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THE DIFFERENTIAL EQUATION $Q = 0$ IN WHICH Q IS A QUADRATIC FORM IN y'', y', y HAVING MEROMORPHIC COEFFICIENTS

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ABSTRACT. A simple necessary and sufficient condition is given for the solutions of $Q = 0$ to be free of movable branch points. And, when the condition is satisfied, all the solutions of $Q = 0$ can be obtained by solving linear differential equations of order ≤ 2. There are four mutually exclusive cases. We shall relate Case 4 to less convenient conditions P. Appell had introduced. We shall also show how Cases 3 and 4 together motivated our discovery of an identity that is essential for a satisfactory theory of relative invariants for homogeneous linear differential equations.

1. INTRODUCTION

Throughout, we suppose the coefficients $a(z), \dots, f(z)$ of

$$a(z)y''^2 + b(z)y''y' + c(z)y''y + d(z)y'^2 + e(z)y'y + f(z)y^2 = 0 \tag{1.1}$$

are meromorphic functions of a complex variable z on a region Ω of the complex plane such that at least one of $a(z), \dots, f(z)$ is not identically zero. For later reference, we introduce

$$\Delta \equiv \begin{vmatrix} a & b/2 & c/2 \\ b/2 & d & e/2 \\ c/2 & e/2 & f \end{vmatrix},$$

$$A_2 \equiv 4ad - b^2, \qquad A_3 \equiv 4ae - 2bc, \qquad A_4 \equiv 4af - c^2,$$

and $D \equiv (A_3)^2 - 4A_2A_4 \equiv -2^6 a\Delta$ on Ω.

Theorem 1.1. *The solutions of* (1.1) *are free of movable branch points in the precise sense of* [5, *pp.* 91, 112; 10, *p.* 133] *if and only if either*

$$\Delta(z) \equiv 0 \tag{1.2}$$

or

$$a(z) \equiv b(z) \equiv 0 \not\equiv c(z) \quad \textit{and} \quad d(z) \equiv -c(z) \tag{1.3}$$

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or

$$a(z) \equiv b(z) \equiv 0 \not\equiv c(z) \quad \textit{and} \quad d(z) \equiv \left[\frac{1-m}{m}\right] c(z), \tag{1.4}$$

for some integer $m \neq 0, 1$, *or the conditions*

$$A_2(z)D(z) \not\equiv 0 \tag{1.5}$$

and

$$b(z) \equiv \left[\frac{2A_2'(z) + 2A_3(z)}{A_2(z)} - \frac{D'(z)}{D(z)}\right] a(z) \tag{1.6}$$

and

$$c(z) \equiv \left[\frac{A_3'(z) + 2A_4(z)}{A_2(z)} - \frac{A_3(z)D'(z)}{2A_2(z)D(z)}\right] a(z) \tag{1.7}$$

are satisfied. (*Of course,* (1.5) *requires* $a(z) \not\equiv 0$.)

Proof. This is a restatement of [10, Theorem 3.1] included here to conveniently introduce four mutually exclusive classes of differential equations (1.1) to be considered next.

Case 1. Suppose (1.2) is satisfied. Regarding the left member Q of (1.1) as a quadratic form in y'', y', y, we see that Δ is its determinant. And, due to (1.2), Q is expressible as a product of two nonzero homogeneous linear polynomials in y'', y', y having meromorphic coefficients on some subregion U of Ω.

Case 2. Suppose (1.3) is satisfied. Then, the substitution $y' = yw$, $y'' = y(w' + w^2)$ given for [17, p. 38, (6)] relates the nonzero solutions $y(z)$ of (1.1) to the solutions $w(z)$ of

$$w' + \left[\frac{e(z)}{c(z)}\right] w + \left[\frac{f(z)}{c(z)}\right] = 0.$$

Case 3. Suppose (1.4) is satisfied for some integer $m \neq 0, 1$. Then, the substitution $y = u^m$ given for [17, p. 35, (1)] relates the nonzero solutions $y(z)$ of (1.1) to the nonzero solutions $u(z)$ of

$$u'' + \left[\frac{e(z)}{c(z)}\right] u' + \left[\frac{f(z)}{(mc(z))}\right] u = 0. \tag{1.8}$$

And, in terms of linearly independent solutions $\phi(z)$, $\psi(z)$ of (1.8) on a subregion U of Ω, the nonzero solutions of (1.1) on U are given by

$$y(z) \equiv [C_1\phi(z) + C_2\psi(z)]^m, \tag{1.9}$$

for constants C_1, C_2 not both zero.

