PRINCIPAL EIGENVALUES FOR INDEFINITE-WEIGHT ELLIPTIC PROBLEMS IN $\mathbb{R}^n$

W. ALLEGRETTO

(Communicated by Barbara L. Keyfitz)

Abstract. We consider the problem $-\Delta u = \lambda gu$ in $\mathbb{R}^n$, $u \to 0$ at $\infty$ with $g$ a function that changes sign. Under suitable growth conditions on $g$ we show that this problem has an eigenvalue $\lambda$ with a positive solution $u$, as well as countably many other eigenvalues.

1. Introduction

Consider the problem

$$
\begin{align*}
-\Delta u &= \lambda gu \quad \text{in } \mathbb{R}^n, \\
u &\to 0 \quad \text{as } |x| \to \infty
\end{align*}
$$

where $\Delta$ denotes the formal Laplace operator and $g$ is a smooth bounded function that changes sign, i.e., $g$ is an indefinite weight function. A principal eigenvalue of (1) is a $\lambda$ for which (1) has a positive solution. The problem of the existence of such $\lambda$ was recently considered by Brown, Cosner, and Fleckinger [5]. They showed, in particular, the following results:

Theorem. (a) Suppose $n \geq 2$ and there exist $R > 0$, $K > 0$ such that $g(x) \leq -K$ for $|x| > R$. Then there exists a positive principal eigenvalue of (1) with eigenvector in $L^2(\mathbb{R}^n)$.

(b) Suppose $n \geq 3$ and there exist $K > 0$, $\alpha > 1$ such that $|g(x)| \leq K(1 + |x|^2)^{-\alpha}$. Then there exists a positive principal eigenvalue of (1).

In this paper we employ Sobolev’s embedding theorems and eigencurve arguments [2, 4] to show the following extensions of the above results:

Theorem 1. Suppose $n \geq 3$ and $g_+ \in L^{n/2}$. Then there exist infinitely many eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ of (1) with $\lambda_1$ a principal eigenvalue.

Corollary 1. If $n \geq 3$ and $g \in L^{n/2}$ then there exist two principal eigenvalues of (1) of opposite sign and infinitely many others $\lambda_j^\pm \to \pm \infty$ as $j \to \infty$. 

Received by the editors November 21, 1990 and, in revised form, March 18, 1991.

1991 Mathematics Subject Classification. Primary 35J20, 35P15.

Key words and phrases. Elliptic equation, indefinite weight, eigenvalue.

Research supported by NSERC (Canada).

©1992 American Mathematical Society

0002-9939/92 $1.00 + .25$ per page

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Next, following the definition in [3], we set

\[ M_{\alpha,1}(\omega) = \sup_{x \in \mathbb{R}^n} \int_{|y| < 1} \frac{|\omega(x-y)|}{|y|^{\alpha-n}}, \]

\[ M_{\alpha,1}(\omega, \Omega) = M_{\alpha,1}(\omega|_{\Omega}), \]

with \( \omega|_{\Omega}(x) = \omega(x) \) if \( x \in \Omega \) or \( \omega|_{\Omega}(x) = 0 \) if \( x \notin \Omega \). Let \( B_R = \{x| |x| < R\} \). We will show

**Theorem 2.** Suppose \( n \geq 3 \), \( g = g_1 - g_2 \) with \( g_1 > 0 \), \( g_2 \geq \varepsilon > 0 \), and \( M_{\alpha,1}(g_1, \mathbb{R}^n - B_R) \to 0 \) as \( R \to \infty \) for some \( 0 < \alpha < 2 \). Then the conclusion of Theorem 1 holds.

The procedure employed to obtain these results is very briefly as follows: Let \( \lambda > 0 \) denote a compact map on the space with norm \((-\Delta \varphi + \lambda g_2 \varphi, \varphi)\) for any \( \lambda > 0 \). This structure ensures the existence of eigenfunctions \( u \) and eigenvalues \( \lambda(X) : -\Delta u + \lambda g_2 u = \lambda(X) g_1 u \). We next show that \( k \) is continuous and that for some \( 0 < \lambda < \mu \) we have \( \lambda(X) - \lambda > 0 > \mu(X) - \mu \), whence \( \lambda(X) = \lambda \) for some \( \lambda \). Observe that we do not require that \( A \) be compact on the space associated with \((-\Delta \varphi, \varphi)\), although this may be the case in some specific examples. Consequently, results for \( n = 2 \) can also be obtained by these arguments. Specifically

**Theorem 3.** Suppose \( n = 2 \) and \( g = g_1 - g_2 \) with \( g_1 > 0 \), \( g_2 \geq \varepsilon > 0 \), \( g_1 \in L^1(\mathbb{R}^2) \). Then the conclusion of Theorem 1 holds.

