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SMOOTH PERTURBATIONS OF REGULAR DIRICHLET FORMS

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ABSTRACT. Given a regular Dirichlet form $\mathfrak{h}$, we prove that a measure μ is smooth iff the domain of $\mathfrak{h} + \mu$ is dense in the domain of $\mathfrak{h}$ with respect to the form norm. The latter condition is in turn equivalent to the convergence of $\mathfrak{h} + a\mu$ to $\mathfrak{h}$ as $a \downarrow 0$.

INTRODUCTION AND PRELIMINARIES

The study of Schrödinger operators of the form $-\Delta + \mu$, where μ is a measure, has attracted the attention of various authors in recent years; we only mention [1–8, 14].

If μ is a nonnegative measure that does not charge sets of zero capacity, one can define this operator by means of quadratic forms, as we will see in §1.

Smooth measures have turned out to be especially important, since one has an analogue for the Feynman-Kac formula for the semigroup generated by $-\Delta + \mu$ (see [9, Chapter 5; 15], where the perturbation of a Hunt process by additive functionals has been analyzed in detail).

The main result of this note, Theorem 2.2, is a characterization of smooth measures in terms of very natural operator theoretic conditions; namely, μ is smooth iff $-\Delta + a\mu$ converge to $-\Delta$ in the strong resolvent sense as $a \downarrow 0$.

Let us now introduce the general framework of the present article. The monograph [9] is our standard reference for the following notions. (Nevertheless, we use the notation from [10] as far as forms are concerned.) We fix a locally compact σ-compact Hausdorff space $\mathbf{X}$ and a Radon measure $\mathbf{m}$ on $(\mathbf{X}, \mathfrak{B})$ (where $\mathfrak{B}$ denotes the Borel σ-algebra) such that $\operatorname{supp} \mathbf{m} = \mathbf{X}$. We assume that a *regular Dirichlet form* $\mathfrak{h}$ with domain $\mathbf{D}(\mathfrak{h}) = \mathbf{D}$ on $(\mathbf{L}_2(\mathbf{X}, \mathbf{m}), (\cdot|\cdot))$ is given, so that $\mathbf{D}$, equipped with the inner product $(\cdot|\cdot)_{\mathfrak{h}} := (\mathfrak{h} + 1)[\cdot, \cdot] = \mathfrak{h}[\cdot, \cdot] + (\cdot|\cdot)$, is a Hilbert space. Consequently, there exists a nonnegative self-adjoint operator H in $L_2(\mathbf{X}, \mathbf{m})$ that corresponds to $\mathfrak{h}$ (H plays the role of $-\Delta$).

We are mainly interested in the form sum $\mathfrak{h} + \mu$ that can be defined whenever μ is a measure that does not charge sets of zero *capacity*. This will be shown in §1 where we also give a necessary and sufficient condition for μ in order that $\mathfrak{h} + \mu$ be densely defined. The main result of the second section is the fact that

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μ is *smooth* if and only if the domain $\mathbf{D}(\mathfrak{h}+\mu)$ is dense in $(\mathbf{D}, (\cdot|\cdot)_{\mathfrak{h}})$, which, in turn, is equivalent to $\mathfrak{h}+a\mu \to \mathfrak{h}$ in the strong resolvent sense for $a \downarrow 0$. In particular, this answers a question in [3] and relates the *regular potentials* of [12, 13, 16] to smooth measures; namely, V is *regular* if and only if $V\,dx$ is *smooth.* Some of the material in §§1 and 2 is obtained by a straightforward generalization of results in [12].

1. Measures that do not charge sets of zero capacity

By $\mathrm{Cap}(A) = \mathrm{Cap}_{\mathfrak{h}}(A)$ we denote the capacity of a set $A \subset X$, which is defined according to [9, §3.1]. If f is an element in $\mathbf{D}$ then there exist quasi-continuous modifications of f [loc. cit.], the set of which will be denoted by $[f]_q$. The notation $f^\sim$ will be used for a quasi-continuous representative of f. In this section we shall consider the class

$$\mathscr{M}_0 := \mathscr{M}_0(\mathfrak{h}) := \{\mu\colon \mathfrak{B} \to [0, \infty];\ \mu \text{ is a } \sigma\text{-additive measure such that } \mu(B) = 0 \text{ for every } B \in \mathfrak{B} \text{ with } \mathrm{Cap}(B) = 0\}$$

of *measures that do not charge sets of zero capacity.* It will turn out that "$\mu \in \mathscr{M}_0$" is the adequate condition for the definition of $\mathfrak{h}+\mu$, if one is willing to consider nondensely defined forms.