Case 4. Suppose (1.5), (1.6), and (1.7) are satisfied. Then, Theorem 2.1 provides a convenient method to obtain the solutions of (1.1) by solving linear differential equations of order ≤ 2.

In view of Theorem 1.1, Cases 1, 2, 3, and 4 yield the following result.

Theorem 1.2. *If the solutions of* (1.1) *are free of movable branch points, then all the solutions of* (1.1) *can be found by solving linear differential equations of order* ≤ 2.

During the years 1887–1889, Appell introduced in [1–4] the condition that the left member Q of (1.1) satisfy $a(z)\Delta(z)A_2(z) \not\equiv 0$ and

$$Q' + \lambda(z)Q \equiv S(z, y, y', y'')T(z, y, y', y'', y'''), \tag{1.10}$$

for some meromorphic function $\lambda(z)$ on Ω and some homogeneous linear polynomials S in y'', y', y and T in y''', y'', y', y having meromorphic coefficients on Ω. And, when (1.1) satisfies his condition, he showed how its nonsingular solutions could be deduced from those of the third-order homogeneous linear differential equation $T = 0$. In [8] of 1960, we developed a transformation for any (1.1) subject to $a(z)\Delta(z) \not\equiv 0$. And, our results showed that: *if* (1.1) *satisfies* $a(z)\Delta(z) \not\equiv 0$ *and possesses a pair of linearly independent singular solutions, then the solutions of* (1.1) *can be obtained by solving a Riccati differential equation and first-order homogeneous linear differential equations.* In view of [9, Theorem 3.5, Proposition 3.6], (1.1) satisfies $a(z)\Delta(z) \not\equiv 0$ and possesses a pair of linearly independent singular solutions if and only if (1.1) satisfies Appell's condition. Due to [9, Theorems 2.2 and 2.3], (1.1) satisfies Appell's condition if and only if (1.1) satisfies the conditions (1.5), (1.6), and (1.7) of Case 4. Thus, all the equations (1.1) that satisfy Appell's condition are obtained parametrically in the equivalent form

$$[2a(z)y'' + b(z)y' + c(z)y]^2 + A_2(z)y'^2 + A_3(z)y'y + A_4(z)y^2 = 0 \tag{1.11}$$

by selecting $a(z)$, $A_2(z)$, $A_3(z)$, $D(z)$ as any meromorphic functions on Ω such that $a(z)A_2(z)D(z) \not\equiv 0$ and defining $A_4(z)$, $b(z)$, $c(z)$ in terms of them through

$$A_4(z) \equiv \frac{[(A_3(z))^2 - D(z)]}{4A_2(z)},$$

(1.6), and (1.7). Of course, an expansion of (1.11) and a division by $4a(z)$ yields (1.1).

Various nonlinear differential equations whose solutions can be found by solving linear differential equations were cited in [18, 21, 15, 14, 19, 20, 16]. However, each of these papers overlooked the equations (1.1) that satisfy Appell's condition.

An important equation (1.1) that satisfies Appell's condition is given by [11, p. 1509, (10.10)]. It plays an essential role in Cosgrove's axisymmetric solutions of Einstein's field equations. Transformations for it and any other (1.1) satisfying Appell's condition are described in [12, pp. 2402–2403]. Other results about (1.1) can be found in [23; 13; 6; 7; 22, pp. 629–631].

2. Solutions of (1.1) for Case 4

Theorem 2.1. *Suppose* (1.1) *satisfies* (1.5), (1.6), *and* (1.7). *Let* τ, η, ϕ, ψ *be meromorphic functions on a subregion* U *of* Ω *such that*:

$$[\tau(z)]^2 \equiv \frac{D(z)}{4[A_2(z)]^2}; \tag{2.1}$$

$$\eta'(z) + \left\{\frac{A_2'(z)}{A_2(z)} + \frac{A_3(z)}{2A_2(z)} - \tau(z)\right\}\eta(z) \equiv 0; \tag{2.2}$$

