PROJECTIONS IN KAC-MOODY LIE ALGEBRAS

KAILASH C. MISRA AND MOHAN S. PUTCHA

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Abstract. Let \( g \) be a Kac-Moody Lie algebra and \( \mathcal{B} \) its set of positive Borel subalgebras. If \( b \in \mathcal{B} \) and \( p \) is a parabolic subalgebra, let \( \text{proj}_p(b) = p \cap b + r_n(p) \) where \( r_n(p) \) denotes the nilradical of \( p \). In this paper we consider the idempotent maps \( E_{p,p^-} = \text{proj}_p \circ \text{proj}_{p^-} : \mathcal{B} \to \mathcal{B} \), where \( p \) and \( p^- \) are opposite parabolic subalgebras with \( p \) being of positive type. We consider the semigroup \( M = M(g) \) generated (with respect to composition) by the maps \( E_{p,p^-} \). In particular we show that the maximal subgroups of \( M \) are closely related to proper Levi subgroups of the Kac-Moody group associated with \( g \).

Introduction

Projections in algebraic groups were introduced and used by Tits [6] in his theory of buildings. If \( G \) is a simple algebraic group, \( B \) a Borel subgroup of \( G \), and \( P \) a parabolic subgroup of \( G \), then \( \text{proj}_P(B) = (P \cap B)RU(P) \) where \( RU(P) \) denotes the unipotent radical of \( P \). In this way, we have an idempotent map \( \text{proj}_P : G/B \to G/B \). One of the authors [5] studied the monoid \( M \) (with respect to composition) generated by these projections. In particular, it was shown in [5] that the maximal subgroups of \( M \) are closely related to proper Levi subgroups of \( G \).

In this paper we study projections in a Kac-Moody Lie algebra. Let \( g \) be a Kac-Moody Lie algebra associated to an indecomposable symmetrizable generalized Cartan matrix and let \( G \) be the corresponding Kac-Moody group [2, 3, 4]. Let \( b_+ \) and \( b_- \) denote the standard positive and negative Borel subalgebras of \( g \). It was proved in [4] that any Borel subalgebra of \( g \) is \( \text{Ad} G \)-conjugate to \( b_+ \) or \( b_- \). We say a Borel subalgebra \( b \) of \( g \) is of positive type if it is conjugate to \( b_+ \) and that a parabolic subalgebra \( p \) of \( g \) is of positive type if it contains a Borel subalgebra of positive type. Let \( \mathcal{B} \) (resp. \( \mathcal{P} \)) denote the set of all Borel (resp. parabolic) subalgebras of \( g \) of positive type. The parabolic subalgebra \( p^- \) is said to be an opposite of the parabolic subalgebra \( p \) if \( p \cap p^- \) is a common Levi factor. For \( b \in \mathcal{B} \) and \( p \) any parabolic subalgebra, we define \( \text{proj}_p(b) = p \cap b + r_n(p) \), where \( r_n(p) \) denotes the nilradical of \( p \). In general \( \text{proj}_p(b) \) does not belong to \( \mathcal{B} \) (unless \( p \in \mathcal{P} \)). We consider the idempotent maps \( E_{p,p^-} : \mathcal{B} \to \mathcal{B} \), where \( E_{p,p^-} = \text{proj}_p \circ \text{proj}_{p^-} \) and \( p \in \mathcal{P}, p^- \nexists \mathcal{B} \).
an opposite of \( p \). Let \( M = M(g) \) be the monoid generated by \( E_{p, p^-}, p \in \mathcal{R} \), with respect to composition. Let \( I = p \cap p^- \), \( \text{St}_G(I) = \{ x \in G | \text{Ad} x(l) = I \} \), \( H = \{ x \in \text{St}_G(I) | \text{Ad} x(b) = b, \text{ for all } b \in \mathcal{R}(I) \} \), and \( \tilde{L} = \text{St}_G(I)/H \). In this paper it is shown (Theorem 2.8) that the maximal subgroup of \( M \) with identity element \( E_{p, p^-} \) is isomorphic to \( \tilde{L} \). Since the Dynkin diagram of any Kac-Moody Lie algebra can be extended by a single node, we see that a group closely related to any Kac-Moody group arises as a maximal subgroup of some \( M \).

1. Preliminaries

In this section we will recall some basic definitions and facts about Kac-Moody algebras and groups associated to a symmetrizable generalized Cartan matrix. For more details the reader is referred to [2, 3, 4].

Let \( A = (a_{ij})_{i,j=1}^{n} \) be an \( n \times n \) symmetrizable generalized Cartan matrix, i.e., an integral matrix satisfying \( a_{ii} = 2, a_{ij} \leq 0 \) for \( i \neq j \), \( a_{ij} = 0 \) \( \Leftrightarrow a_{ji} = 0 \), and \( DA \) is symmetric for some nondegenerated diagonal matrix \( D \). We assume for simplicity that \( A \) is indecomposable. Let \( C \) denote the field of complex numbers and \( C^* \) denote the set of nonzero complex numbers. Consider the Kac-Moody algebra \( g = g(A) \) over \( C \) (see [2]) with generators \( e_i, f_i, h_i, i \in I = \{1, 2, \ldots, n\} \).

