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# THE TYPE OF THE MAXIMAL OPERATORS OF A CLASS OF WALSH CONVOLUTION OPERATORS

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ABSTRACT. This paper discusses the properties of a class of $p$-adic Walsh convolution operators. The class consists of those 1-parameter sets of operators with kernels that can be represented as the $p$-adic Walsh-Fourier integral of a uniformly quasi-convex function. The paper proves that the maximal operators associated with each 1-parameter set are all of strong type $(\infty, \infty)$ and of weak type $(1, 1)$.

## 1. INTRODUCTION

Properties of the Walsh-Fourier series and their various summations have been discussed often, for example, in [1, 3, 6, 8, 9, 13, 14]. In 1955 Selfridge [11] developed the Walsh transform on $L(\mathbb{R}^+)$ (beyond that initially introduced by Fine [4]) and discussed some of its properties. Among other things, he discussed the modified Fejér mean

$$\lim_{\rho\to\infty} \int_0^\rho \left(1 - \frac{[u]}{\rho}\right) \hat{f}(u)\psi_x(u)\,du, \tag{1}$$

where $\hat{f}$ is the Walsh transform of $f$, $[u]$ is the greatest integer less than or equal to $u$, and $\psi_x$ is the $p$-adic Walsh function with index $x$ $(x \in \mathbb{R}^+)$, and showed that for some integrable functions (1) does *not* converge a.e.

There has been very little further discussion on the properties of the various means of the inverse Walsh transform on $L(\mathbb{R}^+)$, but that field seems a more mathematically interesting field than that merely of the summation of Walsh-Fourier series.

The usual definition of the Fejér mean $T_\rho f$ is

$$\forall q \in [1, 2], \forall f \in L^q(\mathbb{R}^+) \qquad T_\rho f = \int_0^\rho \left(1 - \frac{u}{\rho}\right) \hat{f}(u)\psi_\bullet(u)\,du, \tag{2}$$

and the integral in this can be rewritten as

$$T_\rho f = f * \int_0^\rho \left(1 - \frac{u}{\rho}\right) \psi_\bullet(u)\,du, \tag{2'}$$

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where $*$ is the ($p$-adic) Walsh convolution operation. In this paper we adopt (2′) as the defining integral because it is well defined for $f \in L^q(\mathbb{R}^+)$, not only for $q \in [1, 2]$ but also for $q \in [1, \infty]$.

In this paper we shall discuss the *type* of the maximal operator $T$ where

$$Tf = \sup_{\rho \in \mathbb{R}^+} |T_\rho f|, \tag{3}$$

and then by the representation theorem of [7] one can extend the result to more general operators. The property of convergence a.e. follows immediately.

The notation in this paper is standard and follows that (for example) of [2, 11].

## 2. The first theorem

**Theorem 1.** *Let* $Tf(x) = \sup_{\rho \in \mathbb{R}^+} |f * \int_0^\rho (1 - u/\rho)\psi_x(u)\,du|$. *Then* $T$ *is of strong type* $(\infty, \infty)$ *and of weak type* $(1, 1)$, *and so also of strong type* $(p, p)$ *for* $1 < p < \infty$.

*Proof*. Our plan is to expand $u$ $(u \in \mathbb{R}^+)$ in a $p$-adic Walsh series and then to obtain a majorant function of $\int_0^\rho (1 - u/\rho)\psi_x(u)\,du$ from which we get that $T$ is of strong type $(\infty, \infty)$. From this base we use a Calderón-Zygmund decomposition to prove that $T$ is of weak type $(1, 1)$.

First we find the $p$-adic Walsh series expansion of $x \in \mathbb{R}^+$ (cf. [5]). Let $x \in [0, 1)$ have the $p$-adic expansion

$$x = \sum_{k=1}^{\infty} x_k p^{-k}, \qquad x_k \in \{0, 1, 2, \dots, p-1\}. \tag{4}$$

