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REGULAR AND PURELY IRREGULAR BOUNDED CHARGES: A DECOMPOSITION THEOREM

BRUNO GIROTTO AND SILVANO HOLZER

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ABSTRACT. We introduce the notions of regular and purely irregular charges with respect to a pair of pavings and study their structural properties. Moreover, we link regularity and σ-additivity, obtaining some generalizations of well-known theorems. Finally, when the pavings satisfy some reasonable weak conditions, we can decompose any bounded charge into regular and purely irregular decomposants; this decomposition becomes the Hewitt-Yosida one, whenever the charges are defined on the Baire σ-field of a countably compact space.

1. INTRODUCTION

Several decompositions of finitely additive probabilities are given in the literature: the Hewitt-Yosida decomposition (into σ-additive and pure decomposants), the Sobczyk-Hammer decomposition (into atomic and nonatomic decomposants), the diffuse-discrete decomposition (into diffuse and discrete decomposants), the Lebesgue decomposition (into singular and absolutely continuous decomposants w.r.t. a given probability), the de Finetti decomposition (into discrete, continuous and agglutinated decomposants), and the Armstrong-Sudderth decomposition (into nearly strategic and purely nonstrategic decomposants).

In this paper we give another decomposition theorem. Precisely, by a general notion of regularity, we show that any bounded charge can be expressed uniquely as a sum of its regular and purely irregular decomposants. This decomposition becomes the Hewitt-Yosida one whenever the charges are defined on the Baire σ-field of a countably compact space. Our decomposition theorem, as well as other ones, is derived by means of a basic and elementary algebraic tool, i.e., the Riesz Decomposition Theorem for vector lattices. We remark that an alternative geometric approach to decompositions of finitely additive probabilities is given by the split face decompositions of the simplex of finitely additive probabilities [3, §2].

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Now, we briefly describe the contents of the following sections. In §2 we study properties of any normal subspace decomposition (S_1, S_2) of $\mathrm{ba}(\Omega, \mathscr{F})$. Moreover, we prove that any 0-k valued charge must belong to S_1 or S_2. In §3 we introduce the notion of regularity w.r.t. a pair of pavings. We show that the set of regular bounded charges is a normal vector sublattice of $\mathrm{ba}(\Omega, \mathscr{F})$ whenever the pavings satisfy some reasonable weak conditions. We also link regularity and σ-additivity and obtain general results, having some well-known regularity theorems as particular cases. In §4 we introduce purely irregular charges w.r.t. a pair of pavings and give some simple characterizations of them. Moreover we link pure irregularity and antiregularity obtaining, in this way, a known decomposition theorem as a particular case.

2. Preliminaries

Throughout this paper we adopt set theoretic and topological notation used in Preliminaries of [4] (particularly §1.5).

The letter $\mathscr{F}$ always denotes a field on a nonempty set Ω; sets from $\mathscr{F}$ are denoted by F, with or without indices. As usual, $\mathrm{ba}(\Omega, \mathscr{F})$ is the set of bounded charges on $\mathscr{F}$; elements from it are denoted by μ, ν, with or without indices. It is known that $\mathrm{ba}(\Omega, \mathscr{F})$, with the usual pointwise addition, scalar multiplication, and ordering $\leq$, is a boundedly complete real vector lattice [4, 2.2.1(9)]. Moreover, the structure $(\mathrm{ba}(\Omega, \mathscr{F}), \leq, \|\cdot\|)$ is a Banach lattice [4, 2.2.1(11)], where $\|\cdot\|$ denotes the usual norm on $\mathrm{ba}(\Omega, \mathscr{F})$ (i.e., $\|\mu\| = |\mu|(\Omega)$, with $|\mu| = \mu^+ + \mu^-$).

As usual, a vector sublattice S of $\mathrm{ba}(\Omega, \mathscr{F})$ is called *normal* if the following two conditions hold: S contains along with μ any element ν such that $0 \leq |\nu| \leq |\mu|$; the supremum of any nonempty family in S belongs to S, whenever it exists. It is known that any normal vector sublattice of $(\mathrm{ba}(\Omega, \mathscr{F}), \leq, \|\cdot\|)$ is a closed set [4, 1.5.19]. Given $T \subset \mathrm{ba}(\Omega, \mathscr{F})$, the *orthogonal complement* of T is the set $T^\perp = \{\nu : |\nu| \wedge |\mu| = 0 \text{ for all } \mu \in T\}$; it is known that $T^\perp$ is a normal vector sublattice and that, whenever T is a normal vector sublattice, $T = (T^\perp)^\perp$ [4, 1.5.8, 1.5.11].

By a *normal subspace decomposition* of $\mathrm{ba}(\Omega, \mathscr{F})$ (direct decomposition in [5]) is meant a choice of two complementary normal vector sublattices, i.e., a normal vector sublattice S and the orthogonal complement $S^\perp$.

The next theorem gives some interesting properties of any normal subspace decomposition (S_1, S_2). The first proposition gives a characterization of any element of S_i in terms of its positive and negative variations or of its total variation. The third, fourth, and fifth ones point out some algebraic and convergence closure properties of S_i. The sixth one shows that S_i is a convex set with respect to $\leq$. The eighth one points out that any positive charge in S_1 and any positive charge in S_2 are singular. The last one shows that elements of S_1 and S_2 are, so to say, as "unlike" as possible.

2.1. **Theorem.** *Let* (S_1, S_2) *be a normal subspace decomposition of* $\mathrm{ba}(\Omega, \mathscr{F})$. *Then the following propositions hold* $(i, j = 1, 2)$:

(i) $\mu \in S_i$ *iff* $\mu^+, \mu^- \in S_i$ *iff* $|\mu| \in S_i$.

(ii) *Let* μ *be a positive charge. Then* $\mu \in S_i$ *iff* $\mu \wedge \nu = 0$ *for any positive* $\nu \in S_j$ $(j \neq i)$.

(iii) *Let* $\mu_1, \mu_2 \in S_i$ *and* α *be any real number. Then* $\mu_1 + \mu_2$, $\mu_1 \vee \mu_2$, $\mu_1 \wedge \mu_2$, *and* $\alpha\mu_1$ *are elements of* S_i.

(iv) *Let* (μ_n) *be a sequence in* S_i *such that* $\lim_{n\to\infty} \|\mu_n - \mu\| = 0$. *Then* $\mu \in S_i$.

(v) *The pointwise limit of any bounded increasing net in* S_i *is an element of* S_i.

(vi) *Let* $\mu_1, \mu_2 \in S_i$ *and* ν *be such that* $\mu_1 \le \nu \le \mu_2$. *Then* $\nu \in S_i$.