The Lie algebra \( g \) admits a gradation \( g = \bigoplus_{\alpha \in Q} g_{\alpha} \) by the free abelian group \( Q \) on symbols \( \alpha_i, i \in I \), which is called the root lattice, such that \( h = g_0 = \bigoplus_{i \in I} C h_i, g_{\alpha_i} = C e_i \), and \( g_{-\alpha_i} = C f_i \). Call \( h \) the standard Cartan subalgebra of \( g \). Let \( \Delta = \{ \alpha \in Q | g_\alpha \neq 0, \alpha \neq 0 \} \) be the set of roots of \( g(A) \). Define \( r_i \in \text{Aut}_C(h) i \in I \) by \( r_i(h) = h - \alpha_i(h) h_i \) and take \( S = \{ r_i | i \in I \} \). \( S \) generates the Weyl group \( W \), and \( (W, S) \) is a Coxeter system. Let \( \Gamma \) be the Coxeter graph for \( W \). Since \( A \) is indecomposable, \( \Gamma \) is connected. \( W \) preserves the root system \( \Delta \). A real (resp. imaginary) root is an element of \( \Delta^r := \{ w \cdot \alpha_i | w \in W, i \in I \} \) (resp. \( \Delta^i := \Delta \backslash \Delta^r \)). Elements of \( \Delta^+ := Q^+ \cap \Delta \) are called positive roots and \( \Delta^- := \Delta \backslash \Delta^+ \) are called negative roots. Define \( n^\pm = \bigoplus_{\alpha \in \Delta^\pm} g_\alpha \). Then \( g = n^- \oplus h \oplus n^+ \) and \( b^\pm = h \oplus n^\pm \) are the standard Borel subalgebras (positive if '+', negative if '−').

For \( t \in C^* \) and \( u \in C \), define the following elements of \( \text{SL}_2(C) \):

\[
h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad y(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.
\]

The following axioms (1.1)–(1.3) determine (uniquely up to isomorphism) a group \( G = G(A) \) and homomorphisms \( \phi_i: \text{SL}_2(C) \to G(A) \) for \( i \in I \) [3]. Here and further on, \( \phi_i(h(t)), \phi_i(x(u)), \) and \( \phi_i(y(u)) \) are denoted by \( h_i(t), \) \( x_i(u), \) and \( y_i(u), \) respectively, for short.

(1.1) There exists a faithful \( G \)-module \((V, \pi)\) over \( C \) such that each \( \text{SL}_2(C) \)-module \((V, \pi \circ \phi_i)\) is a direct sum of rational finite-dimensional submodules.

(1.2) \( h_i(t)x_j(u)h_i(t)^{-1} = x_j(t^{a_{ji}}u) \) and \( h_i(t)y_j(u)h_i(t)^{-1} = y_j(t^{-a_{ji}}u) \) for all \( i, j \in I, t \in C^* \), and \( u \in C \).

(1.3) If a group \( G' \) and homomorphisms \( \phi_i': \text{SL}_2(C) \to G' \), \( i \in I \), satisfy (1.2) and (1.3), then there exists a unique homomorphism \( \psi: G \to G' \) such that \( \phi_i' = \psi \circ \phi_i \) for all \( i \in I \).

Put \( G_i = \phi_i(\text{SL}_2(C)) \), \( i \in I \). It follows from the axioms that the subgroups
$G_i, \ i \in I,$ generate the group $G$. Put $H_i = \{h_i(t) | t \in \mathbb{C}^*\},$ and let $H$ be the subgroup of $G$ generated by the subgroups $H_i$. Then $H$ is an abelian subgroup of $G$. Let $N_i$ be the normalizer of $H_i$ in $G_i$ and let $N$ be the subgroup of $G$ generated by the $N_i$'s. Then $H$ is an abelian normal subgroup of $N$ and the Weyl group $W$ is isomorphic to the quotient group $N/H$ [3, 4].

To the integrable $g$-module $(g, \text{ad})$ associate the homomorphism $\text{Ad} : G \rightarrow \text{Aut}_\mathbb{C}(g)$ satisfying $\text{Ad}(\exp x) = \exp(\text{ad} x)$ for all $x \in g_\alpha, \ \alpha \in \Delta^\text{re}$. Then the kernel of $\text{Ad}$ is the center $C(G)$ of $G$. For $i \in I$ consider the one-parameter subgroups $U_{\alpha_i} = \{x_i(u) | u \in \mathbb{C}\}, \ U_{-\alpha_i} = \{y_i(u) | u \in \mathbb{C}\}$ of $G$. For a real root $\alpha = w\alpha_i$, take $n \in N$ such that $w = nH$ and put $U_{\pm \alpha} = nU_{\pm \alpha_i}n^{-1}$. Then $U_\alpha = \exp g_\alpha$ for all $\alpha \in \Delta^\text{re}$ [3]. Let $U_+$ (resp. $U_-$) be the subgroup of $G$ generated by the subgroups $U_\alpha$ (resp. $U_{-\alpha}$), $\alpha \in \Delta^+_\text{re}$. Define the standard Borel subgroups $B_{\pm} = HU_{\pm}$. Then we have [3, 4]

(1.4) $G = \bigcup_{w \in W} B_+ w B_+$ (Bruhat decomposition),

(1.5) $G = \bigcup_{w \in W} B_- w B_+$ (Birkhoff decomposition).