We show that there exists a unique ordered set of $p$ numbers $(a_0, a_1, \dots, a_{p-1})$ such that

$$\sum_{l=0}^{p-1} a_l \varphi_{k-1}^l(x) = x_k, \tag{5}$$

where $\varphi_k(x)$ are the $p$-adic Rademacher functions. On using the definition $\varphi_k(x) = e^{(2\pi i/p)x_{k+1}}$, equation (5) becomes

$$\sum_{l=0}^{p-1} a_l e^{(2\pi i/p)x_k l} = x_k \qquad (x_k = 0, 1, \dots, p-1); \tag{6}$$

that is, in detail,

$$\begin{aligned} a_0 + a_1 + \cdots + a_{p-1} &= 0; \\ a_0 + a_1 e^{2\pi i/p} + \cdots + a_{p-1} e^{(2\pi i/p)(p-1)} &= 1; \\ &\vdots \\ a_0 + a_1 e^{(2\pi i/p)(p-1)} + \cdots + a_{p-1} e^{(2\pi i/p)(p-1)^2} &= p-1. \end{aligned} \tag{7}$$

This is a Vandermonde system of equations and the determinant of its coefficient matrix is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{2\pi i/p} & \cdots & e^{(2\pi i/p)(p-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{(2\pi i/p)(p-1)} & \cdots & e^{(2\pi i/p)(p-1)^2} \end{vmatrix}, \tag{8}$$

which clearly does not vanish, and so it has a unique solution, which one easily gets as

$$a_0 = \frac{p-1}{2}, \qquad a_l = \frac{1}{e^{-(2\pi i/p)l} - 1} \qquad (l = 1, 2, \dots, p-1). \tag{9}$$

Substituting (5) into (4), one gets the $p$-adic Walsh series expansion of $x \in [0, 1)$

$$\begin{aligned} x &= \sum_{k=1}^{\infty}\sum_{l=0}^{p-1} a_l \varphi_{k-1}^l(x) p^{-k} = \sum_{k=1}^{\infty}\sum_{l=0}^{p-1} a_l \psi_{lp^{k-1}}(x) p^{-k} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{p-1} a_l \psi_{lp^k}(x) p^{-k-1} \qquad (x \in [0, 1)). \end{aligned} \tag{10}$$

For $x \in [0, p^n)$, one has $x/p^n \in [0, 1)$ so

$$\frac{x}{p^n} = \sum_{k=0}^{\infty}\sum_{l=0}^{p-1} a_l \psi_{lp^k}\left(\frac{x}{p^n}\right) p^{-k-1};$$

that is,

$$x = \sum_{k=0}^{\infty}\sum_{l=0}^{p-1} a_l \psi_{lp^{k-n}}(x) p^{-k-1+n};$$

that is,

$$x = \sum_{k=-n}^{\infty}\sum_{l=0}^{p-1} a_l \psi_{lp^k}(x) p^{-k-1}, \tag{11}$$

the corresponding expansion for $x \in [0, p^n)$. Since $x$ is real, one also has

$$x = \sum_{k=-n}^{\infty}\sum_{l=0}^{p-1} \overline{a}_l \overline{\psi}_{lp^k}(x) p^{-k-1} \qquad (x \in [0, p^n)). \tag{12}$$

Now if $x \in \mathbb{R}^+$ then there is an $m \in \mathbb{N}$ such that $x \in [0, p^m)$, and then

$$\begin{aligned} \sum_{k=-\infty}^{-m-1}\sum_{l=0}^{p-1} a_l \psi_{lp^k}(x) p^{-k-1} &= \sum_{k=-\infty}^{-m-1}\sum_{l=0}^{p-1} a_l \psi_l(p^k x) p^{-k-1} \\ &= \sum_{k=-\infty}^{-m-1}\left(\sum_{l=0}^{p-1} a_l\right) p^{-k-1} = 0, \end{aligned}$$

and so we have the expansion for $x \in \mathbb{R}^+$ as

$$x = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{p-1} a_l \psi_{lp^k}(x) p^{-k-1} \qquad (x \in \mathbb{R}^+) \tag{13}$$

and

$$x = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{p-1} \overline{a}_l \overline{\psi}_{lp^k}(x) p^{-k-1} \qquad (x \in \mathbb{R}^+). \tag{14}$$

Having found the Walsh series expansion of $u$, we can get the majorant function of $\int_0^\rho (1-u/\rho)\psi_t(u)\,du$. Let

$$\rho = \sum_{q=1}^{\infty} \alpha_q p^{n_q},$$

where $n_1 > n_2 > \cdots > n_q > \cdots$ $(n_q \in \mathbb{Z})$ and $\alpha_q \in \{1, 2, \ldots, p-1\}$. Then