(vii) *Let* $\mu \in S_i$ *and* $\nu \in S_j$ $(j \neq i)$ *and* α, β *be arbitrary positive real numbers. Then there is* F *such that* $|\mu|(F) < \alpha$ *and* $|\nu|(F^c) < \beta$.

(viii) *Let* μ *be a positive charge. Then* $\mu \in S_i$ *iff for every positive* $\nu \in S_j$ $(j \neq i)$, *for every* F *and* $\varepsilon > 0$, *there is* $F_1 \subset F$ *such that* $\nu(F_1) < \varepsilon$ *and* $\mu(F_1) > \mu(F) - \varepsilon$.

(ix) $\mu \in S_i$ *iff there is no nonzero* $\nu \in S_j$ $(j \neq i)$ *such that* $0 \le \nu \le |\mu|$.

Proof. (i)–(iv) are obvious. (v) immediately follows on noting that, by the properties of Moore-Smith limits, the pointwise limit of the net is a bounded charge and hence it is equal to the supremum of the net in $(\mathrm{ba}(\Omega, \mathscr{F}), \le)$. (vi) follows from 2.2.2(6) of [4]. (vii) and (viii) can be proved by noting that $(\mu \wedge \nu)(F) = \mathrm{Inf}\{\mu(F_1) + \nu(F - F_1): F_1 \subset F\}$ for all F [4, 2.2.1(6)].

(ix) Let $\mu \in S_i$ and $\nu \in S_j$ $(j \neq i)$ such that $0 \le \nu \le |\mu|$. Then we have $\nu = |\mu| \wedge |\nu| = 0$. Conversely, let μ be such that, for any $\nu \in S_j$, $0 \le \nu \le |\mu|$ implies $\nu = 0$. Let $\nu \in S_j$. Since $0 \le |\mu| \wedge |\nu| \le |\nu|$, we get $|\mu| \wedge |\nu| \in S_j$ and hence $|\mu| \wedge |\nu| = 0$, on noting that $0 \le |\mu| \wedge |\nu| \le |\mu|$. Therefore $\mu \in S_i$. □

The next *normal decomposition theorem* immediately follows from the Riesz Decomposition Theorem [4, 1.5.10] in which we take $L = \mathrm{ba}(\Omega, \mathscr{F})$.

2.2. **Theorem.** *Let* (S_1, S_2) *be a normal subspace decomposition of* $\mathrm{ba}(\Omega, \mathscr{F})$. *Then any charge* μ *can be written uniquely in the form* $\mu = \mu_1 + \mu_2$, *where* $\mu_i \in S_i$ $(i = 1, 2)$. *Further* $\mu_i = (\mu^+)_i - (\mu^-)_i$ *for all* μ $(i = 1, 2)$. *Moreover,* μ_1 *and* μ_2 *are both positive whenever* μ *is a positive charge.*

Finally, we prove that in any normal subspace decomposition, one decomposant of a 0-k valued charge must be zero.

2.3. **Theorem.** *Let* (S_1, S_2) *be a normal subspace decomposition of* $\mathrm{ba}(\Omega, \mathscr{F})$. *Moreover, let* μ *be a* 0-k *valued charge. Then the following propositions hold*:

(i) μ *is either an element of* S_1 *or an element of* S_2.

(ii) $\mu \in S_i$ *whenever there is a nonzero charge* $\nu \in S_i$ *such that* $0 \le \nu \le |\mu|$ $(i = 1, 2)$.

Proof. (i) In view of 2.1(iii), we can consider 0-1 valued charges, only. Let μ be a 0-1 valued charge such that $\mu \notin S_1$. From 2.2, $\mu = \mu_1 + \mu_2$ with $0 \le \mu_i \in S_i$ $(i = 1, 2)$. Plainly $\|\mu_2\| \neq 0$. Now, we prove that $\mu_1 = 0$, i.e., $\mu \in S_2$. Given F, first we assume that $\mu_2(F) = 0$. Then $\mu_2(F^c) > 0$ and hence $\mu(F^c) = 1$; consequently, $\mu(F) = 0$ and whence $\mu_1(F) = 0$. Now, we assume $\mu_2(F) > 0$. Let $0 < \varepsilon < \mu_2(F)$. Since $0 = (\mu_2 \wedge \mu_1)(F) = \mathrm{Inf}\{\mu_2(F_1) + \mu_1(F - F_1): F_1 \subset F\}$ [4, 2.2.1(6)], there is $F_1 \subset F$ such that $\alpha = \mu_2(F_1) + \mu_1(F - F_1) < \varepsilon$. Then $\mu_2(F_1) < \varepsilon$ and hence $\mu_2(F - F_1) > 0$, on noting that $\varepsilon < \mu_2(F) = \mu_2(F_1) + \mu_2(F - F_1)$. Therefore $\mu(F - F_1) = 1$; consequently, $\mu(F) = 1 + \mu(F_1)$ and whence $\mu(F_1) = 0$, seeing that μ is a 0-1 valued charge. Then $\mu_1(F_1) = 0$ and hence, keeping in mind $\alpha < \varepsilon$,

$\mu_1(F) = \mu_1(F - F_1) < \varepsilon$. Since ε is arbitrarily chosen, we get $\mu_1(F) = 0$. Since F is arbitrary, it follows that $\mu_1 = 0$.

(ii) It easily follows from (i) and 2.1(ix). □

3. Regular charges w.r.t. a pair of pavings

The letters $\mathscr{H}$, $\mathscr{K}$ always denote *pavings of* $\mathscr{F}$ (i.e., nonempty subsets of $\mathscr{F}$); sets from $\mathscr{H}, \mathscr{K}$ are denoted by H, K, with or without indices, respectively. To point out the closure properties of set theoretic operations in a paving, we adopt Topsøe's notation [9, p. IX].

Now we introduce a notion of regularity based on the possibility of approximating a bounded charge by means of its values on two given pavings of $\mathscr{F}$.

3.1. **Definition.** The charge μ is a $(\mathscr{H}, \mathscr{K})$*-regular charge* iff for all $\varepsilon > 0$ and for all F there are $H \subset F$ and $K \supset F$ such that $|\mu(F) - \mu(H)| < \varepsilon$ and $|\mu(F) - \mu(K)| < \varepsilon$. We denote by $(\mathscr{H}, \mathscr{K})\text{-ra}(\Omega, \mathscr{F})$ the set of $(\mathscr{H}, \mathscr{K})$-regular charges. Moreover, adopting the usual terminology, we call *inner* $\mathscr{H}$*-regular charges* and *outer* $\mathscr{K}$*-regular charges* the elements of $\mathscr{H}\text{-ra}_*(\Omega, \mathscr{F}) = (\mathscr{H}, \mathscr{F})\text{-ra}(\Omega, \mathscr{F})$ and $\mathscr{K}\text{-ra}^*(\Omega, \mathscr{F}) = (\mathscr{F}, \mathscr{K})\text{-ra}(\Omega, \mathscr{F})$, respectively.

