{p,p^-} \circ \text{Ad } u \circ \text{Ad } v = E_{p,p^-} \circ E_{p,p^-} \circ \text{Ad } u \circ \text{Ad } v \]

\[ = E_{p,p^-} \circ (\text{Ad } u \circ E_{p,p^-}) \circ \text{Ad } v \quad \text{since } p^u = p, \]

\[ = E_{p,p^-} \circ E_{p,p^-} \circ \text{Ad } v \quad \text{since } u \in U, \]

\[ = (E_{p,p^-} \circ \text{Ad } v) \circ E_{p,p^-} \]

\[ = E_{p,p^-} \circ E_{p,p^-} \in M \quad \text{since } v \in U^- \]

Hence \( U U^- \subseteq \mathfrak{A} \). Similarly, since \( E_{p,p^-} \circ \text{Ad } v = E_{p,p^-} \) for \( v \in U^- \), it follows that \( U^- U U^- \subseteq \mathfrak{A} \).

Since \( p \) is maximal, by Proposition 2.5 we have \( L \subseteq \mathfrak{A}_1 \). But for \( x \in L \), we have

\[ E_{p,p^-} \circ \text{Ad } x \circ E_{p,p^-} = E_{p,p^-} \circ E_{p,p^-} \circ \text{Ad } x = E_{p,p^-} \circ \text{Ad } x. \]

Hence it follows that \( L \subseteq \mathfrak{A} \). Now since \( (L \cap \mathfrak{A}) \mathfrak{A} \subseteq \mathfrak{A} \), we have \( L U^- U U^- \subseteq \mathfrak{A} \). Hence \( G \subseteq \mathfrak{A} \). So \( \mathfrak{A} = G \). □

**Theorem 2.7.** \( M = \{1\} \cup \{E \circ \text{Ad } x \circ E \circ E' | E, E' \in \mathcal{G} \setminus \{1\}, \ x \in G\} \).

**Proof.** Let \( N = \{1\} \cup \{E \circ \text{Ad } x \circ E \circ E' | E, E' \in \mathcal{G} \setminus \{1\}, \ x \in G\} \). By Proposition 2.6, we have \( N \subseteq M \). By Corollary 2.3, we have

\[ M = \{E_1 \circ E_2 \circ \cdots \circ E_k | E_1 \leftrightarrow E_2 \leftrightarrow \cdots \leftrightarrow E_k, \ E_i \in \mathcal{G}\}. \]
Let $E_i = E_{p_i, p_i^-}$. Consider $E_1 \circ E_2 \circ E_3$ with $E_1 \leftrightarrow E_2 \leftrightarrow E_3$. Then $p_1^- \leftrightarrow p_2$ and $p_2^- \leftrightarrow p_3$. Let $l_1 \subseteq p_1^- \cap p_2$, $l_2 = p_2 \cap p_2^-$, and $l_3 \subseteq p_2^- \cap p_3$ be the common Levi factors. Let $P_2$ and $P_2^-$ denote the parabolic subgroups associated with $p_2$ and $p_2^-$, respectively. Let $L_2 = L_2 \cap P_2^-$, $P_2 = L_2 U_2$, and $P_2^- = L_2 U_2^-$, where $U_2 = R_u(P_2)$ and $U_2^- = R_u(P_2^-)$. Then there exists $u \in U_2$ and $v \in U_2^-$ such that $l_1 = u l_2$ and $l_2 = v l_3$. Let $b \in \mathcal{B}$ and $b_1 = E_{p_3, p_3^-}(b)$. Then

$$E_{p_1, p_1^-} \circ E_{p_2, p_2^-}(b_1) = E_{p_1, p_1^-} \circ \text{proj}_{p_2^\perp}(b_1)$$

$$= E_{p_1, p_1^-} \circ \text{proj}_{p_2^\perp}(l_2 \cap v l_1 + r_n(p_2^-))$$

$$= E_{p_1, p_1^-}(l_1 \cap w v b_1 + r_n(p_2))$$

$$= (E_{p_1, p_1^-} \circ \text{Ad}(uv))(b_1).$$

Hence

$$E_{p_1, p_1^-} \circ E_{p_2, p_2^-} \circ E_{p_3, p_3^-} = E_{p_1, p_1^-} \circ \text{Ad}(uv) \circ E_{p_3, p_3^-}.$$