We observe that Corollary 1 is identical in form to the classical result for bounded domains (see, e.g., [2]), but Theorems 2 and 3 are not. Indeed, under the conditions of Theorem 2 or 3, problem (1) cannot have a principal eigenvalue \( \lambda \leq 0 \) since, in this case, \(-\Delta u - \lambda g_1 u = 0 \) is oscillatory and there are no positive solutions of (1) near \( |x| = \infty \).

There clearly is a connection between these results and the criticality of \(-\Delta u - \lambda_1 g_1 u\). We comment on this briefly at the end of the paper.

In conclusion, we observe that applications and further references on indefinite weight problems may be found in [2, 5]. We have considered \(-\Delta\) in \( \mathbb{R}^n \) for convenience. Extension of the results to the more general case of a uniformly elliptic operator in unbounded domains is immediate.

### 2. Results

We consider first the case \( n \geq 3 \). Without loss of generality, set \( g = g_1 - g_2 \) with \( g_1, g_2 > 0 \) and \( g_1, g_2 \in L^\infty \). Let \( \lambda > 0 \) be chosen arbitrarily and define \( E_\lambda \) to be the completion of \( C_0^\infty(\mathbb{R}^n) \) with respect to the norm

\[ \|u\|_2^2 = \int_{\mathbb{R}^n} |\nabla u|^2 + \max(\lambda g_2, \omega) u^2 \]

where \( \omega = K/(1 + |x|^2) \) with \( K \) small and positive. As is well known, Hardy's Inequality implies \( \| \lambda \| \sim \| \lambda \|_{l_1} \) where

\[ \| \lambda \|_{l_1}^2 = \int_\Omega |\nabla u|^2 + \lambda g_2 u^2. \]

It is apparent that \( \| \lambda_1 \| \sim \| \lambda_2 \| \) for any positive \( \lambda_1, \lambda_2 \), so we denote \( E_\lambda, \| \lambda \| \) merely by \( E, \| \lambda \| \) in the following. Finally, we let \( L_2^{\omega} \) represent the weighted
$L^2$ space with weight $\omega$ and represent by $(\ ,\ )_i, (\ ,\ )_0$ the inner products associated with $\|\|$, $\|\|_i, L^2(R^n)$, respectively.

We recall the following preliminary results

**Lemma 0.** (1) If $u_n \to u$ in $E$ then $u_n \to u$ in $H^1(B_R)$ for any ball $B_R \subset R^n$.

(2) $E \leftarrow L^{2n/(n-2)}$, $E \leftarrow L^2_\omega$.

(3) Let $g_1 \in L^{n/2}$. The operator $A: E \to E^* \leftarrow E$ given by $A(\varphi) = g_1 \varphi$, i.e., $(g_1 \varphi, \psi)_0 = (A \varphi, \psi)_i$ for all $\psi \in E$, is compact and selfadjoint.

(4) If $u \in E$ satisfies $(u, \psi)_i = k(g_1 u, \psi)_0$ for some constant $k$ and any $\psi \in C_0^\infty(R^n)$, then $u \in C^2$, $-\Delta u + \lambda g_2 u = k g_1 u$, and $u \to 0$ at $\infty$.