$\eta(z) \not\equiv 0$; and $\phi(z)$, $\psi(z)$ are linearly independent solutions on U of

$$w'' + \left\{\frac{a'(z)}{a(z)} - \frac{A_2'(z)}{2A_2(z)} + \tau(z)\right\} w' + \left\{\frac{A_2(z)}{16[a(z)]^2}\right\} w = 0. \tag{2.3}$$

Then, the nonsingular solutions on U of (1.1) *are given by*

$$\begin{aligned} y(z) \equiv\ & A_2(z)\eta(z)[C_1\phi(z) + C_2\psi(z)]^2 \\ & + 16[a(z)]^2\eta(z)[C_1\phi'(z) + C_2\psi'(z)]^2, \end{aligned} \tag{2.4}$$

where C_1, C_2 are constants not both zero. And, when (2.4) *is written as*

$$y(z) \equiv C_1^2 v_1(z) + C_1C_2v_2(z) + C_2^2v_3(z), \tag{2.5}$$

the meromorphic functions v_1, v_2, v_3 on U are linearly independent and satisfy

$$[v_2(z)]^2 - 4v_1(z)v_3(z) \not\equiv 0. \tag{2.6}$$

Moreover, in terms of

$$r_1(z) \equiv -\frac{A_3(z)}{2A_2(z)} + \tau(z) \quad \textit{and} \quad r_2(z) \equiv -\frac{A_3(z)}{2A_2(z)} - \tau(z),$$

the solutions of

$$y' - r_1(z)y = 0 \quad \textit{or} \quad y' - r_2(z)y = 0 \tag{2.7}$$

are the singular solutions of (1.1) *relative to U.*

Proof. Equations having the same solutions as (1.1) include (1.11) and

$$[S(z, y, y', y'')]^2 + A_2(z)[y' - r_1(z)y][y' - r_2(z)y] = 0, \tag{2.8}$$

where $S \equiv 2ay'' + by' + cy$ is the formal partial derivative with respect to y'' of the left member Q of (1.1). Using (1.5), (1.6), (1.7), and (2.8), we apply the results in [9, pp. 86–87, 93–94] to see that the solutions of (2.7) are the solutions of both $S = 0$ and $Q = 0$. Therefore, the singular solutions of (1.1) relative to U are the solutions of (2.7).

There are meromorphic functions ρ, θ on a subregion U_0 of U such that:

$$[\rho(z)]^2 \equiv \frac{-A_2(z)}{16[a(z)]^2} \quad \text{on } U_0; \tag{2.9}$$

$$2A_2(z)\theta'(z) + A_3(z)\theta(z) \equiv 0 \quad \text{on } U_0; \tag{2.10}$$

and $\theta(z) \not\equiv 0$. Using (2.9) and (1.5), we deduce $\rho(z) \not\equiv 0$ and

$$\frac{\rho'(z)}{\rho(z)} \equiv \frac{A_2'(z)}{2A_2(z)} - \frac{a'(z)}{a(z)} \tag{2.11}$$

on U_0 so that (2.3) is given on U_0 by

$$w'' + \left\{\tau(z) - \frac{\rho'(z)}{\rho(z)}\right\} w' - [\rho(z)]^2 w = 0. \tag{2.12}$$

Under the substitution $w'/w = \rho t$, $w''/w = \rho t' + \rho' t + \rho^2 t^2$, (2.12) is transformed into

$$t' + \rho(z)t^2 + \tau(z)t - \rho(z) = 0 \quad \text{on } U_0. \tag{2.13}$$

Moreover, three distinct solutions on U_0 of (2.13) are given by

$$t_1 \equiv \frac{\phi'}{\rho\phi}, \qquad t_2 \equiv \frac{\psi'}{\rho\psi}, \qquad t_3 \equiv \frac{(\phi+\psi)'}{\rho(\phi+\psi)}. \tag{2.14}$$