Also note that $l_1 \subseteq p_1^- \cap w v p_3$, hence $p_1^- \leftrightarrow w v p_3$. Assume that $E_1 \leftrightarrow E_2 \dots \leftrightarrow E_{k-1}$ and $E_1 \circ E_2 \circ \cdots \circ E_{k-1} = E_1 \circ \text{Ad} x \circ E_{k-1}$, with $p_1^- \leftrightarrow x p_{k-1}$, hence $p_1^- \perp \leftrightarrow p_{k-1}$. Consider $E_1 \circ E_2 \circ \cdots \circ E_k$ with $E_1 \leftrightarrow E_2 \leftrightarrow \cdots \leftrightarrow E_k$. Then using Lemma 2.1,

$$E_1 \circ E_2 \circ \cdots \circ E_{k-1} \circ E_k = E_1 \circ \text{Ad} x \circ E_{k-1} \circ E_k$$

$$= E_{p_1, p_1^-} \circ \text{Ad} x \circ E_{p_{k-1}, p_{k-1}^-} \circ E_{p_k, p_k^-}$$

$$= \text{Ad} x \circ E_{p_1, p_1^-} \circ E_{p_{k-1}, p_{k-1}^-} \circ E_{p_k, p_k^-}$$

$$= \text{Ad} x \circ E_{p_1, p_1^-} \circ \text{Ad} y \circ E_{p_k, p_k^-} \quad \text{since } E_{p_1, p_1^-} \leftrightarrow E_{p_{k-1}, p_{k-1}^-} \leftrightarrow E_{p_k, p_k^-}$$

$$= E_{p_1, p_1^-} \circ \text{Ad}(xy) \circ E_{p_k, p_k^-} \quad \text{with } p_1^- \leftrightarrow x y p_k.$$

Hence by induction $E_1 \circ E_2 \circ \cdots \circ E_k = E_1 \circ \text{Ad} x \circ E_k$, $x \in G$, for all $k$. This proves that $M \subseteq N$. Hence $M = N$. \hfill \Box

Fix $p \in \mathcal{P}$ with opposite $p^-$. Let $E = E_{p, p^-}$ and $l = p \cap p^-$. Let $P$ and $P^-$ denote the parabolic subgroups associated with $p$ and $p^-$, respectively, and $L = P \cap P^-$. Let $\text{St}_G(l)$ denote the stabilizer of $l$ in $G$ and let $H = \{x \in \text{St}_G(l) \mid b = x b \text{ for all } b \in \mathcal{B}(l)\}$. Let $\tilde{L} = \text{St}_G(l)/H$. Then we have the following

**Theorem 2.8.** The maximal subgroup of $M$ with identity element $E = E_{p, p^-}$ is $\{E \circ \text{Ad} x \circ E \mid x \in \text{St}_G(l)\}$ which is isomorphic to $\tilde{L}$ as above.

**Proof.** The maximal subgroup of $M$ with identity element $E = E_{p, p^-}$ is the same as the group of units $K$ say, of $EME$. Let $a \in K$. Then as in the proof of Theorem 2.7, we have $a = E_{q, q^-} \circ \text{Ad} y \circ E_{q_1, q_1^-}$ for some $y \in G$, where $q \subseteq p$, $q^- \subseteq p^-$, $q_1 \subseteq p$, $q_1^- \subseteq p^-$, and $q^- \leftrightarrow y q_1$. Since $a \in K$ is invertible and $E$ is the identity in $K$, there exists $b \in K$ such that $a \circ b = E$ and $b \circ a = E$. Hence, by Corollary 2.4,

$$E_{q, q^-} = E_{q, q^-} \circ E = E_{q, q^-} \circ a \circ b = a \circ b = E.$$