$$\begin{aligned}
&\int_0^\rho \left(1-\frac{u}{\rho}\right)\psi_t(u)\,du \\
&= \left(\int_0^{\alpha_1 p^{n_1}} + \int_{\alpha_1 p^{n_1}}^{\alpha_1 p^{n_1}+\alpha_2 p^{n_2}} + \cdots + \int_{\alpha_1 p^{n_1}+\cdots+\alpha_{q-1}p^{n_{q-1}}}^{\alpha_1 p^{n_1}+\cdots+\alpha_q p^{n_q}} + \cdots\right) \\
&\qquad \times \left(1-\frac{u}{\rho}\right)\psi_t(u)\,du,
\end{aligned}$$

and we now estimate each term of the last expression. By (12)

$$\begin{aligned}
&\int_0^{p^{n_1}} \left(1-\frac{u}{\rho}\right)\psi_t(u)\,du \\
&= p^{n_1}\chi_{[0,p^{-n_1})}(t) - \frac{1}{\rho}\sum_{k=-n_1}^{\infty} \overline{a}_l p^{-k-1}\int_0^{p^{n_1}} \overline{\psi}_{lp^k}(u)\psi_t(u)\,du \\
&= p^{n_1}\chi_{[0,p^{-n_1})}(t) - \frac{1}{\rho}\sum_{k=-n_1}^{\infty}\sum_{l=0}^{p-1} \overline{a}_l p^{-k-1+n_1}\int_0^{1} \overline{\psi}_{lp^{k+n_1}}(u)\psi_{tp^{n_1}}(u)\,du \qquad (15)\\
&= p^{n_1}\chi_{[0,p^{-n_1})}(t) - \frac{1}{\rho}\sum_{k=-n_1}^{\infty}\sum_{l=0}^{p-1} \overline{a}_l p^{-k-1+n_1}\chi_{[lp^k,\,lp^k+p^{-n_1})}(t),
\end{aligned}$$

where $\chi_S(t)$ denotes the characteristic function of the set $S$. So

$$\begin{aligned}
\int_0^{\alpha_1 p^{n_1}} \left(1-\frac{u}{\rho}\right)\psi_t(u)\,du &= \sum_{r=0}^{\alpha_1-1}\int_{rp^{n_1}}^{(r+1)p^{n_1}} \left(1-\frac{u}{\rho}\right)\psi_t(u)\,du \\
&= \sum_{r=0}^{\alpha_1-1}\int_0^{p^{n_1}} \left(1-\frac{rp^{n_1}}{\rho}-\frac{u}{\rho}\right)\psi_t(u+rp^{n_1})\,du.
\end{aligned}$$

Because $0 < u < p^{n_1}$, it follows that $\psi_t(u+rp^{n_1}) = \psi_t(u\oplus rp^{n_1}) = \psi_t(u)\psi(rp^{n_1})$ a.e. and so

$$\begin{aligned}
&\sum_{r=0}^{\alpha_1-1}\int_0^{p^{n_1}} \left(1-\frac{rp^{n_1}}{\rho}-\frac{u}{\rho}\right)\psi_t(u+rp^{n_1})\,du \\
&= \sum_{r=0}^{\alpha_1-1}\left[\int_0^{p^{n_1}}\left(1-\frac{rp^{n_1}}{\rho}\right)\psi_t(u)\,du - \frac{1}{\rho}\int_0^{p^{n_1}} u\psi_t(u)\,du\right]\psi_t(rp^{n_1}) \qquad (16)\\
&= \frac{1}{\rho}\sum_{r=0}^{\alpha_1-1}\left[(\rho-rp^{n_1})p^{n_1}\chi_{[0,p^{-n_1})}(t) - \sum_{k=-n_1}^{\infty}\sum_{l=0}^{p-1}\overline{a}_l p^{-k-1+n_1}\chi_{[lp^k,\,lp^k+p^{-n_1})}(t)\right] \\
&\qquad \times \psi_t(rp^{n_1}).
\end{aligned}$$

Similarly

$$\int_{\alpha_1 p^{n_1}}^{\alpha_1 p^{n_1}+\alpha_2 p^{n_2}} \left(1-\frac{u}{\rho}\right)\psi_t(u)\,du$$

$$(17) \quad = \int_0^{\alpha_2 p^{n_2}} \left(1-\frac{u+\alpha_1 p^{n_1}}{\rho}\right)\psi_t(u+\alpha_1 p^{n_1})\,du$$