We set $\omega(z) \equiv \phi(z)\psi'(z) - \phi'(z)\psi(z)$ on U. Since $\phi(z)$, $\psi(z)$ are linearly independent solutions on U of (2.3), we obtain $\omega(z) \not\equiv 0$ and

$$\omega'(z) + \left\{\frac{a'(z)}{a(z)} - \frac{A_2'(z)}{2A_2(z)} + \tau(z)\right\}\omega(z) \equiv 0 \tag{2.15}$$

on U. Applying [9, Theorem 4.8] with $m = n = 1$ by writing (2.14) as

$$t_1 \equiv t_p, \qquad t_2 \equiv t_p + \frac{1}{u_p}, \qquad t_3 \equiv t_p + \frac{1}{u_p + v_p},$$

for $t_p \equiv \phi'/(\rho\phi)$, $u_p \equiv \rho\phi\psi/\omega$, and $v_p \equiv \rho\phi^2/\omega$, we find the nonsingular solutions of (1.1) are given on U_0 by

$$y(z) \equiv K_1^2 u_1(z) + K_1K_2u_2(z) + K_2^2u_3(z), \tag{2.16}$$

where

$$u_1 \equiv \theta v_p[t_p^2 - 1] \equiv \left[\frac{\theta}{(\rho\omega)}\right][\phi'^2 - \rho^2\phi^2],$$

$$u_2 \equiv 2\theta[t_p^2u_p + t_p - u_p] \equiv \left[\frac{\theta}{(\rho\omega)}\right][2\phi'\psi' - 2\rho^2\phi\psi],$$

$$u_3 \equiv \left[\frac{\theta}{v_p}\right][(t_pu_p+1)^2 - u_p^2] \equiv \left[\frac{\theta}{(\rho\omega)}\right][\psi'^2 - \rho^2\psi^2],$$

and K_1, K_2 are constants not both zero. We apply (2.11), (2.15), (2.2), and (2.10) to deduce

$$\frac{[a^2\rho\omega\eta/\theta]'}{[a^2\rho\omega\eta/\theta]} \equiv \frac{2a'}{a} + \frac{\rho'}{\rho} + \frac{\omega'}{\omega} + \frac{\eta'}{\eta} - \frac{\theta'}{\theta} \equiv 0.$$

Therefore, there is a nonzero constant K_0 such that

$$\frac{\theta}{(\rho\omega)} \equiv 16K_0^2a^2\eta \quad \text{on } U_0. \tag{2.17}$$

Setting $C_1 = K_0K_1$ and $C_2 = K_0K_2$, we use (2.9) and (2.17) to rewrite (2.16) as (2.4). And, the character of (2.4) as a solution of (1.1) on U_0 extends to U. Due to [9, Theorem 4.8], u_1, u_2, u_3 are linearly independent and satisfy $u_2^2 - 4u_1u_3 \not\equiv 0$. Therefore, v_1, v_2, v_3 for (2.5) are linearly independent and satisfy (2.6). This completes the proof.

Example 2.2. Let (1.1) be the differential equation

$$\begin{aligned} &3z^2y''^2 + 12zy''y' - 8z^2y''y \\ &\qquad + (z^2+12)y'^2 - 14zy'y + (4z^2+1)y^2 = 0, \end{aligned} \tag{2.18}$$

where Ω is the complex plane. In terms of

$$A_2(z) \equiv 12z^4, \qquad A_3(z) \equiv 24z^3,$$
$$A_4(z) \equiv -16z^4 + 12z^2, \qquad D(z) \equiv 768z^8,$$

we find (1.5), (1.6), and (1.7) are satisfied. Selecting $\tau(z) \equiv 2/\sqrt{3}$, we have $r_1(z) \equiv (2/\sqrt{3}) - (1/z)$ and $r_2(z) \equiv (-2/\sqrt{3}) - (1/z)$. Thus, the singular solutions of (2.18) are

$$y = C\left[\frac{1}{z}\right]\exp\left(\left(\frac{2}{\sqrt{3}}\right)z\right) \quad \text{and} \quad y = C\left[\frac{1}{z}\right]\exp\left(\left(\frac{-2}{\sqrt{3}}\right)z\right),$$

where C is any constant. We select $\eta(z) \equiv [1/z]^5\exp((2/\sqrt{3})z)$ to satisfy (2.2). Since (2.3) is $w'' + (2/\sqrt{3})w' + (1/12)w = 0$, we select

$$\phi(z) \equiv \exp\left[\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)z\right] \quad \text{and} \quad \psi(z) \equiv \exp\left[\left(-\frac{1}{2} - \frac{1}{\sqrt{3}}\right)z\right].$$

Then, for the nonsingular solutions of (2.18), (2.4) yields

$$y(z) \equiv \left[\frac{48}{z}\right][C_1^2(2-\sqrt{3})\exp(z) + C_1C_2 + C_2^2(2+\sqrt{3})\exp(-z)],$$

where C_1, C_2 are constants not both zero. Selecting

$$K_1 = [48(2-\sqrt{3})]^{1/2}C_1 \quad \text{and} \quad K_2 = [48(2+\sqrt{3})]^{1/2}C_2,$$

we see that the nonsingular solutions of (2.18) are given by

$$y(z) \equiv K_1^2\left(\frac{e^z}{z}\right) + K_1K_2\left(\frac{1}{z}\right) + K_2^2\left(\frac{e^{-z}}{z}\right),$$

where K_1, K_2 are constants not both zero.

