WEIGHTED NORM INEQUALITIES
FOR BOCHNER-RIESZ OPERATORS
AND SINGULAR INTEGRAL OPERATORS

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Abstract. Weighted norm inequalities for the Bochner-Riesz operator at the
critical index \( \frac{1}{2}(n-1) \) are investigated. We also give some weighted norm
inequalities for a class of singular integral operators introduced by Fefferman
and Namazi.

1. Introduction and statements

The Bochner-Riesz operators in \( \mathbb{R}^n \) are defined as

\[
(T^R_\lambda f)(x) = (1 - R^2|x|^2)\frac{1}{4}\hat{f}(x)
\]

and the associated maximal operator is defined as

\[
T^*_\lambda f(x) = \sup_{R > 0} |T^R_\lambda f(x)|
\]

for \( \lambda > 0 \), where \( \hat{\cdot} \) denotes the Fourier transform. It is well known by the
works of Carleson and Sjölin [4], Fefferman [8, 9], Tomas [19], and Christ
[6] that \( T^\lambda f \) is bounded on \( L^p(\mathbb{R}^n) \) if and only if \( |1/p - 1/2| < (1 + 2\lambda)/2n \)
provided \( \lambda > 0 \) in dimension 2 and \( \lambda \geq (n-1)/2(n+1) \) in dimension greater
than two. Rubio [16] and Hirschman [12] studied the weighted norm inequality
for the Bochner-Riesz operator \( T^\lambda f \) and showed that \( T^\lambda f \) is bounded on \( L^2(|x|^a) \)
provided \( |a| < 1 + 2\lambda < n \). In 1988 Andersen [1] gave a sufficient condition
and a necessary condition on radial weight \( w(|x|) \) such that the inequality

\[
\int_{\mathbb{R}^n} |T^\lambda f(x)|^2 w(|x|) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(|x|) \, dx
\]

holds for all radial functions \( f \) in \( L^p(w(|x|)) \).

Notice that the Bochner-Riesz operator is a summation operator and \( T^R_\lambda f(x) \)
tends to \( f(x) \) as \( R \) tends to infinity for all Schwartz functions \( f \). Hence it is
meaningful to consider the almost everywhere convergence of \( T^R_\lambda f \) as \( R \) tends
to infinity for some appropriate function \( f \). In 1986 Lu [14] proved that

\[
T^R_\lambda f(x) \to f(x) \quad \text{a.e. as } j \to \infty
\]

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for all \( f \in L^2(|x|^a) \) provided \( 0 < a < \min(2, 2\lambda) < n-1 \) and \( \{R_j\}_{j=1}^{\infty} \) being a Hadamard lacunary sequence, i.e., \( \lim_{j \to \infty} (R_{j+1}/R_j) > 1 \). In addition we can reduce almost everywhere convergence of the operator \( T_{\lambda}^R \) to some maximal inequality. In [14] Lu proved

**Theorem L** [14]. Let \( 0 < \lambda < \frac{1}{2}(n-1) \), \( 0 < a < \min(2, 2\lambda) \), and \( \{R_j\}_{j=1}^{\infty} \) be a Hadamard lacunary sequence. Then

\[
\int \left( \sup_j |T_{\lambda}^R f(x)| \right)^2 |x|^a \, dx \leq C \int |f(x)|^2 |x|^a \, dx.
\]

In 1988 Carbery, Rubio, and Vega proved

**Theorem CRV** [3]. Let \( |a| < 1 + 2\lambda < n \). Then

\[
\int |T_{a}^* f(x)|^2 |x|^a \, dx \leq C \int |f(x)|^2 |x|^a \, dx.
\]

Hence they improved Theorem L.