The following proposition easily follows from the previous definition.

3.2. **Proposition.** (i) $(\mathscr{H}, \mathscr{K})\text{-ra}(\Omega, \mathscr{F}) = \mathscr{H}\text{-ra}_*(\Omega, \mathscr{F}) \cap \mathscr{K}\text{-ra}^*(\Omega, \mathscr{F})$.

(ii) *Let* $\mathscr{H}^c$ *be the set of complements of the elements of* $\mathscr{H}$. *Then* $\mathscr{H}\text{-ra}_*(\Omega, \mathscr{F}) = \mathscr{H}^c\text{-ra}^*(\Omega, \mathscr{F}) = (\mathscr{H}, \mathscr{H}^c)\text{-ra}(\Omega, \mathscr{F})$ *and* $\mathscr{H}\text{-ra}^*(\Omega, \mathscr{F}) = \mathscr{H}^c\text{-ra}_*(\Omega, \mathscr{F}) = (\mathscr{H}^c, \mathscr{H})\text{-ra}(\Omega, \mathscr{F})$ (*duality property*).

The next theorem points out that the set of regular charges is a normal vector sublattice whenever the pavings satisfy some reasonable weak conditions.

3.3. **Theorem.** *Let* $\mathscr{H}$ *be a* $(\varnothing, \bigcup f)$*-paving and* $\mathscr{K}$ *a* $(\Omega, \bigcap f)$*-paving. Then the set* $S = (\mathscr{H}, \mathscr{K})\text{-ra}(\Omega, \mathscr{F})$ *is a normal vector sublattice of* $\text{ba}(\Omega, \mathscr{F})$. *Consequently,* S *is a closed subspace of* $(\text{ba}(\Omega, \mathscr{F}), \leq, \|\cdot\|)$ *and is closed under pointwise convergence of bounded increasing nets; moreover,* $(S, \leq)$ *is a boundedly complete vector lattice and also a Banach lattice with the norm* $\|\cdot\|$.

Proof. The proof is carried out in the following steps.

1°. We claim that $\mu^+, \mu^- \in T = \mathscr{H}\text{-ra}_*(\Omega, \mathscr{F})$ whenever $\mu \in T$. Let $\mu \in T$. Let $\varepsilon > 0$. Given F, by $\mu^+(F) = \text{Sup}\{\mu(F_1): F_1 \subset F\}$ [4, 2.2.1(5)], there is $F_1 \subset F$ such that $\mu^+(F) - \mu(F_1) < \varepsilon$. Since $\mu \in T$, there is $H \subset F_1$ such that $|\mu(F_1) - \mu(H)| < \varepsilon$. Then

$$0 \leq \mu^+(F) - \mu(H) \leq [\mu^+(F) - \mu(F_1)] + |\mu(F_1) - \mu(H)| < 2\varepsilon$$

and hence $0 \leq \mu^+(F) - \mu^+(H) < 2\varepsilon$, on noting that $\mu(H) \leq \mu^+(H)$. Therefore $\mu^+ \in T$. Replacing in this proof $\mu(F_1)$ with $-\mu(F_1)$ we get $\mu^- \in T$. This proves the claim.

2°. Let $\mathscr{H}_F = \{H: H \subset F\}$. Since $\mathscr{H}$ is a paving closed under finite unions, $(\mathscr{H}_F, \supset)$ is a directed set; therefore, $\{\mu(H): H \in \mathscr{H}_F\}$ is a real net for all F and μ. We claim that $\mu \in T$ iff $\mu(F) = \lim_{H \in \mathscr{H}_F} \mu(H)$ for all F. If μ is positive, the claim immediately follows from the monotonicity of μ. Let μ be an arbitrary inner $\mathscr{H}$-regular charge. Then, by 1°, $\mu^+, \mu^- \in T$ and hence, by 2.2.2(1) of [4], we get

$$\lim_{H \in \mathscr{H}_F} \mu(H) = \lim_{H \in \mathscr{H}_F} \mu^+(H) - \lim_{H \in \mathscr{H}_F} \mu^-(H) = \mu^+(F) - \mu^-(F) = \mu(F).$$

The other implication is obvious. Thus the claim is established.

3°. We claim that T is a vector sublattice of $(\mathrm{ba}(\Omega, \mathscr{F}), \leq)$. It follows from 1°, 2°, and 1.5.4(16),(17) of [4], on noting that the zero charge is inner $\mathscr{H}$-regular ($\varnothing \in \mathscr{H}$!).

4°. We claim that $|\nu| \leq |\mu|$ implies $\nu \in T$ whenever $\mu \in T$. Let $\mu \in T$ and $|\nu| \leq |\mu|$. Let $\varepsilon > 0$. Given F, by 3°, there is $H \subset F$ such that $|\mu|(F) - |\mu|(H) < \varepsilon$. Hence we have

$$|\nu(F) - \nu(H)| = |\nu(F - H)| \leq |\nu|(F - H) \leq |\mu|(F - H) = |\mu|(F) - |\mu|(H) < \varepsilon.$$

Therefore $\nu \in T$. This proves the claim.

5°. Let $(D, \geq^*)$ be a directed set and $\{\mu_d : d \in D\}$ an increasing net in T such that $\mu_d \leq \mu$ for all $d \in D$ and for some μ. We claim that the pointwise limit $\varphi = \lim_{d\in D} \mu_d$ is an inner $\mathscr{H}$-regular charge. Plainly $\varphi \in \mathrm{ba}(\Omega, \mathscr{F})$. Now we prove $\varphi \in T$. First, we assume that each μ_d is a positive charge. Since the real net $\{\mu_d(F) : d \in D\}$ is an increasing net with bounded range, $\varphi(F) = \mathrm{Sup}_{d\in D}\, \mu_d(F)$ for all F. Then, by $0 \leq \mu_d \in T$ $(d \in D)$ and 2°, we get

$$\begin{aligned}\varphi(F) &= \operatorname*{Sup}_{d\in D} \mu_d(F) = \operatorname*{Sup}_{d\in D}\, \operatorname*{Sup}_{H\in\mathscr{H}_F} \mu_d(H) \\ &= \operatorname*{Sup}_{H\in\mathscr{H}_F}\, \operatorname*{Sup}_{d\in D} \mu_d(H) = \operatorname*{Sup}_{H\in\mathscr{H}_F} \varphi(H).\end{aligned}$$

Since F is arbitrarily chosen and φ is positive, by 2°, we get $\varphi \in T$.