The following theorem is known [4, Theorems 2, 3].

**Theorem 1.1** [4]. (i) Every Cartan subalgebra of $g$ is $\text{Ad}(G)$-conjugate to $h_i$.

(ii) Every Borel subalgebra of $g$ is $\text{Ad}(G)$-conjugate to $b_+$ or to $b_-$.

(iii) Every Cartan subalgebra of $b_+$ is $\text{Ad}(U_+)$-conjugate to $h_i$.

The following is an immediate corollary of Theorem 1.1(ii) and equations (1.4), (1.5).

**Corollary 1.2.** If $p_1, p_2$ are any two parabolic subalgebras of $g$, then $p_1 \cap p_2$ contains a Cartan subalgebra of $g$. In particular, if $b_1$ and $b_2$ are any two Borel subalgebras of $g$, then $b_1 \cap b_2$ contains a Cartan subalgebra of $g$.

Let $b$ be any Borel subalgebra and $p$ any parabolic subalgebra of $g$. Define $B = \{g \in G | (\text{Ad} g)(b) = b\}$ and $P = \{g \in G | (\text{Ad} g)(p) = p\}$. Then $B$ is a Borel subgroup and $P$ is a parabolic subgroup of $G$. We define $B$ (resp. $P$) to be the Borel (resp. parabolic) subgroup associated with the Borel (resp. parabolic) subalgebra $b$ (resp. $p$) of $g$. A parabolic subgroup $P^-$ is said to be an opposite of the parabolic subgroup $P$ of $G$ if $L = P \cap P^-$ is a reductive group. Then we have a Levi decomposition $P = LU$ where $U = R_u(P)$ is the unipotent radical of $P$. For any parabolic subalgebra $p$ of $g$ and a Cartan subalgebra $c$, the Levi factor $l$ of $p$ is the subalgebra spanned by $c$ and all pairs of opposite root spaces with respect to $c$ occurring in $p$. Then we have the Levi decomposition $p = l \oplus u$ where $u = r_n(p)$ is the nilradical of $p$. A parabolic subalgebra $p^-$ is said to be an opposite of the parabolic subalgebra $p$ of $g$ if $l = p \cap p^-$ is a Levi factor of $p$ (and $p^-$).

2. Main section

As in §1, let $g$ and $G$ denote the Kac-Moody algebra and Kac-Moody group, respectively, associated to a symmetrizable indecomposable Cartan matrix $A$. We say a Borel subalgebra $b$ of $g$ is of positive (resp. negative) type if $b$ is $\text{Ad} G$-conjugate to the standard positive (resp. negative) Borel subalgebra $b_+$.
A parabolic subalgebra $p$ of $g$ is of positive (resp. negative) type if $p$ contains a Borel subalgebra of positive (resp. negative) type. Let $\mathcal{B} = \mathcal{B}(g)$ denote the set of Borel subalgebras of $g$ of positive type and $\mathcal{P} = \mathcal{P}(g)$ denote the set of parabolic subalgebras of $g$ of positive type. For any subalgebra $\mathfrak{k}$ of $g$, whenever defined we will write $\mathcal{B}(\mathfrak{k})$ and $\mathcal{P}(\mathfrak{k})$ to denote the set of Borel subalgebras and parabolic subalgebras, respectively, of positive type.

For any Borel subalgebra $b$ and parabolic subalgebra $p$ of $g$ define
\[
\text{proj}_p(b) = p \cap b + r_n(p),
\]
where $r_n(p)$ is the nilradical of $p$. For $b \in \mathcal{B}$ and $p \in \mathcal{P}$ define
\[
E_{p,p^{-}} = \text{proj}_p \circ \text{proj}_{p^{-}}.
\]
where $p^{-}$ is an opposite of $p$. Then $E_{p,p^{-}}(b) \in \mathcal{B}$ for all $b \in \mathcal{B}$, $E_{\mathfrak{g},\mathfrak{g}} = 1$, and $E_{p,p^{-}} \circ E_{p,p^{-}} = E_{p,p^{-}}$. Let $\mathcal{E}$ denote the set of idempotent maps $E_{p,p^{-}}$, $p \in \mathcal{P}$. Observe that if $l = p \cap p^{-}$ and $b \in \mathcal{B}$ have a common Cartan subalgebra then $E_{p,p^{-}}(b) = \text{proj}_p(b)$. For a subalgebra $\mathfrak{k} \subseteq g$ and $x \in G$, we will write $x \mathfrak{k}$ to denote $(\text{Ad}x)(\mathfrak{k})$ and $\mathfrak{k}x^{-1}$ to denote $(\text{Ad}x^{-1})(\mathfrak{k})$. Then the next lemma follows immediately from the definitions.

**Lemma 2.1.** For $x \in G$ and $p \in \mathcal{P}$ we have (i) $\text{Ad}x \circ E_{p,p^{-}} = E_{xp,x^{-}p^{-}} \circ \text{Ad}x$, and (ii) $E_{p,p^{-}} \circ \text{Ad}x = \text{Ad}x \circ E_{xp^{-},p^{-}x}$.

For two parabolic subalgebras $p$ and $q$ of $g$, we define $p \leftrightarrow q$ if and only if $p$ and $q$ have a common Levi factor. We say $E_{p,p^{-}} \leftrightarrow E_{q,q^{-}}$ if $p^{-} \leftrightarrow q^{-}$.