On other hand, we observe that if \( \lambda \) exceeds the critical index \( \frac{1}{2}(n-1) \) then \( T_{\lambda}^* f \) is dominated by a multiple of the Hardy-Littlewood maximal function \( Mf \) defined by

\[
Mf(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| \, dy,
\]

where supremum is taken over all cubes with center \( x \) and sides parallel to the coordinate axes. Hence a result of Muckenhoupt [11] showed that \( T_{\lambda}^* \) is bounded on \( L^p(w) \) provided \( w \in A_p \), i.e.,

\[
\left( |Q|^{-1} \int_Q w(x) \, dx \right) \left( |Q|^{-1} \int_Q w(x)^{-\frac{(p-1)}{p-1}} \, dx \right)^{p-1} \leq C
\]

holds for all cubes \( Q \subset \mathbb{R}^n \) with sides parallel to the coordinate axes and some \( C \) independent of \( Q \). Then a natural question is whether \( T_{(n-1)/2}^* \) is bounded on \( L^p(w) \) provided \( w \in A_p \) and \( 1 < p < \infty \). In this paper we prove

**Theorem 1.** Let \( 1 < p < \infty \) and \( w \in A_p \). Then

\[
\int |T_{(n-1)/2}^* f(x)|^p w(x) \, dx \leq C \int |f(x)|^p w(x) \, dx.
\]

We also observe that \( T_{(n-1)/2}^{*1}f \) can be written as

\[
\int h(|y|)|y|^{-n} f(x-y) \, dy,
\]

where

\[
h(t) = (2\pi)^{n/2} 2^{(n-1)/2} \Gamma(\frac{1}{2}(n-1)) J_{n-1/2}(t)^{1/2},
\]

\( \Gamma(t) \) denotes the Gamma function and \( J_{n}(t) \) denotes the Bessel function, which is defined by

\[
J_{n}(t) = \frac{(t/2)^n}{\Gamma(n/2+1/2)\Gamma(n/2)} \int_0^{\pi/2} \cos(t \sin u)(\cos u)^{2n} \, du.
\]

Therefore the weighted norm inequality for \( T_{(n-1)/2}^{*1} \) is closely related to the one for the operator introduced by Fefferman [9] and Namazi [15].
Now let us write the operator introduced by Fefferman and Namazi precisely. Let \( h \in L^\infty([0, \infty)) \) and \( \Omega \) be an integrable function on the unit sphere \( S^{n-1} \) having mean zero, i.e., \( \int_{S^{n-1}} \Omega(x) \, d\sigma(x) = 0 \), \( d\sigma \) is the standard measure on \( S^{n-1} \).

Define
\[
T_\varepsilon f(x) = \int_{|y| > \varepsilon} h(|y|) \Omega \left( \frac{y}{|y|} \right) |y|^{-n} f(x - y) \, dy
\]
for \( \varepsilon > 0 \),
\[
T_0 f(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x),
\]
and the associated maximal operator
\[
T^* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|.
\]

There are many works about the operators \( T_0 \) and \( T^* \) (see [5, 7, 15, 17, 18], etc.). Namazi [15] proved that \( T_0 \) is a bounded operator on \( L^p(\mathbb{R}^n) \) for all \( 1 < p < \infty \) when \( \Omega \in L^q(S^{n-1}) \) for some \( q > 1 \), Chen [5] proved that \( T^* \) is also a bounded operator on \( L^p(\mathbb{R}^n) \) when \( \Omega \in L^q(S^{n-1}) \) for some \( q > 1 \), and the second author [18] proved that \( \Omega \) is a sufficient condition such that \( T_0 \) (resp. \( T^* \)) is a bounded operator on \( L^2(\mathbb{R}^n) \). As weighted norm inequalities for the operators \( T_0 \) and \( T^* \), Duoandikoetxea and Rubio [7] proved that \( T^* \) and \( T_0 \) are bounded operators on \( L^p(w) \) provided \( 1 < p < \infty \), \( w \in A_p \), and \( \Omega \in L^\infty(S^{n-1}) \).

In this paper we prove the following with the complex interpolation method.

**Theorem 2.** Let \( 1 < q \leq \infty \), \( \Omega \in L^q(S^{n-1}) \), \( q(q - 1)^{-1} < p < \infty \), and \( w \in A_{p(1-1/q)} \). Then
\[
\int |T_0 f(x)|^p w(x) \, dx \leq C \int |f(x)|^p w(x) \, dx
\]
holds for all \( f \) in \( L^p(w) \).