Now we treat the general case. Let $d_0 \in D$ and $D_0 = \{d \in D : d \geq^* d_0\}$. Since $\{\mu_d : d \in D\}$ is an increasing net in T, the net $\{\mu_d - \mu_{d_0} : d \in D_0\}$ is an increasing net of positive charges in T. Keeping in mind what we have proved above, the pointwise limit $\lim_{d\in D_0}[\mu_d - \mu_{d_0}] = \lim_{d\in D} \mu_d - \mu_{d_0} = \varphi - \mu_{d_0}$ belongs to T. Hence, by 3°, $\varphi \in T$. This proves the claim.

6°. We claim that T is a normal vector sublattice of $(\mathrm{ba}(\Omega, \mathscr{F}), \leq)$. By what we have proved above, we must verify only that the supremum of any subset of T is an element of T whenever it exists in $\mathrm{ba}(\Omega, \mathscr{F})$. Let $\{\mu_i : i \in I\} \subset T$ such that $\nu = \bigvee_{i\in I} \mu_i \in \mathrm{ba}(\Omega, \mathscr{F})$. Moreover, let $\mathscr{A}$ be the set of finite subsets of I. Plainly, $(\mathscr{A}, \supset)$ is a directed set. For each $A \in \mathscr{A}$, let $\mu_A = \bigvee_{i\in A} \mu_i$; then, by 3°, the net $\{\mu_A : A \in \mathscr{A}\}$ is an increasing net in T that is bounded above. Hence, by 5°, the pointwise limit $\varphi = \lim_{A\in\mathscr{A}} \mu_A \in T$ and whence $v \in T$, on noting that $\nu = \varphi$ (see the proof of 2.2.1(9) of [4]). Thus the claim is established.

7°. We claim that S is normal vector sublattice of $(\mathrm{ba}(\Omega, \mathscr{F}), \leq)$. It easily follows from 6° and 3.2, on noting that $\mathscr{K}^c$ is a $(\varnothing, \bigcup f)$-paving and intersections of normal vector sublattices are normal vector sublattices as well.

Finally, keeping in mind 2.1(v), (vi), we get the last statement of the theorem. □

3.4. *Remark.* (i) We note that in proving 3.3 we have used the hypothesis $\varnothing \in \mathscr{H}$ and $\Omega \in \mathscr{K}$ to assure $(\mathscr{H}, \mathscr{K})\text{-ra}(\Omega, \mathscr{F}) \neq \varnothing$, only. Therefore, in 3.3 and in the sequel, we may replace the hypothesis $\varnothing \in \mathscr{H}$ and $\Omega \in \mathscr{K}$ with $(\mathscr{H}, \mathscr{K})\text{-ra}(\Omega, \mathscr{F}) \neq \varnothing$.

(ii) Let Ω be a topological space. Then the previous theorem assures that the following sets of regular charges, already considered in the literature, though in different contexts, are normal vector sublattices of $\mathrm{ba}(\Omega, \mathscr{F})$:

- the set of *Caratheodory-Alexandroff regular charges,* i.e., the set $\mathscr{H}$-ra$_*(\Omega, \mathscr{F})$, where $\mathscr{H}$ is the $(\varnothing, \bigcup f)$-paving of closed sets [1, Definition 1(3), p. 567];
- the sets of *strongly compact inner regular charges, outer regular charges,* and *compact regular ones,* i.e., the sets $\mathscr{H}$-ra$_*(\Omega, \mathscr{F})$, $\mathscr{K}$-ra$^*(\Omega, \mathscr{F})$, and $(\mathscr{H}, \mathscr{K})$-ra$(\Omega, \mathscr{F})$, where $\mathscr{H}$ is the $(\varnothing, \bigcup f)$-paving of compact sets and $\mathscr{K}$ the $(\Omega, \bigcap f)$-paving of open sets;
- the sets of *inner closed G_δ-regular charges, outer open F_σ-regular charges,* and *inner closed G_δ and outer open F_σ-regular charges,* i.e., the sets $\mathscr{H}$-ra$_*(\Omega, \mathscr{F})$, $\mathscr{K}$-ra$^*(\Omega, \mathscr{F})$ and $(\mathscr{H}, \mathscr{K})$-ra$(\Omega, \mathscr{F})$, where $\mathscr{H}$ is the $(\varnothing, \bigcup f)$-paving of closed G_δ's and $\mathscr{K}$ is the $(\Omega, \bigcap f)$-paving of open F_σ's.

Finally, we note that tightness and radonness may also be seen as regularity conditions w.r.t. suitable pavings in the topological space Ω.

Now we are going to study the relationship between regular charges and measures (i.e., σ-additive charges). The following theorem characterizes measures in the regular charge setting.

3.5. **Theorem.** *Let $\mathscr{H}$, $\mathscr{K}$ be $(\bigcup f, \bigcap f)$-pavings and $\mu \in (\mathscr{H}, \mathscr{K})$-*ra$(\Omega, \mathscr{F})$. *Then the following statements are equivalent.*

(i) *μ is a measure,*
(ii) *μ is σ-smooth at $\varnothing$ w.r.t. $\mathscr{H}$ (i.e., $\lim_{n\to+\infty} \mu(H_n) = 0$, whenever $H_n \downarrow \varnothing$),*
(iii) *μ is σ-tight at Ω w.r.t. $\mathscr{K}$ (i.e., $\lim_{n\to+\infty} \mu(K_n) = \mu(\Omega)$, whenever $K_n \uparrow \Omega$).*

Proof. (i) $\Rightarrow$ (ii) and (iii). It is obvious.

(ii) $\Rightarrow$ (i). We use the criterion for σ-additivity given in 2.3.2(iii) of [4]. Let $F_n \downarrow \varnothing$. We claim $\mu(F_n) \to 0$. Let $\varepsilon > 0$. Given n, by 2.1(i), 3.3, 3.4(i), and regularity of μ, there is $H_n^* \subset F_n$ such that $|\mu|(F_n - H_n^*) < 2^{-n}\varepsilon$. Let $H_n = \bigcap_{i=1}^n H_i^* \in \mathscr{H}$ for all n. Plainly $H_n \subset F_n$ for all n. Since $F_n = \bigcap_{i=1}^n F_i$, we get $F_n - H_n \subset \bigcup_{i=1}^n (F_i - H_i^*)$ and hence $|\mu(F_n) - \mu(H_n)| = |\mu(F_n - H_n)| \leq |\mu|(F_n - H_n) \leq \sum_{i=1}^n |\mu|(F_i - H_i^*) < \varepsilon$. Since $H_n \downarrow \varnothing$ and μ is σ-smooth at $\varnothing$ w.r.t. $\mathscr{H}$, we get $\mu(H_n) \to 0$, and hence there is m such that $|\mu(F_n)| < 2\varepsilon$ for all $n > m$. This shows that $\mu(F_n) \to 0$.