**Lemma 2.2.** Let $p, q \in \mathcal{P}$ with respective opposites $p^{-}$ and $q^{-}$. Suppose $p \cap p^{-} \cap q \cap q^{-}$ contains a Cartan subalgebra of $g$. Then there exists $p_1, q_1 \in \mathcal{P}$ with respective opposites $p_1^{-}$ and $q_1^{-}$ such that $p_1 \subseteq p$, $p_1^{-} \subseteq p^{-}$, $q_1 \subseteq q$, $q_1^{-} \subseteq q^{-}$, $p_1^{-} \leftrightarrow q_1^{-}$ (so $E_{p_1,p_1^{-}} \leftrightarrow E_{q_1,q_1^{-}}$), and $E_{p,p^{-}} \circ E_{q,q^{-}} = E_{p_1,p_1^{-}} \circ E_{q_1,q_1^{-}}$.

**Proof.** Let $l_1 = p \cap p^{-}$, $l_2 = q \cap q^{-}$, and $l = l_1 \cap l_2$. Then we have the Levi decompositions $p = l_1 + u$, $p^{-} = l_1 + u^{-}$, $q = l_2 + v$, $q^{-} = l_2 + v^{-}$, where $u = r_n(p)$, $u^{-} = r_n(p^{-})$, $v = r_n(q)$, $v^{-} = r_n(q^{-})$. Let $P$, $P^{-}$, $Q$, $Q^{-}$ denote the parabolic subgroups associated with the parabolic subalgebras $p$, $p^{-}$, $q$, $q^{-}$, respectively. We have the Levi decomposition of the parabolic subgroups $P = L_1U$, $P^{-} = L_1U^{-}$, $Q = L_2V$, and $Q^{-} = L_2V^{-}$, where $U = R_u(P)$, $U^{-} = R_u(P^{-})$, $V = R_u(Q)$, and $V^{-} = R_u(Q^{-})$. Consider the parabolic subalgebras
\[
p_1 = (p \cap q) + u = l + (l_1 \cap v) + u,
\]
\[
p_1^{-} = (p^{-} \cap q^{-}) + u^{-} = l + (l_1 \cap v^{-}) + u^{-},
\]
\[q_1 = (p \cap q) + v = l + (l_2 \cap u) + v,
\]
\[q_1^{-} = (p^{-} \cap q^{-}) + v^{-} = l + (l_2 \cap u^{-}) + v^{-}.
\]
Observe that $p_1^{-}$ and $q_1$ have $l$ as common Levi factor. Let $b \in \mathcal{B}$. By Corollary 1.2, $b \cap q_1$ contains a Cartan subalgebra $\mathfrak{h}$, say. There exists $x \in L_2 \cap U^{-}$, $y \in V^{-}$ such that $xy\mathfrak{h} \subseteq l$. Therefore, $xy\mathfrak{h} \subseteq l_2$. So $E_{q,q^{-}}(b) = (l_2 \cap xy\mathfrak{b}) + v$. Also $xy\mathfrak{h} \subseteq l$, $x \in L_2 \cap U^{-}$ implies that $xy\mathfrak{h} \subseteq p^{-}$. Hence
\[
E_{p,p^{-}} \circ E_{q,q^{-}}(b) = (l_1 \cap xy\mathfrak{b}) + (l_1 \cap x\mathfrak{v}) + u = (l_1 \cap xy\mathfrak{b}) + (l_1 \cap x\mathfrak{v}) + u.
\]
Now $x = x\mathfrak{v}$ as $x \in L_2 \subseteq Q$. But
\[
E_{q_1,q_1^{-}}(b) = (l_1 \cap xy\mathfrak{b}) + (l_2 \cap u) + v,
\]
so

\[ E_{p_1, p_1^{-}} \circ E_{q_1, q_1^-} (b) = (I \cap x^y b) + (I_1 \cap v) + u = E_{p, p^{-}} \circ E_{q, q^{-}} (b), \]

which proves the lemma. □

**Corollary 2.3.** Let \( p, q \in \mathcal{P} \) with respective opposites \( p^- \) and \( q^- \). Then there exists \( p_1, q_1 \in \mathcal{P} \) with respective opposites \( p_1^- \) and \( q_1^- \) such that

\[ E_{p_1, p_1^-} \leftrightarrow E_{q_1, q_1^-}, \]

\[ p_1 \subseteq p, \quad p_1^- \subseteq p^- , \quad q_1 \subseteq q, \quad q_1^- \subseteq q^- , \quad \text{and} \quad E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-}. \]