Example 1. By $\mathfrak{h} = 0$ a bounded regular Dirichlet form is given; $\mathrm{Cap}(B) = \mathbf{m}(B)$ for every $B \in \mathfrak{B}$ and $\mathscr{M}_0 = \{V\mathbf{m};\ V\colon \mathbf{X} \to [0, \infty],\ V \text{ measurable}\}$.

It is sometimes interesting to consider the above trivial example, however, the typical situation one should think of is

Example 2. $\mathbf{X} = \mathbb{R}^\nu$, $\mathbf{m} = dx$ the Lebesgue measure, $\mathbf{D} = W_2^1(\mathbb{R}^\nu)$, and $\mathfrak{h}[u, v] := \int \nabla u \cdot \nabla \overline{v}\, dx$ define a regular Dirichlet form. For the special case $\nu = 1$, $\mathrm{Cap}(B) > 0$ for every $B \neq \varnothing$. Consequently,

$$\mathscr{M}_0 = \{\mu\colon \mathfrak{B} \to [0, \infty]\colon \mu \text{ a measure}\}.$$

1.1. **Proposition.** *Let* $\mu \in \mathscr{M}_0$. *Then*

$$\mathbf{D}(\mu) := \{f \in \mathbf{D};\ [f]_q \subset \mathscr{L}_2(\mathbf{X}, \mu)\}, \qquad \mu[f, g] := \int f^\sim(x)\overline{g}^\sim \mu(dx)$$

defines a closed form on $(\mathbf{D}, (\cdot|\cdot)_{\mathfrak{h}})$.

Proof. Let (f_n) be a sequence in $\mathbf{D}$, $f_n \to f$ in $(D, (\cdot|\cdot)_{\mathfrak{h}})$, such that $\mu[f_n - f_m] \to 0$ for $n, m \to \infty$; from [9, Theorem 3.1.4] it follows that we can find a subsequence satisfying $f_{n_k}^\sim \to f^\sim$ q.e. Since $\mu \in \mathscr{M}_0$, this implies $f_{n_k}^\sim \to f^\sim$ μ-a.e. By assumption, $(f_{n_k}^\sim)$ is a Cauchy sequence in $\mathscr{L}_2(\mathbf{X}, \mu)$, which implies $f \in \mathscr{L}_2(\mathbf{X}, \mu)$, $\mu[f_n - f] \to 0$. □

As an immediate consequence, we have

1.2. **Theorem.** *Let* $\mu \in \mathscr{M}_0$. *Then*

$$\mathbf{D}(\mathfrak{h}+\mu) := \mathbf{D}(\mu), \qquad (\mathfrak{h}+\mu)[f, g] := \mathfrak{h}[f, g] + \mu[f, g]$$

for $f, g \in \mathbf{D}(\mathfrak{h}+\mu)$, *defines a closed nonnegative form on* $(L_2(\mathbf{X}, \mathbf{m}), (\cdot|\cdot))$.

Even though, in general, $\mathfrak{h} + \mu$ need not be densely defined we may follow Simon [11] and define the resolvent $\mathrm{R}(E, \mathfrak{h} + \mu)$ for each $E > 0$.

Remark. In the situation of Example 2, with $G = \mathbb{R}^\nu$,

$$\mathrm{R}(E, \mathfrak{h} + \mu) = \mathrm{R}_\mu^G(E),$$

where the right-hand side is the "variational resolvent" of [5, Definition 2.3]; see also [5, Proposition 2.1].

Following the ideas of [12], we shall now characterize those measures in $\mathscr{M}_0$ that give rise to a densely defined form $\mathfrak{h} + \mu$.

1.3. **Theorem.** *Let* $\mu \in \mathscr{M}_0$. *Then the following conditions are equivalent*:

(i) $\mathbf{D}(\mathfrak{h} + \mu)$ *is dense in* $L_2(\mathbf{X}, \mathbf{m})$.

(ii) *For each* $\varepsilon > 0$ *there exist a closed set* $A \subset \mathbf{X}$ *and an open set* $U \subset \mathbf{X}$ *such that* $\mathbf{m}(A) = 0$, $\mathrm{Cap}(U) < \varepsilon$, *and* $\chi_{\mathbf{X}\setminus U} \cdot \mu$ *is locally finite on* $\mathbf{X}\setminus A$.

(iii) *For each* $\varepsilon > 0$ *there exist a closed set* $A \subset \mathbf{X}$, $f \in \mathbf{D}$, *and* $\varphi \in [f]_q$, *such that* $\mathbf{m}(A) = 0$, $0 \le f \le 1$, $\|f\|_{\mathfrak{h}} \le \varepsilon$, *and* $(1 - \varphi)\mu$ *is locally finite on* $\mathbf{X}\setminus A$.