3. Identities for Case 3 analogous to (1.10)

Let $v_1(z)$, $v_2(z)$, $v_3(z)$ be linearly independent meromorphic functions on a region U and let $W(z)$ be their Wronskian. Of course, we have $W(z) \not\equiv 0$. To obtain a differential equation (1.1) satisfied by

$$y(z) \equiv C_1^2v_1(z) + C_1C_2v_2(z) + C_2^2v_3(z), \tag{3.1}$$

for any constants C_1, C_2, we differentiate (3.1) two times and then substitute the expressions obtained for $W(z)C_1^2$, $W(z)C_1C_2$, $W(z)C_2^2$ by means of Cramer's rule into $[W(z)C_1C_2]^2 - [W(z)C_1^2][W(z)C_2^2] \equiv 0$. This shows that (3.1) is a solution of the differential equation (1.1) having

$$a \equiv (\alpha_{13})^2 - \alpha_{12}\alpha_{23}, \qquad b \equiv -a', \tag{3.2}$$

$$c \equiv 2\alpha_{13}\gamma_{13} - \alpha_{12}\gamma_{23} - \alpha_{23}\gamma_{12}, \qquad d \equiv (\beta_{13})^2 - \beta_{12}\beta_{23}, \tag{3.3}$$

$$e \equiv -2\beta_{13}\gamma_{13} + \beta_{12}\gamma_{23} + \beta_{23}\gamma_{12}, \qquad f \equiv (\gamma_{13})^2 - \gamma_{12}\gamma_{23}, \tag{3.4}$$

where, for $1 \le i < j \le 3$,

$$\alpha_{ij} \equiv v_iv_j' - v_i'v_j, \qquad \beta_{ij} \equiv \alpha_{ij}', \qquad \gamma_{ij} \equiv v_i'v_j'' - v_i''v_j'. \tag{3.5}$$

Proposition 3.1. *The left member Q of the differential equation* (1.1) *specified by* (3.2) *through* (3.5) *satisfies*

$$Q' + \lambda(z)Q \equiv [2a(z)y'' + b(z)y' + c(z)y][y''' + M], \tag{3.6}$$

where $\lambda(z) \equiv -2W'(z)/W(z)$ and where M is a homogeneous linear polynomial in y'', y', y having meromorphic coefficients on U. Moreover, when

$v_2^2 - 4v_1v_3 \not\equiv 0$, *conditions* (1.5), (1.6), *and* (1.7) *of Case* 4 *are satisfied. And, when* $v_2^2 - 4v_1v_3 \equiv 0$, *condition* (1.4) *of Case* 3 *is satisfied for* $m = 2$.

Proof. Suppose $v_2^2 - 4v_1v_3 \equiv 0$. Then, there are meromorphic functions $\phi(z)$, $\psi(z)$ on a subregion U_0 of U such that

$$(3.7) \qquad v_1(z) \equiv [\phi(z)]^2, \qquad v_2(z) \equiv 2\phi(z)\psi(z), \qquad v_3(z) \equiv [\psi(z)]^2.$$

Setting $\omega(z) \equiv \phi(z)\psi'(z) - \phi'(z)\psi(z)$, we have $W(z) \equiv 4[\omega(z)]^3$ and $\omega(z) \not\equiv 0$. Using (3.2) through (3.5), we obtain $a \equiv 0$, $b \equiv 0$, $c \equiv -8\omega^4$, $d \equiv 4\omega^4$, $e \equiv 8\omega^3\omega'$, and $f \equiv 16\omega^3(\phi''\psi' - \phi'\psi'')$. Thus, (1.4) is satisfied for $m = 2$. We apply [10, Proposition 3.6] to deduce (3.6) for

$$\lambda(z) \equiv \frac{-c'(z) + 2e(z)}{c(z)} \equiv \frac{-6\omega'(z)}{\omega(z)} \equiv \frac{-2W'(z)}{W(z)}.$$