**Proof.** Let \( I_1 = p \cap p^-, \quad I_2 = q \cap q^- \). By Corollary 1.2, \( p^- \cap q \) contains a Cartan subalgebra \( h \), say. Let \( P, P^-, Q, Q^- \) denote the parabolic subgroups associated with \( p, p^-, q, q^- \), respectively. Let \( U = R_u(P), \quad U^- = R_u(P^-), \quad V = R_u(Q), \quad \text{and} \quad V^- = R_u(Q^-) \). Then there exists \( x \in U^- \) and \( y \in V \) such that \( h \subseteq I_1^x \) and \( h \subseteq I_2^y \). Then using Lemma 2.1,

\[ E_{p, p^-} \circ E_{q, q^-} = E_{p, p^-} \circ \text{Ad} x \circ \text{Ad} y \circ E_{q, q^-} = \text{Ad} x \circ E_{p, p^-} \circ E_{q, q^-} \circ \text{Ad} y \]

since \( p_x = p^- \) and \( q_y = q^- \). By Lemma 2.2, since \( h \subseteq p^x \cap p^- \cap q \cap q^- \), we can find \( p_1, q_1 \in \mathcal{P} \) with opposites \( p_1^-, q_1^- \), respectively, such that \( p_1 \subseteq p^x, \quad p_1^- \subseteq p^- , \quad q_1 \subseteq q, \quad q_1^- \subseteq q^- , \quad p_1^- \leftrightarrow q_1 \), and \( E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \). Hence using Lemma 2.1,

\[ E_{p, p^-} \circ E_{q, q^-} = \text{Ad} x \circ E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \circ \text{Ad} y \]

\[ = E_{x p_1, p_1^-} \circ \text{Ad} x \circ \text{Ad} y \circ E_{q_1, q_1^-} = E_{x p_1, p_1^-} \circ E_{q_1, q_1^-}, \]

since \( x \in U^- \), \( y \in V \), so that \( x p_1^- = p_1^- \) and \( q_1^y = q_1^- \). Hence the corollary follows. □

**Corollary 2.4.** Let \( p, q \in \mathcal{P} \) with respective opposites \( p^- \) and \( q^- \). Suppose \( p \subseteq q \) and \( p^- \subseteq q^- \). Then \( E_{p, p^-} \circ E_{q, q^-} = E_{p, p^-} \circ E_{q, q^-} = E_{q, q^-} \circ E_{p, p^-} \).

**Proof.** By Lemma 2.2, we have

\[ E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \]

where \( p_1 = p \cap q + \tau_n(p) = p, \quad p_1^- = p^- \cap q^- + \tau_n(p^-) = p^- , \quad q_1 = p \cap q + \tau_n(q) = p, \quad\text{and} \quad q_1^- = p^- \cap q^- + \tau_n(q^-) = p^- . \) Hence \( E_{p, p^-} \circ E_{q, q^-} = E_{p, p^-} \circ E_{p^-} = E_{p, p^-} \).

Similarly, \( E_{q, q^-} \circ E_{p, p^-} = E_{p, p^-} \). □

Let \( M \) denote the monoid generated by the idempotent maps \( E_{p, p^-} \in \mathcal{P} \) with respect to composition.

**Proposition 2.5.** Let \( p \in \mathcal{P} \) be a maximal parabolic subalgebra of \( g \) with opposite \( p^- \). Let \( P \) and \( P^- \) be the parabolic subgroups of \( G \) associated with \( p \) and \( p^- \), respectively, and \( L = P \cap P^- \). Let \( \mathcal{A}_1 = \{ x \in G | E_{p, p^-} \circ \text{Ad} x \circ E_{p, p^-} \in M \} \). Then \( L \subseteq \mathcal{A}_1 \).

**Proof.** Without loss of generality, we may assume that \( P = P_J \) for some \( J = I \setminus \{ i \}, \quad 1 \leq i \leq n \). Let \( L = L_J, \quad P^- = P_J^-, \quad P = LU, \quad \text{where} \quad U = U_J = R_u(P_J) \) and \( P^- = LU^-, \quad \text{where} \quad U^- = U_J^- = R_u(P_J^-) \). Let \( u \in U, \quad v \in U^- \). Then by
Lemma 2.1, we have
\[ E_{p, p^-} \circ \text{Ad} \, u \circ \text{Ad} \, v \circ E_{p, p^-} = \text{Ad} \, u \circ E_{p, p^-} \circ E_{p, p^-} \circ \text{Ad} \, v \quad \text{since } p^u = p, \quad p^v = p^- \]
\[ = E_{p, p^-} \circ E_{p, p^-} \circ \text{Ad} \, u \circ \text{Ad} \, v \quad \text{since } u \in U, \quad v \in U^- . \]
Hence \( UU^- \subseteq \mathfrak{A}_1 \). Again, since \( E_{p, p^-} \circ \text{Ad} \, v = E_{p, p^-} \) as \( v \in U^- \) and \( \text{Ad} \, u \circ E_{p, p^-} = E_{p, p^-} \) as \( u \in U \), it follows that \( U^- U U^- U \subseteq \mathfrak{A}_1 \). Let \( \sigma = \alpha_i \) and \( U_\sigma, U_{-\sigma} \) be the two root subgroups corresponding to \( \sigma \) and \( -\sigma \), respectively. Consider the group \( G_\sigma = (U_\sigma, U_{-\sigma}) \) that is isomorphic to \( \text{SL}(2, \mathbb{C}) \). One then observes by direct computation that \( G_{\sigma} = U_{-\sigma} U_\sigma U_{-\sigma} U_\sigma \). Since \( \sigma \notin \{ \alpha_j | j \in J \} \) (see [1]) we have \( U_\sigma \subseteq U \) and \( U_{-\sigma} \subseteq U^- \). Thus \( G_{\sigma} \subseteq U^- U U^- U \subseteq \mathfrak{A}_1 \). Hence the maximal torus \( T_{\sigma} \) of \( G_{\sigma} \) is contained in \( \mathfrak{A}_1 \).