The proof of [12, Theorem 1.6, p. 144] can be modified to cover the above case. An analogous modification is worked out in the proof of Theorem 2.2.

2. Smooth measures

Let us call a sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets a *local nest* (for $\mathfrak{h}$) if, for every compact $K \subset \mathbf{X}$,

$$\mathrm{Cap}(K\setminus F_n) \to 0 \quad \text{for } n \to \infty.$$

Remark. Let $(F_n)_{n\in\mathbb{N}}$ be a local nest, $\mu \in \mathscr{M}_0$. Then

$$\mu\left(\mathbf{X}\setminus \bigcup_{n\in\mathbb{N}} F_n\right) = 0.$$

Proof. Fix an increasing sequence $(\mathbf{X}_k)$ of compact subsets of $\mathbf{X}$, such that $\bigcup \mathring{\mathbf{X}}_k = \mathbf{X}$. For every $k \in \mathbb{N}$, $\mathrm{Cap}(\mathbf{X}_k\setminus \bigcup_{n\in\mathbb{N}} F_n) \le \lim_{n\to\infty} \mathrm{Cap}(\mathbf{X}_k\setminus F_n) = 0$. Consequently, $\mu(\mathbf{X}_k\setminus \bigcup_{n\in\mathbb{N}} F_n) = 0$. Together with $\mathbf{X} = \bigcup_{k\in\mathbb{N}} \mathbf{X}_k$, this proves the assertion. □

A measure μ is called *smooth* (cf. [9, p. 72]) if it charges no set of zero capacity and there exists a local nest $(F_n)_{n\in\mathbb{N}}$ consisting of μ-integrable sets. By

$$\mathscr{S} := \mathscr{S}(\mathfrak{h}),$$

we denote the set of smooth measures.

The following density theorem proves the usefulness of local nests.

2.1. **Theorem.** *Let* $(F_n)_{n\in\mathbb{N}}$ *be a local nest. Then*

$$\{f \in \mathbf{D}\,;\, 0 \le \tilde{f} \le \chi_{F_n} \textit{ for some } n \in \mathbb{N}\}$$

is a total set in $(\mathbf{D}, (\cdot|\cdot)_{\mathfrak{h}})$.

Proof. Let $\varphi \in \mathbf{D}\cap \mathbf{C}_c(\mathbf{X})$. Since $\mathfrak{h}$ is regular and $L := \mathrm{lin}\{\dots\}$ (the linear hull of the set that appears in the statement of the theorem) is convex, it is enough to construct a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in L that converges to φ in the weak topology of $(\mathbf{D}, (\cdot|\cdot)_{\mathfrak{h}})$. Since $K := \operatorname{supp}\varphi$ is compact, $\operatorname{Cap}(K\backslash F_n) \to 0$ for $n \to \infty$. Hence, we find quasi-continuous $\psi_n \in \mathbf{D}$, $\chi_{K\backslash F_n} \le \psi_n \le 1$ for $n \in \mathbb{N}$ such that $\psi_n \to 0$ in $(\mathbf{D}, (\cdot|\cdot)_{\mathfrak{h}})$. For $n \in \mathbb{N}$, $\varphi_n := \varphi(1 - \psi_n)$, it follows that $\varphi_n \in \mathbf{D}$ and

$$\|\varphi_n\|_{\mathfrak{h}} \le \|\varphi\|_{\mathfrak{h}} + \|\varphi\psi_n\|_{\mathfrak{h}} \le \|\varphi\|_{\mathfrak{h}} + \|\varphi\|_\infty \|\psi_n\|_{\mathfrak{h}} + \|\varphi\|_{\mathfrak{h}} \|\psi_n\|_\infty$$

(see [9, Theorem 1.4.2(ii), p. 25]). Therefore, $(\varphi_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{D}$. Moreover, for every $\psi \in \mathbf{D}(H)$,

$$\mathfrak{h}[\varphi_n, \psi] = (\varphi_n|H\psi) \to (\varphi|H\psi) = \mathfrak{h}[\varphi, \psi],$$

since $\varphi_n \to \varphi$ in $\mathbf{L}_2(\mathbf{X}, \mathbf{m})$. This implies the weak convergence of (φ_n); by construction, $\varphi_n \in L$ for each $n \in \mathbb{N}$. □

With the last theorem at hand we can now prove the main result of this article. It contains an operator theoretic description of $\mathscr{S}$ that, in particular, gives the following answer to a question raised in [3, Remark, p. 6]: For the potential V constructed in [13], $V\,dx$ is smooth.