(iii) $\Rightarrow$ (i). It easily follows from (ii) $\Rightarrow$ (i) and 3.2, on noting that $\mathscr{K}^c$ is a $(\bigcup f, \bigcap f)$-paving and μ is σ-smooth at $\varnothing$ w.r.t. $\mathscr{K}^c$. □

3.6. *Remark.* (i) If $\mathscr{K} = \mathscr{H}^c$ then (ii) and (iii) of 3.5 are equivalent.

(ii) Plainly, if $\mathscr{H}$ is a σ-compact set system [7, 1.4, p. 198], then any μ is σ-smooth at $\varnothing$. Hence, 1.6 of [7, p. 198], may be seen as a consequence of 3.5 (put $\mathscr{K} = \mathscr{F}$) in the context of σ-compact set systems that are $(\bigcup f, \bigcap f)$-pavings of $\mathscr{F}$.

(iii) Let $\mathscr{H}$ be the set of closed sets of a topological space Ω. Then 3.5 becomes Theorem 2 of [1, p. 587]. Moreover, if Ω is a countably compact space, then $\mathscr{H}$ is a σ-compact set system and hence, by (ii), any $(\mathscr{H}, \mathscr{H}^c)$-regular charge is a measure.

(iv) Let Ω be a topological space, $\mathscr{H}$ a $(\bigcup f, \bigcap f)$-paving of closed countably compact subsets of Ω and $\mathscr{K}$ a $(\bigcup f, \bigcap f)$-paving. Then any $(\mathscr{H}, \mathscr{K})$-regular charge is a measure, on noting that $\mathscr{H}$ is a σ-compact set system.

(v) Theorem 3.5 points out that we may replace in 2.2 of [9] the countable intersection closure property of the paving with the finite intersection one.

Since not any bounded measure is an inner regular charge (e.g., let Ω be the compact Hausdorff space of ordinals less than or equal to the first uncountable ordinal, endowed with the order topology; then the Dieudonné measure defined on the Borel σ-field of Ω is not a strongly compact inner regular charge [8, (9–10)]), we give some conditions assuring the regularity of bounded measures. In what follows we denote, as usual, by $\sigma(\mathscr{H})$ the smallest σ-field on Ω including $\mathscr{H}$.

3.7. **Theorem.** *Let $\mathscr{H}$ be a $\bigcup f$-paving and $\mathscr{K}$ be a $(\bigcap f, \bigcup c)$-paving such that $H-K \in \mathscr{H}$ and $K-H \in \mathscr{K}$ for all H, K. Moreover, let μ be a measure. Then the set $\mathscr{R}^{(\mathscr{H},\mathscr{K})}(\mu) = \{F\colon \operatorname{Sup}_{H\subset F} |\mu|(H) = \operatorname{Inf}_{K\supset F} |\mu|(K)\}$ is a σ-ring whenever $\mathscr{R}^{(\mathscr{H},\mathscr{K})}(\mu) \neq \varnothing$.*

Consequently, $\mathscr{R}^{(\mathscr{H},\mathscr{K})}(\mu)$ is a σ-field iff $\Omega \in \mathscr{K}$ and μ is $\mathscr{H}$-tight at Ω (i.e., $|\mu|(\Omega) = \operatorname{Sup}_H |\mu|(H)$). Therefore, if $\mathscr{F} = \sigma(\mathscr{H})$ $(= \sigma(\mathscr{K}))$, $\mu \in (\mathscr{H}, \mathscr{K})$-ra$(\Omega, \mathscr{F})$ iff $\mathscr{R}^{(\mathscr{H},\mathscr{K})}(\mu) \supset \mathscr{H}$ $(\supset \mathscr{K})$, $\Omega \in \mathscr{K}$ and μ is $\mathscr{H}$-tight at Ω.

Proof. Let μ be a measure such that $\mathscr{R} = \mathscr{R}^{(\mathscr{H},\mathscr{K})}(\mu) \neq \varnothing$. Let $F_1, F_2 \in \mathscr{R}$. In order to prove that $F_1 - F_2 \in \mathscr{R}$, let $\varepsilon > 0$. Then there are $H_i \subset F_i$ and $K_i \supset F_i$ such that $|\mu|(K_i - H_i) < \varepsilon$ $(i = 1, 2)$. Consequently, $H_1 - K_2 \subset F_1 - F_2 \subset K_1 - H_2$ and hence

$$\begin{aligned} &|\mu|(K_1 - H_2) - |\mu|(H_1 - K_2) \\ &\quad = |\mu|[(K_1 - H_2) - (H_1 - K_2)] \leq |\mu|(K_1 - H_1) + |\mu|(K_2 - H_2) < 2\varepsilon. \end{aligned}$$

Since $H_1 - K_2 \in \mathscr{H}$, $K_1 - H_2 \in \mathscr{K}$, and ε is arbitrary, we have $F_1 - F_2 \in \mathscr{R}$. Now let (F_n) be a sequence in $\mathscr{R}$ and let $F = \bigcup_{n=1}^{+\infty} F_n$. Let $\varepsilon > 0$. Then there are $H_n \subset F_n$ and $K_n \supset F_n$ such that $|\mu|(K_n - H_n) < 2^{-n}\varepsilon$ for all n. Since $\bigcup_{i=1}^{n} F_i \uparrow F$ and $|\mu|$ is a measure, there is m such that $|\mu|(F - \bigcup_{i=1}^{m} F_i) < \varepsilon$. Now, let $H = \bigcup_{i=1}^{m} H_i \in \mathscr{H}$ and $K = \bigcup_{n=1}^{+\infty} K_n \in \mathscr{K}$. Then $H \subset F \subset K$ and

$$\begin{aligned} |\mu|(F) - |\mu|(H) &= |\mu|\left(F - \bigcup_{i=1}^{m} F_i\right) + |\mu|\left(\bigcup_{i=1}^{m} F_i - H\right) \\ &< \varepsilon + \sum_{i=1}^{m} |\mu|(F_i - H_i) < 2\varepsilon, \end{aligned}$$

$$|\mu|(K) - |\mu|(F) = |\mu|(K - F) \leq |\mu|\left[\bigcup_{n=1}^{+\infty}(K_n - F_n)\right] \leq \sum_{n=1}^{+\infty} |\mu|(K_n - F_n) < 2\varepsilon.$$

Since ε is arbitrarily chosen, $F \in \mathscr{R}$. Thus $\mathscr{R}$ is a σ-ring.