Now let \( \sigma_1 \) and \( \sigma_2 \) be two simple roots corresponding to two adjacent nodes in the Coxeter graph \( \Gamma \) of \( G \) and suppose that \( G_{a_1} \subseteq \mathfrak{A}_1 \). Then \( T_{\sigma_1} \subseteq L_{\sigma_2} \cap \mathfrak{A}_1 \cap L_{\sigma_1} = C(L_{\sigma_2})G_{a_1} \). But since \( \sigma_1 \) and \( \sigma_2 \) are adjacent, \( T_{\sigma_1} \not\subseteq C(L_{\sigma_2}) \). Hence \( G_{a_2} \subseteq L_{\sigma_2} \cap \mathfrak{A}_1 \), so \( G_{a_2} \subseteq \mathfrak{A}_1 \). Clearly, \( C(L) \subseteq \mathfrak{A}_1 \). Since the generalized Cartan matrix \( A = (a_{ij}) \) is indecomposable, \( \Gamma \) is connected. Hence \( G_\sigma \subseteq \mathfrak{A}_1 \) for all \( \sigma \in \{ \alpha_i | i \in I \} \). Since \( L \) is generated by \( C(L) \) and \( G_\sigma, \sigma \in \{ \alpha_i | i \in I \} \), it follows that \( \mathfrak{L} \subseteq \mathfrak{A}_1 \). □

**Proposition 2.6.** Let \( p \in \mathcal{P}, \ p \neq g, \) with opposite \( p^- \). Let \( \mathfrak{A} = \{ x \in G | E_{p, p^-} \circ \text{Ad} \, x \in M \} \). Then \( \mathfrak{A} = G \).

**Proof.** It is enough to prove the result for \( p \) maximal. For if \( p_1 \in \mathcal{P}, \ p_1 \subseteq p, \ p^-_1 \subseteq p^- \), then by Corollary 2.4,
\[ E_{p_1, p^-_1} \circ \text{Ad} \, x = E_{p_1, p^-_1} \circ (E_{p, p^-} \circ \text{Ad} \, x) \in M \]
if \( E_{p, p^-} \circ \text{Ad} \, x \in M \). Let \( P, P^- \) denote the parabolic subgroups associated with \( p \) and \( p^- \), respectively. Let \( L = P \cap P^- \), \( P = LU \), \( P^- = LU^- \), where \( U = R_u(P) \) and \( U^- = R_u(P^-) \). Let \( u \in U \), \( v \in U^- \). Then by Lemma 2.1,
\[ E_{p, p^-} \circ \text{Ad} \, u \circ \text{Ad} \, v = E_{p, p^-} \circ E_{p, p^-} \circ \text{Ad} \, u \circ \text{Ad} \, v \]
\[ = E_{p, p^-} \circ (\text{Ad} \, u \circ E_{p, p^-}) \circ \text{Ad} \, v \quad \text{since } p^u = p, \]
\[ = E_{p, p^-} \circ E_{p, p^-} \circ \text{Ad} \, u \circ \text{Ad} \, v \quad \text{since } u \in U, \]
\[ = (E_{p, p^-} \circ \text{Ad} \, v) \circ E_{p, p^-} \]
\[ = E_{p, p^-} \circ E_{p, p^-} \in M \quad \text{since } v \in U^- . \]
Hence \( U U^- \subseteq \mathfrak{A} \). Similarly, since \( E_{p, p^-} \circ \text{Ad} \, v = E_{p, p^-} \) for \( v \in U^- \), it follows that \( U^- U U^- \subseteq \mathfrak{A} \).

Since \( p \) is maximal, by Proposition 2.5 we have \( L \subseteq \mathfrak{A}_1 \). But for \( x \in L \), we have
\[ E_{p, p^-} \circ \text{Ad} \, x \circ E_{p, p^-} = E_{p, p^-} \circ E_{p, p^-} \circ \text{Ad} \, x = E_{p, p^-} \circ \text{Ad} \, x . \]
Hence it follows that \( L \subseteq \mathfrak{A} \). Now since \( (L \cap \mathfrak{A}) \mathfrak{A} \subseteq \mathfrak{A} \), we have \( L U^- U U^- \subseteq \mathfrak{A} \). Hence \( G \subseteq \mathfrak{A} \). So \( \mathfrak{A} = G \). □

**Theorem 2.7.** \( M = \{ 1 \} \cup \{ E \circ \text{Ad} \, x \circ E' | E, E' \in \mathcal{B} \backslash \{ 1 \}, \ x \in G \} \).