**Theorem 3.** Let \( \Omega \in L^\infty(S^{n-1}) \) and \( w \in A_p \). Then
\[
\int |T^* f(x)|^p w(x) \, dx \leq C \int |f(x)|^p w(x) \, dx
\]
holds for all \( f \) in \( L^p(w) \).

Hence we prove Duoandikoetxea and Rubio's result in another way. The above results are still interesting even when \( h \equiv 1 \) because in [13] \( \Omega \) satisfies an \( L^r \)-Dini condition for some \( r > 1 \).

### 2. Some lemmas

Define the Bochner-Riesz operator \( T_z^R \) by
\[
(T_z^R f)^-(x) = (1 - R^2|x|^2)^z f^-(x)
\]
for \( \text{Re } z > 0 \). Then we have
Lemma 1 [3]. Let $\Re z > 0$ and $k = 0, 1$. Then
\[
\int \sup_{R > 0} \left| \left( \frac{\partial}{\partial z} \right)^k T_z^R f(x) \right|^2 \, dx \leq C e^{c|\Im z|} (|\Re z|^{-c} + 1) \int |f(x)|^2 \, dx
\]
holds for a constant $C$ independent of $z$.

To prove Theorem 1, we will also use

Lemma 2. Let $\Re z > \frac{1}{2}(n - 1)$. Then the inequality
\[
\sup_{R > 0} |T_z^R f(x)| \leq C (\Re z - \frac{1}{2}(n - 1))^{-c} e^{c|\Im z|} M f(x)
\]
holds for all $f \in L^1_{\text{loc}}(R^n)$ and an absolutely positive constant $C$.

Proof. Write
\[
T_z^R f(x) = \sum_{j=0}^\infty \left( \frac{2^j}{\Gamma(j + 1)} \right) \int J_{n/2+z}(R^{-1}|y|) \cdot |y|^{-(n+1)/2} f(x - y) \, dy.
\]
Also we observe from the asymptotic properties of the Bessel function $J_{n/2+z}(t)$ that
\[
|J_{n/2+z}(t)| \leq C t^{n/2+Re z}
\]
when $t \leq 1$ and
\[
|J_{n/2+z}(t)| \leq C t^{-1/2}
\]
when $t > 1$. Therefore for $\Re z > \frac{1}{2}(n - 1)$ we have
\[
|T_z^R f(x)| \leq C R^{-n/2+Re z} \int |f(x - y)| R^{-n/2-Re z} \, dy
\]
\[
+ C R^{n/2-Re z} \int_{|y| > R} |f(x - y)||y|^{-(n+1)/2} R^{-1/2} \, dy
\]
\[
\leq C M f(x)
\]
and Lemma 2 holds.

Lemma 3 [11]. For $s \in (1, \infty)$ and $w \in A_s$, there exists a positive number $\delta$ such that $w^{1+\delta} \in A_s$.

Lemma 4 (Three-Circles Theorem). Suppose $F$ is a bounded continuous complex-valued function on the closed strip $S = \{x + iy : 0 \leq x \leq 1\}$ that is analytic in the interior of $S$. If $|F(iy)| \leq m_0$ and $|F(1 + iy)| \leq m_1$ for all $y$, then $|F(x + iy)| \leq m_0^{-x} m_1^{-y}$ for all $x + iy \in S$.

To prove Theorems 2 and 3, we need to introduce some notation and use some lemmas. Let $\Omega \subset L^q(S^{n-1})$ for some $q > 1$.