$$= \frac{1}{\rho}\sum_{r=0}^{\alpha_2-1}\left[(\rho-\alpha_1 p^{n_1}+rp^{n_2})p^{n_2}\chi_{[0,p^{-n_2})}(t) - \sum_{k=-n_2}^{\infty}\sum_{l=0}^{p-1}\overline{a}_l p^{-k-1+n_2}\chi_{[lp^k,lp^k+p^{-n_2})}(t)\right]\psi_t(\alpha_1 p^{n_1}+rp^{n_2})$$

and

$$\int_{\alpha_1 p^{n_1}+\cdots+\alpha_{q-1}p^{n_{q-1}}}^{\alpha_1 p^{n_1}+\cdots+\alpha_{q-1}p^{n_{q-1}}+\alpha_q p_q^n} \left(1-\frac{u}{\rho}\right)\psi_t(u)\,du$$

$$(18) \quad = \frac{1}{\rho}\sum_{r=0}^{\rho_{q-1}}\left[(\rho-\alpha_1 p^{n_1}-\cdots-\alpha_{q-1}p^{n_{q-1}}-rp^{n_q})p^{n_q}\chi_{[0,p^{-n_q})}(t) - \sum_{k=-n_q}^{\infty}\sum_{l=0}^{p-1}\overline{a}_l p^{-k-1+n_q}\chi_{[lp^k,lp^k+p^{-n_q})}(t)\right] \times \psi_t(\alpha_1 p^{n_q}+\cdots+\alpha_{q-1}p^{n_{q-1}}+rp^{n_q}).$$

So by (16)–(18) we have

$$(19) \quad \left|\int_0^{\rho}\left(1-\frac{u}{\rho}\right)\psi_t(u)\,du\right| \le \frac{A}{\rho}\sum_{j=1}^{\infty}\sum_{k=-n_j}^{\infty}\sum_{l=0}^{p-1}p^{-k-1+n_j}\chi_{[lp^k,lp^k+p^{-n_j})}(t),$$

where $A$ is a constant. Let

$$(20) \quad K_\rho^*(t) = \frac{A}{\rho}\sum_{j=1}^{\infty}\sum_{k=-n_j}^{\infty}\sum_{l=0}^{p-1}p^{-k-1+n_j}\chi_{[lp^k,lp^k+p^{-n_j})}(t),$$

$$(21) \quad T^*f = \sup_{\rho\in\mathbb{R}^+}|f*K_\rho^*|,$$

and

$$(22) \quad Mf = \sup_{\rho\in\mathbb{R}^+}|f|*K_\rho^*.$$

Clearly

$$(23) \quad (Tf)(x)\le(Mf)(x) \quad \text{and} \quad (T^*f)(x)\le(Mf)(x).$$

By (19)

$$(24) \quad \begin{aligned}\|Mf\|_\infty &\le \sup_\rho\|K_\rho^*\|_1\|f\|_\infty \le \sup_\rho\frac{A}{\rho}\sum_{j=1}^{\infty}\sum_{k=-n_j}^{\infty}p^{-k}\|f\|_\infty \\ &\le \|f\|_\infty\sup_\rho\frac{A}{\rho}\sum_{j=1}^{\infty}\frac{p^{n_j}}{1-\frac{1}{p}} \le \|f\|_\infty A\frac{p}{p-1} = B\|f\|_\infty,\end{aligned}$$

where $B = pA/(p-1)$; so $M$ is of strong type $(\infty, \infty)$ (and, of course, $T$ and $T^*$ are too).

In what follows we prove $T$ is of weak type $(1, 1)$. Clearly it is enough to prove that $M$ is of weak type $(1, 1)$.

By the Calderón-Zygmund method (cf. [12] for the dyadic case—it is similar to the $p$-adic case) we have that for every $\lambda > 0$ there exists a sequence $\{I_m\}$ of open disjoint $p$-adic subintervals such that

$$|f(x)| = g(x) + b(x), \tag{25}$$

where

$$\|g\|_\infty \le p\lambda \tag{26}$$

and

$$b(x) = \sum b_m(x);$$

here

$$b_m(x) = 0 \quad \text{if } x \notin I_m \quad \text{and} \quad \int_{I_m} b_m(x)\,dx = 0, \tag{27}$$

$$\|b\|_1 \le 2\|f\|_1, \tag{28}$$

and

$$|\Omega| = \left|\bigcup I_m\right| \le \frac{1}{\lambda}\|f\|_1. \tag{29}$$

Because

$$\begin{aligned} Mf &= \sup_\rho |f| * K_\rho^* = \sup_\rho (g+b) * K_\rho^* \\ &\le \sup |g * K_\rho^*| + \sup_\rho |b * K_\rho^*| = T^*g + T^*b \end{aligned} \tag{30}$$

and $T^*$ is of strong type $(\infty, \infty)$ and $|g(x)| \le p\lambda$ a.e., it follows that

$$\|T^*g\|_\infty \le B\|g\|_\infty \le C\lambda, \tag{31}$$

where $C = Bp$.