**Proof.** Let \( N = \{ 1 \} \cup \{ E \circ \text{Ad} \, x \circ E' | E, E' \in \mathcal{B} \backslash \{ 1 \}, \ x \in G \} \). By Proposition 2.6, we have \( N \subseteq M \). By Corollary 2.3, we have
\[ M = \{ E_1 \circ E_2 \circ \cdots \circ E_k | E_1 \leftrightarrow E_2 \leftrightarrow \cdots \leftrightarrow E_k, \ E_i \in \mathcal{B} \} . \]
Let \( E_i = E_{p_i, p_i^-} \). Consider \( E_1 \circ E_2 \circ E_3 \) with \( E_1 \leftrightarrow E_2 \leftrightarrow E_3 \). Then \( p_1^- \leftrightarrow p_2 \) and \( p_2^- \leftrightarrow p_3 \). Let \( l_1 \subseteq p_1^- \cap p_2 \), \( l_2 = p_2 \cap p_2^- \), and \( l_3 \subseteq p_2^- \cap p_3 \) be the common Levi factors. Let \( P_2 \) and \( P_2^- \) denote the parabolic subgroups associated with \( p_2 \) and \( p_2^- \), respectively. Let \( L_2 = p_2 \cap p_2^- \), \( P_2 = L_2 U_2 \), and \( P_2^- = L_2 U_2^- \), where \( U_2 = R_u(P_2) \) and \( U_2^- = R_u(P_2^-) \). Then there exists \( u \in U_2 \) and \( v \in U_2^- \) such that \( l_1 = u l_2 \) and \( l_2 = v l_3 \). Let \( b \in B \) and \( b_1 = E_{p_3, p_3^-}(b) \). Then

\[
E_{p_1, p_1^-} \circ E_{p_2, p_2^-} \circ E_{p_1, p_1^-}(b_1) = E_{p_1, p_1^-} \circ \text{proj}_{p_2} \circ \text{proj}_{p_2^-}(b_1)
\]

\[
= E_{p_1, p_1^-} \circ \text{proj}_{p_2}((l_2 \cap \gamma b_1 + \tau_n(p_2)))
\]

\[
= E_{p_1, p_1^-}((l_1 \cap \gamma b_1 + \tau_n(p_2)))
\]

\[
= (E_{p_1, p_1^-} \circ \text{Ad}(uv))(b_1). \]

Hence

\[
E_{p_1, p_1^-} \circ E_{p_2, p_2^-} \circ E_{p_3, p_3^-} = E_{p_1, p_1^-} \circ \text{Ad}(uv) \circ E_{p_3, p_3^-}.
\]

Also note that \( l_1 \subseteq p_1^- \cap p_3 \), hence \( p_1^- \leftrightarrow \gamma p_3 \). Assume that \( E_1 \leftrightarrow E_2 \leftrightarrow \cdots \leftrightarrow E_{k-1} \) and \( E_1 \circ E_2 \circ \cdots \circ E_{k-1} = E_1 \circ \text{Ad} x \circ E_{k-1} \), with \( p_1^- \leftrightarrow \gamma p_{k-1} \), hence \( p_1^- \gamma x \leftrightarrow p_{k-1} \). Consider \( E_1 \circ E_2 \circ \cdots \circ E_k \) with \( E_1 \leftrightarrow E_2 \leftrightarrow \cdots \leftrightarrow E_k \). Then using Lemma 2.1,

\[
E_1 \circ E_2 \circ \cdots \circ E_{k-1} \circ E_k = E_1 \circ \text{Ad} x \circ E_{k-1} \circ E_k
\]

\[
= E_{p_1, p_1^-} \circ \text{Ad} x \circ E_{p_{k-1}, p_{k-1}^-} \circ E_{p_k, p_k^-}
\]

\[
= \text{Ad} x \circ E_{p_1, p_1^-} \circ E_{p_{k-1}, p_{k-1}^-} \circ E_{p_k, p_k^-}
\]

\[
= \text{Ad} x \circ E_{p_1, p_1^-} \circ \text{Ad} y \circ E_{p_{k-1}, p_{k-1}^-} \circ E_{p_k, p_k^-}
\]

\[
= (E_{p_1, p_1^-} \circ \text{Ad}(xy)) \circ E_{p_k, p_k^-} \quad \text{with } p_1^- \leftrightarrow \gamma p_k.
\]

Hence by induction \( E_1 \circ E_2 \circ \cdots \circ E_k = E_1 \circ \text{Ad} x \circ E_k \), \( x \in G \), for all \( k \). This proves that \( M \subseteq N \). Hence \( M = N \). \( \Box \)

Fix \( p \in B \) with opposite \( p^- \). Let \( E = E_{p, p^-} \) and \( l = p \cap p^- \). Let \( P \) and \( P^- \) denote the parabolic subgroups associated with \( p \) and \( p^- \), respectively, and \( L = P \cap P^- \). Let \( St_G(l) \) denote the stabilizer of \( l \) in \( G \) and let \( H = \{ x \in St_G(l) | b = x b \text{ for all } b \in B(l) \} \). Let \( \tilde{L} = St_G(l)/H \). Then we have the following

**Theorem 2.8.** The maximal subgroup of \( M \) with identity element \( E = E_{p, p^-} \) is \( \{ E \circ \text{Ad} x \circ E | x \in St_G(l) \} \) which is isomorphic to \( \tilde{L} \) as above.