Suppose $v_2^2 - 4v_1v_3 \not\equiv 0$. We apply (3.2) through (3.5) to deduce

$$A_2 \equiv [4v_1v_3 - v_2^2]W^2, \qquad A_3 \equiv [v_2^2 - 4v_1v_3]'W^2, \qquad A_4 \equiv [4v_1'v_3' - v_2'^2]W^2,$$

and $D \equiv A_3^2 - 4A_2A_4 \equiv 16aW^4$. This yields $4ad - b^2 \equiv A_2 \not\equiv 0$ and, therefore, $a \not\equiv 0$ because $b \equiv -a'$. Thus, (1.5) is satisfied. Using these expressions for A_2, A_3, A_4, and D, we find the right member of (1.6) reduces to $[-a'/a]a \equiv b$. Thus, (1.6) is satisfied. And, in terms of

$$\Gamma \equiv [2v_2v_2'' - 4v_1''v_3 - 4v_1v_3'']a - [v_2v_2' - 2v_1'v_3 - 2v_1v_3']a',$$

we find the right member of (1.7) is equal to $\Gamma/[4v_1v_3 - v_2^2]$. We apply the identity $\Gamma \equiv [4v_1v_3 - v_2^2]c$ to see that (1.7) is satisfied. Finally, we use [9, Theorem 3.5, (2.23)] to obtain (3.6) for

$$\lambda(z) \equiv \left[\frac{a'(z) + b(z)}{a(z)}\right] - \left[\frac{A_2'^{(z)} + A_3(z)}{A_2(z)}\right] \equiv \frac{-2W'(z)}{W(z)}.$$

This completes the proof.

Of course, when (3.7) is satisfied, (3.1) can be rewritten as (1.9) for $m = 2$.

Restricting Case 3 to $m \geq 2$ and $c(z) \equiv 1$, we see that for any meromorphic functions $\alpha(z)$, $\beta(z)$ on Ω, the transformation $y = u^m$ relates the solutions $u(z)$ of

$$(3.8) \qquad u'' + \alpha(z)u' + \beta(z)u = 0$$

to the solutions $y(z)$ of $Q_m = 0$, where

$$Q_m \equiv y''y + \left[\frac{1-m}{m}\right]y'^2 + \alpha(z)y'y + m\beta(z)y^2.$$

Recalling (3.6) for the situation $v_2^2 - 4v_1v_3 \equiv 0$, we find that

$$[Q_2]' + 2\alpha Q_2 \equiv y\{y''' + 3\alpha y'' + (\alpha' + 2\alpha^2 + 4\beta)y' + (2\beta' + 4\alpha\beta)y\}.$$

It is natural to inquire whether Q_m may satisfy an analogous identity when $m \geq 3$.

For any integer $m \geq 2$ and any meromorphic functions $\alpha(z)$, $\beta(z)$ on Ω, the algorithms presented in [10, Theorem 4.1 and its proof] specify meromorphic functions $a_{m,i}(z)$ on Ω and polynomials $P_{m,j}$ in y, y', ..., $y^{(m-1)}$ having meromorphic coefficients on Ω such that Q_m satisfies

$$\sum_{j=0}^{m-1} [P_{m,j}][Q_m^{(j)}] \equiv [y^m][L_m], \tag{3.9}$$

where

$$L_m \equiv y^{(m+1)} + \sum_{i=0}^{m} a_{m,i}(z) y^{(i)}. \tag{3.10}$$

The identity (3.9) shows that the mth power of each local solution of (3.8) is a local solution of $L_m = 0$. As a consequence of this, we have proved that if $\{\phi(z), \psi(z)\}$ is a fundamental system of solutions for (3.8) on a subregion U of Ω, then

$$\{[\phi(z)]^{m-k}[\psi(z)]^k: \text{ for } k = 0, 1, 2, \ldots, m\}$$

is a fundamental system of solutions for $L_m = 0$ on U. In particular, L_m in the monic form (3.10) is uniquely specified by (3.8). As explained in [10], information provided by (3.9) is essential for a satisfactory theory of relative invariants of homogeneous linear differential equations.

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