By (29), (30), (28), and (26),

$$\begin{aligned} |\{x \in \mathbb{R}^+ | (Mf)(x) > (1+C)\lambda\}| &\le |\{x \in \mathbb{R}^+ | (T^*b)(x) > \lambda\}| \\ &\le |\Omega| + |\Omega^c \cap \{x \in \mathbb{R}^+ | (T^*b)(x) > \lambda\}| \\ &\le \frac{1}{\lambda}\|f\|_1 + \frac{1}{\lambda}\int_{\Omega^c} (T^*b)(x)\,dx \le \frac{1}{\lambda}\|f\|_1 + \frac{1}{\lambda}\sum_m \int_{I_m^c} (T^*b_m)(x)\,dx, \end{aligned} \tag{32}$$

where $A^c$ denotes the complement in $\mathbb{R}^+$ of $A$.

Now we estimate $\int_{I_m^c} (T^*b_m)(x)\,dx = \int_{I_m^c} \sup_\rho |b_m * K_\rho^*(x)|\,dx$. Notice that in this integral $x \notin I_m$ and

$$b_m * K_\rho^*(x) = \int_0^\infty b_m(t) \frac{A}{\rho} \sum_{j=1}^\infty \sum_{k=-n_j}^\infty \sum_{l=0}^{p-1} p^{-k-1+n_j} \chi_{[lp^k, lp^k + p^{-n_j})}(x \ominus t)\,dt.$$

Let $J(l, k, n_j, x)$ denote the support of $\chi_{[lp^k, lp^k+p^{-n_j})}(x \ominus t)$ considered as a function of $t$. It is a $p$-adic interval with length $p^{-n_j}$ and including $x \ominus lp^k$.

There are two cases to consider.

*Case* 1. $\rho \le 1/|I_m|$. Here $p^{n_j} \le 1/|I_m|$ $(j = 1, 2, \dots)$, so $p^{-n_j} \ge |I_m|$. Then for each $l \in \{0, 1, \dots, p-1\}$, $j = 1, 2 \dots$, and $k \ge -n_j$, $I_m$ (the support of $b_m(t)) \subset J(l, k, n_j, x)$ or $I_m \cap J(l, k, n_j, x) = \varnothing$, so $b_m * K_\rho^*(x) = 0$.

*Case* 2. $\rho > 1/|I_m|$. Let the integers $j_0$ and $k_0$ be such that $p^{-n_{j_0+1}} \ge |I_m| > p^{-n_{j_0}}$ and $p^{k_0+1} \ge |I_m| > p^{k_0}$. Then, by the proof of Case 1, we see that

$$b_m * K_\rho^*(x) = \int_0^\infty b_m(t)\frac{A}{\rho}\sum_{j=1}^{j_0}\sum_{k=-n_j}^{\infty}\sum_{l=0}^{p-1} p^{-k-1+n_j}\chi_{[lp^k, lp^k+p^{-n_j})}(x \ominus t)\,dt.$$

Because $x \notin I_m$ and for $k < k_0$, $x \ominus lp^k \notin I_m$, it follows that $J(l, k, n_j, x)$, which is a $p$-adic interval including $x \ominus lp^k$ with length $p^{-n_j} \le p^{-n_{j_0}} < |I_m|$, is disjoint from $I_m$, and it is easy to see that $-n_{j_0} \le k_0$,

$$\begin{aligned} |b_m * K^*\rho(x)| &= \left|\int_0^\infty b_m(t)\frac{A}{\rho}\sum_{j=1}^{j_0}\sum_{k=k_0}^{\infty}\sum_{l=0}^{p-1} p^{-k-1+n_j}\chi_{[lp^k, lp^k+p^{-n_j})}(x \ominus t)\,dt\right| \\ &\le A\int_0^\infty |b_m(t)|\sum_{k=k_0}^{\infty}\sum_{l=0}^{p-1} p^{-k-1}\chi_{[lp^k, lp^k+p^{k_0})}(x \ominus t)\frac{1}{\rho}\sum_{j=1}^{j_0} p^{n_j}\,dt \\ &\le A\int_0^\infty |b_m(t)|\sum_{k=k_0}^{\infty}\sum_{l=0}^{p-1} p^{-k-1}\chi_{[lp^k, lp^k+p^{k_0})}(x \ominus t)\,dt. \end{aligned}$$