**Proof.** The proof of (1) is immediate since $\|\| \sim \|\|_i$ and, in any ball $B_R$, $\max(\lambda g_2, \omega) > \omega > c > 0$ for some constant $c = c(R)$. Similarly for (2), we first employ Sobolev’s Inequality [6, Theorem 7.10, equation (7.26)] and obtain for $\varphi \in C_0^\infty(R^n)$

$$\|\varphi\|_{2n/(n-2)} \leq C_1 \|\nabla \varphi\|_2 \leq C_1 \|\varphi\|.$$ 

Since $C_1$ only depends on $n$, the same estimate holds for $u \in E$ and we have $E \hookrightarrow L^{2n/(n-2)}$. The second embedding follows in the same way from Hardy’s Inequality, specifically,

$$\int_{R^n} \omega \phi^2 \leq C_2 \int_{R^n} |\nabla \phi|^2$$

with $C_2$ also only a function of $n$ and $\varphi \in C_0^\infty(R^n)$. Next for (3), observe that the Holder Inequality

$$\int_{R^n} g \phi \psi \leq \|g\|_{n/2} \|\phi\|_{2n/(n-2)} \|\psi\|_{2n/(n-2)}$$

is valid for $g \in L^{n/2}(R^n)$, $\phi, \psi \in E$, and thus $A$ is defined. Suppose first that $g_1$ has compact support. (1) and (2) imply that $A(\varphi)$ is compact since $H^1$ is compactly embedded in $L^p(p < 2n/(n-2))$ for any ball. In general, set $g_r = g_1$ if $|x| < r$ and $g_r = 0$ otherwise and let $A_r(\varphi) \triangleq g_r \varphi$. Then $\|A_\varphi - A \varphi\| \leq K \|g_r - g_1\|_{n/2} \|\varphi\|$, i.e., $A_r \to A$ in the operator norm and $A$ is also compact. Note that, since $A$ is obviously symmetric and defined on the whole of $E$, it is selfadjoint. Finally for (4), if $u \in E$ then $u \in H_0^{1,loc}$ and thus $u$ is a weak solution of $-\Delta u + \lambda g_2 u = k g_1 u$. We apply [6, p. 243] and conclude that $u \in C^2$ since $g_1, g_2$ are smooth. In conclusion we note that by (2), $u \in L^{2n/(n-2)}$, and we recall [6, p. 194]

$$|u(x)| \leq K[\|u\|_{L^p(B_2(x))}]$$

for some constant $K$ and that $p = 2n/(n-2)$, whence $u \to 0$ at $\infty$.

**Proof of Theorem 1.** We show in detail only the existence of $\lambda_1$. The existence of the other $\lambda_i$ follows with obvious changes. We apply the Spectral Theorem and conclude for $\lambda > 0$ that

$$\sup_{\varphi \in E} \frac{(g_1 \varphi, \varphi)_0}{\|\varphi\|_i^2} \leq [k(\lambda)]^{-1}$$

is achieved by an eigenfunction $u \in E$ corresponding to the largest eigenvalue of $A$. Since $g_1 > 0$, we also have

$$k(\lambda) = \inf_{\varphi \in E} \frac{\|\varphi\|_i^2}{(g_1 \varphi, \varphi)_0} = \inf_{\varphi \in C_0^\infty} \left[ \frac{\|\varphi\|^2_i}{(g_1 \varphi, \varphi)_0} \right].$$
By Lemma 0(4), we conclude that \( u \) satisfies \( L_\lambda u = k(\lambda) g_1 u \), \( u \to 0 \) at \( \infty \), where \( L_\lambda \) denotes the formal differential expression associated with \( l_\lambda \). Since the functional being minimized is quadratic, and observing that if \( u \) belongs to \( E \) then so do \( u^+ \), \( u^- \), we may assume \( u \geq 0 \), for otherwise we replace \( u \) by \( |u| \), whence \( u > 0 \) by the maximum principle since zero cannot be a minimum of a nonnegative nontrivial solution of \( L_\lambda u = k(\lambda) g_1 u \). Observe that \( (g_1 \varphi, \varphi) = C(\|g_1\|_{\infty/2} \|\nabla \varphi\|_2^2) \leq C(\|g_1\|_{\infty/2} \|\varphi\|^2_2) \), where \( C \) is the Sobolev Embedding constant. From this, we obtain that \( k(\lambda) > \lambda \) for \( \lambda \) small. Next we observe that since

\[
k(\lambda) - \lambda = \inf_{\varphi \in C_0^\infty} \frac{(-\Delta \varphi + \lambda g_1 \varphi, \varphi)}{(g_1 \varphi, \varphi)}\]

and \( g < 0 \) in some subset, \( k(\lambda) < \lambda \) for \( \lambda \) large. We also note that \( k(\lambda) \) is locally Lipschitz in \( R^+ \). To see this, let \( h > 0 \) and observe

\[
k(\lambda + h) = \inf_{\varphi \in C_0^\infty} \frac{(-\Delta \varphi + (\lambda + h) g_2 \varphi, \varphi)}{(g_1 \varphi, \varphi)} = \frac{(\lambda + h)}{\lambda} \inf_{\varphi \in C_0^\infty} \frac{(-\Delta \varphi + \lambda g_2 \varphi, \varphi)}{(g_1 \varphi, \varphi)} = \left(1 + \frac{h}{\lambda}\right) k(\lambda).
\]