Finally, keeping in mind 2.1(i), 3.3 and 3.4(i), we get the other statements of the theorem. □

The follwoing corollary follows quite easily from the previous theorem.

3.8. **Corollary.** *Let $\mathscr{F} = \sigma(\mathscr{H})$ and $\mathscr{H}$ be a $(\varnothing, \Omega, \bigcup f, \bigcap c)$-paving such that for any H there is a sequence (H_n^c) such that $H_n^c \downarrow H$. Moreover, let $\mathscr{H}^*$ be a paving such that $H^* \cap H \in \mathscr{H}^*$ for all $H^* \in \mathscr{H}^*$ and all H. Then any measure μ is a $(\mathscr{H}^*, \mathscr{H}^c)$-regular charge whenever μ is $\mathscr{H}^*$-tight at Ω.*

3.9. *Remark.* (i) By suitable choices of topological spaces Ω and pavings $\mathscr{H}$, $\mathscr{K}$ we get, by the previous theorem and corollary, generalizations of some well-known regularity theorems regarding positive measures only. For example, let Ω be:

• an arbitrary topological space and $\mathscr{H}$ the $(\varnothing, \Omega, \bigcup f, \bigcap c)$-paving of closed Baire sets. Since any closed Baire set is a countable intersection of open Baire sets and $\sigma(\mathscr{H})$ is the Baire σ-field, we get that bounded Baire measures are $(\mathscr{H}, \mathscr{H}^c)$-regular;

• a normal space (e.g., compact Hausdorff) and $\mathscr{H}$ the $(\varnothing, \Omega, \bigcup f, \bigcap c)$-paving of closed G_δ's. Since any closed G_δ set is a countable intersection of open F_σ's and $\sigma(\mathscr{H})$ is the Baire σ-field, we get that bounded Baire measures are $(\mathscr{H}, \mathscr{H}^c)$-regular;

• a perfectly normal space (e.g., metric space) and $\mathscr{H}$ the $(\varnothing, \Omega, \bigcup f, \bigcap c)$-paving of closed sets. Then, any bounded Borel measure is $(\mathscr{H}, \mathscr{H}^c)$-regular. Hence in a Polish space, denoting by $\mathscr{H}^*$ the $(\varnothing, \bigcup f)$-paving of compact sets, bounded Borel measures are $(\mathscr{H}^*, \mathscr{H}^c)$-regular, on noting that bounded Borel measures are $\mathscr{H}^*$-tight at Ω;

• a locally compact and σ-compact Hausdorff space, $\mathscr{H}$ the $(\varnothing, \bigcup f)$-paving of compact G_δ's, and $\mathscr{K}$ the $(\Omega, \bigcap f, \bigcup c)$-paving of open Baire sets. Since compact Baire sets are G_δ's and compact G_δ's are countable intersections of open sets that are countable unions of compact G_δ's and any measure is $\mathscr{H}$-tight at Ω, we get that bounded Baire measures are $(\mathscr{H}, \mathscr{K})$-regular charges, on noting that $\sigma(\mathscr{H})$ is the Baire σ-field.

(ii) Let $\mathscr{B}_0$ be the restricted Baire σ-ring of a locally compact Hausdorff space Ω, $\mathscr{H}$ the $(\varnothing, \bigcup f)$-paving of compact G_δ's, and $\mathscr{K}$ the $(\bigcap f, \bigcup c)$-paving of open σ-compact Baire sets. Since the properties recalled in the last example of (i) hold also in $\mathscr{B}_0$, we get, by 3.7, that any bounded measure on $\mathscr{B}_0$ is a $(\mathscr{H}, \mathscr{K})$-regular charge.

4. Purely irregular charges w.r.t. a pair of pavings

We start with the following definition.

4.1. **Definition.** The charge μ is a $(\mathscr{H}, \mathscr{K})$*-purely irregular charge* iff $\mu \in (\mathscr{H}, \mathscr{K})\text{-ra}(\Omega, \mathscr{F})^\perp$. We denote by $(\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$ the set of $(\mathscr{H}, \mathscr{K})$-purely irregular charges. Moreover, we call *inner* $\mathscr{H}$*-purely irregular charges* and *outer* $\mathscr{K}$*-purely irregular charges* the elements of $\mathscr{H}\text{-pia}_*(\Omega, \mathscr{F}) = (\mathscr{H}, \mathscr{F})\text{-pia}(\Omega, \mathscr{F})$ and $\mathscr{K}\text{-pia}^*(\Omega, \mathscr{F}) = (\mathscr{F}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$, respectively.

The following theorem easily follows from the normality of $(\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$ without any hypothesis on the structure of the pavings.

4.2. **Theorem.** (i) *$\mu \in (\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$ iff $\mu^+, \mu^- \in (\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$ iff $|\mu| \in (\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$.*

(ii) *Let $\mu_1, \mu_2 \in (\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$ and α be any real number. Then $\mu_1 + \mu_2$, $\mu_1 \vee \mu_2$, $\mu_1 \wedge \mu_2$, and $\alpha\mu_1$ are $(\mathscr{H}, \mathscr{K})$-purely irregular charges.*

(iii) *Let (μ_n) be a sequence in $(\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$ such that*

$$\lim_{n\to+\infty} \|\mu_n - \mu\| = 0.$$

Then μ is a $(\mathscr{H}, \mathscr{K})$-purely irregular charge.

(iv) *The pointwise limit of any bounded increasing net in $(\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F})$ is a $(\mathscr{H}, \mathscr{K})$-purely irregular charge.*

(v) *Let* $\mu_1, \mu_2 \in (\mathscr{H}, \mathscr{K})$-pia$(\Omega, \mathscr{F})$ *and* ν *be such that* $\mu_1 \le \nu \le \mu_2$. *Then* ν *is a* $(\mathscr{H}, \mathscr{K})$*-purely irregular charge.*

Now, let $\mathscr{H}$ be a $(\varnothing, \bigcup f)$-paving and $\mathscr{K}$ be a $(\Omega, \bigcap f)$-paving. Then, by 3.3, $((\mathscr{H}, \mathscr{K})\text{-ra}(\Omega, \mathscr{F}), (\mathscr{H}, \mathscr{K})\text{-pia}(\Omega, \mathscr{F}))$ is a normal subspace decomposition of $\text{ba}(\Omega, \mathscr{F})$. Therefore, from results given in §2, we easily get the following theorems. The first one gives a decomposition of a bounded charge into its regular and purely irregular decomposants.