Therefore

$$\begin{aligned} (T^*b_m)(x) &= \sup_{\rho\in\mathbb{R}^+} |b_m * K_\rho^*(x)| \\ &\le A\int_0^\infty |b_m(t)|\sum_{k=k_0}^{\infty}\sum_{l=0}^{p-1} p^{-k-1}\chi_{[lp^k, lp^k+p^{k_0}]}(x \ominus t)\,dt \qquad (x \notin I_m), \end{aligned}$$

and so

$$\begin{aligned} \int_{I_m^c}(T^*b_m)(x)\,dx &\le A\int_0^\infty\left[|b_m(t)|\sum_{k=k_0}^{\infty}\sum_{l=0}^{p-1} p^{-k-1}\int_0^\infty \chi_{[lp^k, lp^k+p^{k_0})}(x \ominus t)\,dx\right]dt \\ &= A\int_0^\infty |b_m(t)|\sum_{k=k_0}^{\infty} p^{-k+k_0}\,dt = \frac{Ap}{p-1}\|b_m\|_1 \qquad (33) \\ &= B\|b_m\|_1, \end{aligned}$$

where $B = Ap/(p-1)$. By (31) and (32) we have

$$|\{x \in \mathbb{R}^+ | (Mf)(x) > (1+C)\lambda\}| \le \frac{1}{\lambda}\|f\|_1 + \frac{B}{\lambda}\|b\|_1 \le \frac{1+2B}{\lambda}\|f\|_1,$$

so

$$|\{x \in \mathbb{R}^+ | (Mf)(x) > \lambda\}| \le \frac{(1+2B)(1+C)}{\lambda} \|f\|_1 .$$

This completes the proof.

## 3. The second theorem

Having proved Theorem 1, we can get the same properties for some general operators by the results of [7].

We call $g_\rho(u)$ $(u \in \mathbb{R}^+)$ "uniformly quasi-convex" if and only if for each $\rho > 0$, $g_\rho(u) \in AC_{\mathrm{loc}}(\mathbb{R}^+)$, $g'_\rho(u) \in BV_{\mathrm{loc}}(\mathbb{R}^+)$, and there exists a constant $C$ independent of $\rho$ such that $\int_0^\infty u|dg'_\rho(u)| < C$. For example, if $g(u)$ is quasi-convex then $g(u/\rho)$ is uniformly quasi-convex.

**Theorem 2.** *If $K_\rho(t) = \int_0^\infty g_\rho(u)\psi_t(u)\,du$ where $g_\rho(u)$ is uniformly quasi-convex and $g_\rho(\infty) = 0$ then $T$, where $Tf = \sup_{\rho\in\mathbb{R}^+} |f * K_\rho|$, is of strong type $(\infty, \infty)$ and of weak type $(1, 1)$.*

*Proof.* By [7] $K_\rho(t) = \int_0^\infty g_\rho(u)\psi_t(u)\,du \in L(\mathbb{R}^+)$ so $f \to f * K_\rho$ is an operator from $L^q(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$ for each $q \in [1, \infty]$. By (2) in [7]

$$\begin{aligned}
\sup_{\rho\in\mathbb{R}^+} \left| f * \int_0^\infty g_\rho(u)\psi_t(u)\,du \right| &= \sup_{\rho\in\mathbb{R}^+} \left| f * \int_0^\infty u\,dg'_\rho(u) \int_0^u \left(1 - \frac{v}{u}\right) \psi_t(v)\,dv \right| \\
&= \sup_{\rho\in\mathbb{R}^+} \left| \int_0^\infty u\,dg'_\rho(u) \left[ f * \int_0^u \left(1 - \frac{v}{u}\right) \psi_t(v)\,dv \right] \right| \\
&\le C \sup_{\rho\in\mathbb{R}^+} \left| f * \int_0^u \left(1 - \frac{v}{u}\right) \psi_t(v)\,dv \right| ,
\end{aligned}$$

where $C = \sup_{\rho\in\mathbb{R}^+} \int_0^\infty u|dg'_\rho(u)|$.

By Theorem 1 we get that $T$ is of strong type $(\infty, \infty)$ and of weak type $(1, 1)$.

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