Since \( k(\lambda + h) \geq k(\lambda) \) by definition, we obtain

\[|k(\lambda_1) - k(\lambda_2)| \leq k(\lambda_1)|\lambda_1 - \lambda_2|/\lambda_1.
\]

We conclude that \( k(\lambda) = \lambda \) for some \( \lambda > 0 \) and the result follows.

**Proof of Theorem 2.** In view of the results given in [3], most of the arguments in the proofs of Lemma 0 and Theorem 1 need only to have \( \|g_1\|_{L^{n/2}(R^n - B)} \), \( \|g_1\|_{L^{n/2}(R^n - B)} \) substituted by \( M_{\alpha,1}(g_1, B_R) \), \( M_{\alpha,1}(g_1, R^n - B_R) \), respectively. Note that \( E \sim H^1 \) here and that for \( \lambda > 0 \) compactness of \( g_1 \) and continuity of \( k(\lambda) \) follow as before, as does \( k(\lambda) - \lambda \to -\infty \) as \( \lambda \to \infty \). To see that \( k(\lambda) > \lambda \) for some \( \lambda > 0 \), observe that the embedding theorems given in [3, Theorem 2.3] show for \( \varphi \in C_0^\infty(R^n) \)

\[
(g_1 \varphi, \varphi) = \int_{B_R} g_1 \varphi^2 + \int_{R^n - B_R} g_1 \varphi^2 \\
\leq C(\|\nabla \varphi\|_{L^2(B_R)}^2 + \|\varphi\|_{L^2(B_R)}^2) \\
+ C M_{\alpha,1}(g_1, R^n - B_R)^2(\|\nabla \varphi\|_{L^2(R^n)}^2 + \|\varphi\|_{L^2(R^n)}^2) \\
\leq C (1 + R^2) + \frac{C M_{\alpha,1}(g_1, R^n - B_R)^2}{\lambda} \|\varphi\|_{L^2}^2
\]

for some constant \( C \) independent of \( \lambda \) and \( 0 < \lambda \) small. Here we have employed the equivalence \( \|\| \sim \|\|_\lambda \). We thus conclude:

\[
k(\lambda) \geq \left[ C(1 + R^2) + \frac{C M_{\alpha,1}(g_1, R^n - B_R)^2}{\lambda} \right]^{-1}.
\]

Finally, we select \( R \) so that \( C M_{\alpha,1}(g_1, R^n - B_R)^2 < 1 \) and, consequently, obtain \( k(\lambda) > \lambda \) for some \( 0 < \lambda \) small enough. To see the decay of the eigenfunction observe that \( u \in L^2 \) and \( L_\lambda u = -\Delta u - \lambda g_1 u = 0 \). Since the
essential spectrum of the selfadjoint operator $L : L^2 \rightarrow L^2$ thus generated is contained in the interval $[\lambda e, \infty)$, we conclude that $u$ is an eigenvector of $L_\lambda$ with eigenvalue below the essential spectrum and the decay follows.

**Proof of Theorem 3.** Let $\varphi \in C_0^\infty(R^2)$, $\psi \in C_0^\infty(-a, a)$ with $\int_{-a}^a \psi^2 = 1$, and $a > 0$ to be chosen below. Set

$$\tau(x, y, z) = \varphi(x, y)\psi(z), \quad h(x, y, z) = g_1(x, y) \cdot \chi_{(-a, a)}(z)$$

where $\chi_{(-a, a)}(z) = 1$ if $|z| < a$ and 0 otherwise, and note $\tau \in C_0^\infty(R^2 \times (-a, a))$. We apply the embedding estimates for $R^3$ and conclude

$$\langle h\tau, \tau \rangle_0 \leq C \| h \|_{L^{3/2}(R^3)} \| \nabla \tau \|^2_{L^2(R^3)}.$$  