4.3. **Theorem.** *Let* $\mathscr{H}$ *be a* $(\varnothing, \bigcup f)$*-paving and* $\mathscr{K}$ *be a* $(\Omega, \bigcap f)$*-paving. Then any charge* μ *can be written uniquely in the form* $\mu = \mu_{\text{r}} + \mu_{\text{pi}}$, *where* μ_r *is a* $(\mathscr{H}, \mathscr{K})$*-regular charge and* μ_{pi} *is a* $(\mathscr{H}, \mathscr{K})$*-purely irregular charge. Further* $\mu_r = (\mu^+)_r - (\mu^-)_{\text{r}}$ *and* $\mu_{\text{pi}} = (\mu^+)_{\text{pi}} - (\mu^-)_{\text{pi}}$ *for all* μ. *Moreover* μ_{r} *and* μ_{pi} *are both positive whenever* μ *is positive.*

4.4. **Theorem.** *Let* $\mathscr{H}$ *be a* $(\varnothing, \bigcup f)$*-paving and* $\mathscr{K}$ *be a* $(\Omega, \bigcap f)$*-paving. Then the following statements hold.*

(i) *Let* μ *be a positiive charge. Then* μ *is a* $(\mathscr{H}, \mathscr{K})$*-purely irregular (regular) charge iff* $\mu \wedge \nu = 0$ *for any positive* $(\mathscr{H}, \mathscr{K})$*-regular (purely irregular) charge* ν.

(ii) *Let* μ *be a* $(\mathscr{H}, \mathscr{K})$*-purely irregular (regular) charge,* ν *a* $(\mathscr{H}, \mathscr{K})$*-regular (purely irrregular) charge, and* α, β *arbitrary positive real numbers. Then there is* F *such that* $|\mu|(F) < \alpha$ *and* $|\nu|(F^c) < \beta$.

(iii) *Let* μ *be a positive charge. Then* μ *is a* $(\mathscr{H}, \mathscr{K})$*-purely irregular (regular) charge iff for every positive* $(\mathscr{H}, \mathscr{K})$*-regular (purely irregular) charge* ν, *for every* F *and,* $\varepsilon > 0$, *there is* $F_1 \subset F$ *such that* $\nu(F_1) < \varepsilon$ *and* $\mu(F_1) > \mu(F) - \varepsilon$.

(iv) μ *is a* $(\mathscr{H}, \mathscr{K})$*-purely irregular (regular) charge iff there is no nonzero* $(\mathscr{H}, \mathscr{K})$*-regular (purely irregular) charge* ν *such that* $0 \le \nu \le |\mu|$.

In the following two statements we assume that μ *is a* 0-k *valued charge.*

(v) μ *is either a* $(\mathscr{H}, \mathscr{K})$*-regular charge or a* $(\mathscr{H}, \mathscr{K})$*-purely irregular one;*

(vi) μ *is* $(\mathscr{H}, \mathscr{K})$*-purely irregular (regular) charge whenever there is a nonzero* $(\mathscr{H}, \mathscr{K})$*-purely irregular (regular) charge* ν *such that* $0 \le \nu \le |\mu|$.

4.5. *Remark.* (i) Let Ω be a countable compact space and $\mathscr{H}$ the paving of:

• closed sets. Then, by 3.6(iii), any pure charge is a $(\mathscr{H}, \mathscr{H}^c)$-purely irregular charge;

• closed Baire sets. Then, by 3.6(iii) and 3.9(i), the previous normal decomposition theorem is the Hewitt-Yosida decomposition one in the bounded Baire charge setting.

(ii) In the context of Gardner paper [6, pp. 44–46], any 0-k valued measure that is strongly compact irregular (e.g., Dieudonné measure) is a strongly compact inner purely irregular charge.

(iii) Let $\mathscr{H}$ be the paving of closed sets of a countable compact space Ω having cardinality less than the first real-valued measurable cardinal. We claim that any diffuse positive bounded charge on the power set 2^Ω $(= \mathscr{F})$ is a $(\mathscr{H}, \mathscr{H}^c)$-purely irregular charge. Let μ be a diffuse positive charge. Moreover let $\nu \in (\mathscr{H}, \mathscr{H}^c)$-ra$(\Omega, \mathscr{F})$ such that $0 \le \nu \le \mu$. Hence, by 3.6(iii), ν is a diffuse positive measure and hence ν is the null-charge. Consequently, by 4.4(iv), we get the claim.

(iv) We assume the Continuum Hypothesis. Then $\Omega = [0, 1]$ has cardinality less than the first real-valued measurable cardinal. Let $\mathscr{F} = 2^\Omega$ and $\mathscr{H}$ the usual paving of closed sets. Moreover let λ be the Lebesgue measure on the Borel σ-field on Ω. Since λ is a diffuse charge, by (iii), any positive bounded extension of λ to $\mathscr{F}$ is a $(\mathscr{H}, \mathscr{H}^c)$-purely irregular charge.

(v) Let Ω be the set of real numbers and $\mathscr{H}$ the paving of bounded real sets. Moreover let $\mu \geq 0$. Then, we easily get that μ is an inner $\mathscr{H}$-regular charge iff μ is $\mathscr{H}$-tight at Ω (i.e., without any adherence at infinity in de Finetti's terminology). Moreover, we claim that μ is an inner $\mathscr{H}$-purely irregular charge iff $\mu(H) = 0$ for all H. First, let $\mu(H) = 0$ for all H. Moreover let $\nu \in \mathscr{H}$-$\mathrm{ra}_*(\Omega, \mathscr{F})$ such that $0 \leq \nu \leq \mu$. Then $\nu(H) = 0$ for all H and hence, by the inner $\mathscr{H}$-regularity of ν, ν is the null-charge. Consequently, by 4.4(iv), $\mu \in \mathscr{H}$-$\mathrm{pia}_*(\Omega, \mathscr{F})$. Now let $\mu \in \mathscr{H}$-$\mathrm{pia}_*(\Omega, \mathscr{F})$. Given H, we consider the charge $\nu(F) = \mu(F \cap H)$ for all F. Plainly, $0 \leq \nu \leq \mu$ and $\nu \in \mathscr{H}$-$\mathrm{ra}_*(\Omega, \mathscr{F})$; consequently, by 4.4(iv), ν is the null-charge and hence $\mu(H) = \nu(H) = 0$.

(vi) By means of 4.3 and the Riesz Decomposition Theorem, we may write a given charge μ as a sum of many decomposants. For example, let Ω be a topological space and $\mathscr{H}$, $\mathscr{K}$ the pavings of compact and open sets, respectively. Moreover, let S be the normal vector sublattice of bounded meaures, $T_1 = S \cap (\mathscr{H}, \mathscr{K})$-$\mathrm{ra}(\Omega, \mathscr{F})$, and T_2 be the orthogonal complement of T_1 *in* S. Then (T_1, T_2) is a normal subspace decomposition of S and hence any charge μ may be written as $\mu = \nu_{\mathrm{r}} + \nu_{\mathrm{pi}} + \mu_{\mathrm{p}}$, where ν_{r} is a compact regular measure, ν_{pi} is a compact purely irregular measure, and μ_{p} is a pure charge. In this way we may obtain decompositions of bounded charges similar to the one given in 25.3 of [6].