**Proof.** The maximal subgroup of \( M \) with identity element \( E = E_{p, p^-} \) is the same as the group of units \( K \) say, of \( EME \). Let \( a \in K \). Then as in the proof of Theorem 2.7, we have \( a = E_{q, q^-} \circ \text{Ad} y \circ E_{q_1, q_1^-} \) for some \( y \in G \), where \( q \subseteq p \), \( q^- \subseteq p^- \), \( q_1 \subseteq p \), \( q_1^- \subseteq p^- \), and \( q^- \leftrightarrow \gamma q_1 \). Since \( a \in K \) is invertible and \( E \) is the identity in \( K \), there exists \( b \in K \) such that \( a \circ b = E \) and \( b \circ a = E \). Hence, by Corollary 2.4,

\[
E_{q, q^-} = E_{q, q^-} \circ E = E_{q, q^-} \circ a \circ b = a \circ b = E.
\]

Similarly,

\[
E_{q_1, q_1^-} = E \circ E_{q_1, q_1^-} = b \circ a \circ E_{q_1, q_1^-} = b \circ a = E.
\]
Hence \( a = E \circ \text{Ad} y \circ E \) for some \( y \in G \), and \( p^- \leftrightarrow y p^- \). Let \( l_1 = p^- \cap y p^- \). Let \( U = R_u(\mathcal{P}) \) and \( U^- = R_y(\mathcal{P}^-) \). Then there exists \( u \in U \) and \( v \in U^- \) such that \( l_1 = v l \) and \( l_1^- = v l^- \), hence \( yu l = l_1 \). Therefore, \( x l = l \), where \( x = v^{-1} y u \) so \( y = v x u^{-1} \). Hence \( x \in \text{St}_G(l) \) and

\[
a = E \circ \text{Ad} y \circ E = (E_{p, p^-} \circ \text{Ad} v) \circ \text{Ad} x \circ (\text{Ad} u^{-1} \circ E_{p, p^-})
\]

\[
= E_{p, p^-} \circ \text{Ad} x \circ E_{p, p^-} = E \circ \text{Ad} x \circ E,
\]

since \( v \in U^- \) and \( u^{-1} \in U \). Observe that \( l \subseteq p \cap p^{-x} \cap x p^- \). Now define the map \( \theta: \text{St}_G(l) \to K \), by \( \theta(x) = E \circ \text{Ad} x \circ E \). By the above discussion \( \theta \) is onto. Furthermore, for \( x, y \in \text{St}_G(l) \).

\[
\theta(x) \circ \theta(y) = E \circ \text{Ad} x \circ E \circ E \circ \text{Ad} y \circ E
\]

\[
= E_{p, p^-} \circ \text{Ad} x \circ E_{p, p^-} \circ \text{Ad} y \circ E_{p, p^-}
\]

\[
= \text{Ad}(xy) \circ E_{p^{xy}, p^{-xy}} \circ E_{p^y, p^{-y}} \circ E_{p, p^-} \quad \text{by Lemma 2.1},
\]

\[
= \text{Ad}(xy) \circ E_{p^{xy}, p^{-xy}} \circ E_{p, p^-}
\]

since \( p^{xy}, p^{-xy}, p^y, p^{-y}, p, p^- \) have common Levi factor \( l \),

\[
= E_{p, p^-} \circ \text{Ad}(xy) \circ E_{p, p^-} \quad \text{by Lemma 2.1}
\]

\[
= \theta(xy).
\]

Hence \( \theta \) is a surjective homomorphism. Clearly, \( \text{Ker} \theta = H \). Hence \( K \cong \text{St}_G(l)/H = \tilde{L} \). \( \square \)

We note that \( \tilde{L} \) is closely related to the Levi subgroup \( L \) of \( G \). We therefore see that the monoid \( M \) gives rise to groups closely related to Kac-Moody groups.

3. Examples

When \( g \) is finite dimensional, \( M \) is exactly the monoid generated by \( \text{proj}_p(p \in \mathcal{P}) \) and is isomorphic to the monoid studied in [5, Theorem 2] for the associated simple algebraic group. The situation is quite different when \( g \) is infinite dimensional. For example, consider the affine Kac-Moody Lie algebras \( g = A_1^{(1)} \) with Dynkin diagram

\[
(3.1)
\]

Then \( g \) has four standard positive parabolics, but the set \( \mathcal{P} \) of all positive parabolics is of course infinite. The monoid generated by \( \text{proj}_p(p \in \mathcal{P}) \) is not that interesting in that the subgroups of this monoid are trivial. We, therefore, are naturally led to considering our monoid \( M \) generated by \( E_{p, p^-} = \text{proj}_p \circ \text{proj}_p^{-} \). The only nontrivial subgroups of \( M \) are those corresponding to parabolic subalgebras \( p \) obtained by deleting a node of the diagram (3.1). The associated Levi factor \( l \) is a finite-dimensional Lie algebra with the simple part being \( \text{sl}_2(\mathbb{C}) \). The subgroup of \( M \) with identity element \( E_{p, p^-} \) is just the projective group \( \text{PGL}(2, \mathbb{C}) \).
To get finite-dimensional Kac-Moody groups, we consider the hyperbolic Kac-Moody Lie algebra $\mathfrak{g}_1$ with Dynkin diagram

(3.2)

Let $M_1 = M(\mathfrak{g}_1)$ denote the corresponding monoid. Let $\mathfrak{p}$ denote the parabolic subalgebra obtained by deleting the right-most node in (3.2). Then the maximal subgroup of $M_1$ with identity $E_{\mathfrak{p},\mathfrak{p}^-}$ is 'almost' the Kac-Moody group corresponding to (3.1).

REFERENCES


DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27695-8205