Similarly,

$$E_{q_1, q_1^-} = E \circ E_{q_1, q_1^-} = b \circ a \circ E_{q_1, q_1^-} = b \circ a = E.$$
Hence \( a = E \circ \text{Ad} y \circ E \) for some \( y \in G \), and \( p^- \leftrightarrow \gamma p \). Let \( I_1 = p^- \cap \gamma p \). Let \( U = R_y(P) \) and \( U^- = R_y(P^-) \). Then there exists \( u \in U \) and \( v \in U^- \) such that \( I_1 = v I \) and \( I_1^v = v I \), hence \( y u I = I_1 \). Therefore, \( x I = I \), where \( x = v^{-1} y u \) so \( y = v x u^{-1} \). Hence \( x \in St_G(I) \) and

\[
a = E \circ \text{Ad} y \circ E = (E_{p, p^-} \circ \text{Ad} v) \circ \text{Ad} x \circ (\text{Ad} u^{-1} \circ E_{p, p^-})
\]

\[
= E_{p, p^-} \circ \text{Ad} x \circ E_{p, p^-} = E \circ \text{Ad} x \circ E,
\]

since \( v \in U^- \) and \( u^{-1} \in U \). Observe that \( I \subseteq p \cap p^{-x} \cap \gamma p^- \). Now define the map \( \theta : St_G(I) \to K \), by \( \theta(x) = E \circ \text{Ad} x \circ E \). By the above discussion \( \theta \) is onto. Furthermore, for \( x, y \in St_G(I) \).

\[
\theta(x) \circ \theta(y) = E \circ \text{Ad} x \circ E \circ E \circ \text{Ad} y \circ E = E_{p, p^-} \circ \text{Ad} x \circ E_{p, p^-} \circ \text{Ad} y \circ E_{p, p^-} = \text{Ad}(xy) \circ E_{p, x, p^-} \circ E_{p, p^-} \circ E_{p, x, p^-} \circ E_{p, p^-} \quad \text{by Lemma 2.1},
\]

\[
= \text{Ad}(xy) \circ E_{p, x, p^-} \circ E_{p, p^-} \quad \text{by Lemma 2.1}
\]

Hence \( \theta \) is a surjective homomorphism. Clearly, \( \text{Ker} \theta = H \). Hence \( K \cong St_G(I)/H = \tilde{L} \). □

We note that \( \tilde{L} \) is closely related to the Levi subgroup \( L \) of \( G \). We therefore see that the monoid \( M \) gives rise to groups closely related to Kac-Moody groups.

3. Examples

When \( g \) is finite dimensional, \( M \) is exactly the monoid generated by \( \text{proj}_p (p \in \mathcal{P}) \) and is isomorphic to the monoid studied in [5, Theorem 2] for the associated simple algebraic group. The situation is quite different when \( g \) is infinite dimensional. For example, consider the affine Kac-Moody Lie algebras \( g = A_1^{(1)} \) with Dynkin diagram

\[
\text{(3.1)}
\]

Then \( g \) has four standard positive parabolics, but the set \( \mathcal{P} \) of all positive parabolics is of course infinite. The monoid generated by \( \text{proj}_p \ (p \in \mathcal{P}) \) is not that interesting in that the subgroups of this monoid are trivial. We, therefore, are naturally led to considering our monoid \( M \) generated by \( E_{p, p^-} = \text{proj}_p \circ \text{proj}_{p^-} \). The only nontrivial subgroups of \( M \) are those corresponding to parabolic subalgebras \( p \) obtained by deleting a node of the diagram (3.1). The associated Levi factor \( I \) is a finite-dimensional Lie algebra with the simple part being \( \mathfrak{sl}_2(\mathbb{C}) \). The subgroup of \( M \) with identity element \( E_{p, p^-} \) is just the projective group \( \text{PGL}(2, \mathbb{C}) \).
To get finite-dimensional Kac-Moody groups, we consider the hyperbolic Kac-Moody Lie algebra $g_1$ with Dynkin diagram

(3.2)

Let $M_1 = M(g_1)$ denote the corresponding monoid. Let $p$ denote the parabolic subalgebra obtained by deleting the right-most node in (3.2). Then the maximal subgroup of $M_1$ with identity $E_{p,p^{-}}$ is 'almost' the Kac-Moody group corresponding to (3.1).

REFERENCES


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