Expanding (2), we obtain

$$\left( \int_{R^2} g_1 \varphi^2 \right) \left( \int_{-a}^a \psi^2 \right) \leq C \left( \int_{R^2} g_1^{3/2} \right)^{2/3} (2a)^{2/3} \left( \int_{R^2} \nabla \varphi \right)^{2/3} \left( \int_{R^2} \psi^2 \right)^{2/3} + \left( \int_{R^2} \varphi^2 \right) \left( \int_{-a}^a \psi^2 \right),$$

i.e.,

$$\int_{R^2} g_1 \varphi^2 \leq C \left( \int_{R^2} g_1^{3/2} \right)^{2/3} a^{2/3} \left( \int_{R^2} \nabla \varphi \right)^{2/3} + \left( \int_{R^2} \varphi^2 \right) \left( \int_{-a}^a \psi^2 \right)^{2/3}.$$

This inequality clearly also applies to any smooth normalized (in $L^2(-a, a)$) function $\psi$ with $\psi(-a) = \psi(a) = 0$. In particular, we may choose $\psi$ to be the first eigenfunction of $-\psi'' = \theta \psi$, $\psi(-a) = \psi(a) = 0$, i.e., $\psi = \sin[\pi(x + a)/2a]/(\sqrt{a})$. Substituting this $\psi$ into equation (3) yields in $R^2$

$$\langle g_1 \varphi, \varphi \rangle_0 \leq C \| g_1 \|_{L^{3/2}(R^2)} \cdot a^{2/3} \left( \int_{R^2} \nabla \varphi \right)^{2} + \frac{\pi^2}{4a^2} \varphi^2.$$

We now choose $\lambda e = \pi^2/4a^2$ and obtain

$$\langle g_1 \varphi, \varphi \rangle_0 \leq C \| g_1 \|_{L^{3/2}(R^2)} \lambda^{-1/3} \| \varphi \|^2_{L^2}.$$

Compactness arguments, continuity of $k(\lambda)$ and $k(\lambda) - \lambda \rightarrow -\infty$, and decay of any eigenfunction follow as before. Finally we observe that $k(\lambda) \geq [C \| g_1 \|_{L^{3/2}(R^2)}]^{-1} \lambda^{1/3}$, whence $k(\lambda) > \lambda$ for $0 < \lambda$ small, and, therefore, $k(\lambda) = \lambda$ for some $0 < \lambda$.

We conclude by briefly and heuristically connecting our results to the question of the criticality of the operator formally given by $L \varphi = -\Delta \varphi - \lambda_1 g \varphi$, see [1, 7, 8] for definitions and background discussion of this and related concepts. Specifically, we point out that above we have constructed critical operators $L$ with decaying positive solutions. We recall that critical or subcritical operators need not have such solutions, e.g., $-\Delta$ in $R^n$, so that no conclusions can be drawn in general about the decay of positive solutions of critical or subcritical operators. To see the criticality of $L$, observe that $\langle L \varphi, \varphi \rangle_0 \geq 0$ for any $\varphi \in C_0^\infty$, whence $L$ is either critical or subcritical. Suppose that $L$ satisfies the
“λ-property”: for some function \( h > 0 \) and any \( \phi \in C_0^\infty \) we have \((L\phi, \phi)_0 \geq (h\phi, \phi)_0\). In such a case, we construct a positive function \( v \) such that \( Lv - hv = 0 \) and apply Picone’s Identity to get
\[
\int_{R^n} v^2 \left| \nabla \left( \frac{\phi}{v} \right) \right|^2 = \int_{R^n} \phi L\phi - \int_{R^n} \frac{\phi^2}{v} Lv,
\]
i.e.,
\[
\int_B \phi^2 h \leq \int_{R^n} \phi^2 h + \int_{R^n} v^2 \left| \nabla \left( \frac{\phi}{v} \right) \right|^2 = \int_{R^n} \phi L\phi
\]
for any fixed ball \( B \) and \( \phi \in C_0^\infty \).

We let \( \phi \to u \) in \( E \) and conclude
\[
\int_B u^2 h \leq \int_{R^n} u Lu = 0,
\]
i.e., \( u \equiv 0 \). This shows that \( L \) does not satisfy the “λ-property” and thus must be critical.

**References**


**Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1 Canada**