In order to give some characterizations of purely irregular charges defined on σ-fields, we denote by $\ll_{\mathrm{w}}$, $\ll$, and $\perp_{\mathrm{s}}$ the relations of weak absolute continuity, absolute continuity and strong singularity, respectively [4, 6.1.1, 6.1.2, 6.1.15].

4.6. **Theorem.** *Let $\mathscr{F}$ be a σ-field and $\mathscr{H}$, $\mathscr{K}$ be $(\bigcup f, \bigcap f)$-pavings. Moreover, we assume that any positive $(\mathscr{H}, \mathscr{K})$-regular charge is σ-smooth at $\varnothing$ w.r.t. $\mathscr{H}$. Finally, let μ be a positive charge. Then $\mu \in (\mathscr{H}, \mathscr{K})$-$\mathrm{pia}(\Omega, \mathscr{F})$ iff for every positive $\nu \in (\mathscr{H}, \mathscr{K})$-$\mathrm{ra}(\Omega, \mathscr{F})$ and $\varepsilon > 0$ there is F such that $\mu(F) = 0$ and $\nu(F^c) < \varepsilon$. Consequently, if μ is a measure, then $\mu \in (\mathscr{H}, \mathscr{K})$-$\mathrm{pia}(\Omega, \mathscr{F})$ iff $\mu \perp_{\mathrm{s}} \nu$ for every positive $\nu \in (\mathscr{H}, \mathscr{K})$-$\mathrm{ra}(\Omega, \mathscr{F})$.*

Proof. Let $\mu \in (\mathscr{H}, \mathscr{K})$-$\mathrm{pia}(\Omega, \mathscr{F})$ and $0 \leq \nu \in (\mathscr{H}, \mathscr{K})$-$\mathrm{ra}(\Omega, \mathscr{F})$. Let $\varepsilon > 0$. Given n, by 4.4(iii) (put $F = \Omega$), there is F_n such that $\mu(F_n) < 2^{-n}\varepsilon$ and $\nu(F_n^c) < 2^{-n}\varepsilon$. Now let $F = \bigcap_{n=1}^{+\infty} F_n \in \mathscr{F}$. Then $\mu(F) \leq \mu(F_n) < 2^{-n}\varepsilon$ for all n and hence $\mu(F) = 0$. On the other hand, $\nu(F^c) = \nu(\bigcup_{n=1}^{+\infty} F_n^c) \leq \sum_{n=1}^{+\infty} \nu(F_n^c) \leq \sum_{n=1}^{+\infty} 2^{-n}\varepsilon = \varepsilon$, where the first inequality holds seeing that, by 3.5, ν is a measure.

The converse implication follows easily from 4.4(iii). The last statement of the theorem follows easily from 6.1.17 of [4]. □

4.7. **Theorem.** *Let $\mathscr{F}$ be a σ-field and $\mathscr{H}$, $\mathscr{K}$ be $(\bigcup f, \bigcap f)$-pavings. Moreover, we assume that any positive $(\mathscr{H}, \mathscr{K})$-regular charge is σ-smooth at $\varnothing$ w.r.t. $\mathscr{H}$. Let $0 \leq \nu \in (\mathscr{H}, \mathscr{K})$-$\mathrm{ra}(\Omega, \mathscr{F})$ and $\mu \ll_{\mathrm{w}} \nu$. Then $\mu \in (\mathscr{H}, \mathscr{K})$-$\mathrm{pia}(\Omega, \mathscr{F})$ iff there is decreasing sequence (F_n) such that $\nu(F_n) \downarrow 0$ and $|\mu|(F_n^c) = 0$ for all n.*

Proof. The necessity easily follows from 4.2(i) and 4.6. To prove the sufficiency let $\nu' \in (\mathscr{H}, \mathscr{K})$-ra$(\Omega, \mathscr{F})$ such that $0 \leq \nu' \leq |\mu|$. Then, by 3.5, ν' is a measure. Since $\mu \ll_{\mathrm{w}} \nu$, by 6.1.4 and 6.1.5(ii) of [4], we have $|\mu| \ll_{\mathrm{w}} |\nu| = \nu$. Then $\nu' \ll_{\mathrm{w}} \nu$ and hence, by 6.1.6 of [4] and 3.5, $\nu' \ll \nu$. Now let F_n be the given decreasing sequence such that $\nu(F_n) \downarrow 0$ and $|\mu|(F_n^c) = 0$ for all n. Since $\nu' \ll \nu$, we get $\nu'(F_n) \downarrow 0$ and hence $\nu'(\bigcap_{n=1}^{+\infty} F_n) = 0$. Since $|\mu|(F_n^c) = 0$, we get $\nu'(F_n^c) = 0$ for all n and hence $\nu'(\bigcup_{n=1}^{+\infty} F_n^c) = 0$. Consequently $\nu'(\Omega) = \nu'(\bigcap_{n=1}^{+\infty} F_n) + \nu'(\bigcup_{n=1}^{+\infty} F_n^c) = 0$ and whence $\nu' = 0$. Then, by 4.4(iv), $\mu \in (\mathscr{H}, \mathscr{K})$-pia$(\Omega, \mathscr{F})$. This completes the proof. □

4.8. *Remark.* (i) Keeping in mind 3.6(ii), the previous two theorems hold whenever $\mathscr{H}$ is a σ-compact set system.

(ii) Let Ω be a countably compact space and $\mathscr{H}, \mathscr{K}$ be the pavings of compact and open sets, respectively. Then, adopting Gardner's terminology [6, §25], we get, by 4.6, that any positive measure μ is $(\mathscr{H}, \mathscr{K})$-purely irregular iff μ is an antiregular measure. Consequently, the decomposition Theorem 25.2 of [6] is a consequence of 4.6, in the context of σ-additivity.

(iii) Assume the hypothesis of 4.6. Then, keeping in mind the notion of strong singularity given by Armstrong [2, p. 567], μ is $(\mathscr{H}, \mathscr{K})$-purely irregular iff every positive $\nu \in (\mathscr{H}, \mathscr{K})$-ra$(\Omega, \mathscr{F})$ is strongly singular to μ.

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Dipartimento di Matematica Applicata Alle Scienze Economiche, Statistiche e Attuariali "Bruno de Finetti", Universita' di Trieste, Piazzale Europa 1, 34127 Trieste, Italia