2.2. **Theorem.** *Let $\mu \in \mathscr{M}_0$. Then the following conditions are equivalent:*

(i) *μ is smooth.*
(ii) *$\mathbf{D}(\mathfrak{h} + \mu)$ is dense in $(\mathbf{D}, (\cdot|\cdot)_{\mathfrak{h}})$.*
(iii) *In the strong resolvent sense, $\mathfrak{h} + a\mu \to \mathfrak{h}$ for $a \downarrow 0$.*

Remark. For $\mu \in \mathscr{M}_0$ and $a > 0$ it follows that $a\mu \in \mathscr{M}_0$ so that the condition in (iii) makes sense.

Proof of Theorem 2.2. (i) ⇒ (ii) follows from Theorem 2.1 since

$$\{f \in \mathbf{D};\, 0 \le \tilde{f} \le \chi_{F_n} \text{ for some } n \in \mathbb{N}\} \subset \mathbf{D}(\mathfrak{h} + \mu)$$

whenever $(F_n)_{n\in\mathbb{N}}$ is a local nest consisting of μ-integrable sets.

(ii) ⇒ (i) (cf. [12, Proof of Theorem 1.5, p. 143]). Let $(\mathbf{X}_k)$ be as in the proof of the remark. Since $\mathfrak{h}$ is regular, there exist functions $\psi_n \in \mathbf{D} \cap \mathbf{C}_c(\mathbf{X})$, $\psi_n \ge \chi_{\mathbf{X}_n}$ for every $n \in \mathbb{N}$. By assumption there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbf{D}(\mathfrak{h} + \mu)$ satisfying $f_n \le \psi_n$ for $n \in \mathbb{N}$ and $\|\varphi_n - \psi_n\|_{\mathfrak{h}} \to 0$ for $n \to \infty$. According to [9, Theorem 3.1.3] there is an open set $U_n \subset \mathbf{X}$ and a quasi-continuous $\varphi_n \in [f_n]_q$ such that $\operatorname{Cap}(U_n) \le 2^{-n}$ and φ_n/U_n is continuous. Let

$$F_n := (X_n\backslash U_n) \cap \{y \in \mathbf{X};\, \varphi_n(y) \ge n^{-1}\}$$

for $n \in \mathbb{N}$. Since $D_n := \{y \in \mathbf{X};\, \varphi_n(y) \ge n^{-1}\}$ is relatively closed in $X_n\backslash U_n$, F_n is compact for every $n \in \mathbb{N}$. $F_n \subset D_n$ implies that $\mu(F_n) < \infty$ for every $n \in \mathbb{N}$. Let $K \subset X$ be compact. Then $K \subset X_n$ for n large enough. It follows that

$$K\backslash F_n \subset U_n \cup (D_n \cap (X_n\backslash U_n)),$$

and since $\operatorname{Cap}(U_n) \to 0$, it remains to show

$$\operatorname{Cap}(D_n \cap (X_n\backslash U_n)) \to 0.$$

To verify this, observe that

$$0 \le (1 - n^{-1})\chi_{D_n \cap (X_n\backslash U_n)} \le \psi_n - \varphi_n,$$

which yields

$$\operatorname{Cap}(D_n \cap (X_n\backslash U_n)) \le (1 - n^{-1})^{-2} \|\varphi_n - \psi_n\|_{\mathfrak{h}}^2 \to 0,$$

where we used [9, Lemma 3.1.5, p. 66] for the last inequality.

(ii) $\Leftrightarrow$ (iii). By [11, Theorem 3.2, p. 382f], $\mathfrak{h} + a\mu \to (\mathfrak{h}|\mathbf{D}(\mathfrak{h}+\mu))_r^-$, where the last symbol stands for the closure of the regular part (see [11]) of $\mathfrak{h}|\mathbf{D}(\mathfrak{h}+\mu)$, which simply equals the closure of $\mathfrak{h}|\mathbf{D}(\mathfrak{h}+\mu)$. □

In our point of view, the equivalent conditions (ii) and (iii) in the above theorem are very natural. In terms of the underlying process, they can be interpreted as follows: μ is not smooth iff "too many particles" are totally absorbed by μ, which forces elements of $\mathbf{D}(\mathfrak{h}+\mu)$ to vanish on some set.

To present a typical example, let us take $V(x) = |x_1|^{-1}$ and $\mu = V\,dx$ in the situation of Example 2. Then $\mathbf{D}(\mathfrak{h}+\mu) = \overset{\circ}{W}{}_2^1(\mathbb{R}^\nu\backslash\{0\}\times\mathbb{R}^{\nu-1})$, i.e., the limit of $\mathfrak{h}+a\mu$ as $a \downarrow 0$ is the form corresponding to the Dirichlet-Laplacian on $\mathbb{R}^\nu\backslash\{0\}\times\mathbb{R}^{\nu-1}$. This means that typical Brownian particles cannot pass the barrier that is given by V.

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