Define
\[
T_{z,\varepsilon} f = \frac{\pi^{(z-1)/2}\Gamma((n-z)/2)}{\Gamma(z/2)} |y|^{-n-z} h(|y|) \Omega \left( \frac{y}{|y|} \right) \chi_{|y| > \varepsilon} * |y|^{-n+z} f, \quad \varepsilon > 0,
\]
for $-\frac{1}{2}(1 - 1/q) < \Re z < 1$, where $*$ is the convolution operation. Denote the kernel function of the operator $T_{z,0}$ by $K_z$. Therefore we have
Lemma 5 [18]. For $-\frac{1}{2}(1 - 1/q) < \text{Re} z < 1$ and $k = 0, 1$, the inequality

$$\int \sup_{\varepsilon > 0} \left| \left( \frac{\partial}{\partial z} \right)^k T_z, \varepsilon f(x) \right|^2 \, dx$$

$$\leq C \left( |\text{Re} z - 1|^{-c} + |\text{Re} z + \frac{1}{2} \left( 1 - \frac{1}{q} \right)|^{-c} \right) e^{c|\text{Im} z|} \int |f(x)|^2 \, dx$$

holds.

Lemma 6. For $0 < \text{Re} z < 1$ and $1 < p < \infty$, the inequality

$$\int \sup_{\varepsilon > 0} |T_z, \varepsilon f(x)|^1 \, dx \leq C(|\text{Re} z|^{-c} + |\text{Re} z - 1|^{-c})e^{c|\text{Im} z|} \int |f(x)|^p \, dx$$

holds.

Lemma 7. For $0 < \text{Re} z < 1$, the inequality

$$\left( R^{-n} \int_{R<|x|<2R} |K_z(x+y) - K_z(x)|^q \, dx \right)^{1/q} \leq C(z) R^{-n} \left( \frac{|y|}{R} \right)^{\text{Re} z}$$

holds for all $R > 0$ and $|y| < \frac{1}{2} R$, where

$$C(z) \leq C(|\text{Re} z|^{-c} + |\text{Re} z - 1|^{-c}) \exp(C|\text{Im} z|).$$

Lemma 8 [13]. Let $K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, $q > 1$, and $\delta > 0$. Suppose

$$\left( R^{-n} \int_{R<|x|<2R} |K(x+y) - K(x)|^q \, dx \right)^{1/q} \leq CR^{-n} \left( \frac{|y|}{R} \right)^{\delta}$$

holds for all $R > 0$ and $|y| < \frac{1}{2} R$. Suppose $\overline{T}$ be defined as $\overline{T} f = K * f$ and $\overline{T}$ be bounded on $L^2(\mathbb{R}^n)$. Then the operator $\overline{T}$ is bounded on $L^p(w)$ for all $q(q-1)^{-1} < p < \infty$ and $w \in A_p(1-1/q)$.

Lemma 9. Let $\Omega \in L^q(S^{n-1})$ for some $q > n$. For $n/q < \text{Re} z < 1$, the inequalities

(i) $|K_z(x)| \leq C(z)|x|^{-n}$,

(ii) $|K_z(x+y) - K_z(x)| \leq C(z)|x|^{-n-(\text{Re} z - n/q)|y|\text{Re} z - n/q)}$

hold for all $x \neq 0$ and $|y| \leq \frac{1}{2} |x|$, where

$$C(z) \leq C(|\text{Re} z - n/q|^{-c} + |\text{Re} z - 1|^{-c}) \exp(C|\text{Im} z|).$$

Lemma 10 [11]. Let $K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Suppose

(i) $|K(x)| \leq C|x|^{-n}$,

(ii) $|K(x+y) - K(x)| \leq C|y|^\delta |x|^{-n-\delta}$

hold for all $x \neq 0$, $|y| < \frac{1}{2} |x|$, and some $\delta > 0$. Suppose an operator $\overline{T}^*$ defined as $\overline{T}^* f = K * f$ is bounded on $L^2(\mathbb{R}^n)$. Then the operator $\overline{T}^*$ defined as

$$\overline{T}^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|y|>\varepsilon} k(y)f(x-y) \, dy \right|$$

is bounded on $L^p(w)$ provided $1 < p < \infty$ and $w \in A_p$.
The proofs of Lemmas 7 and 9 are elementary and the proof of Lemma 6 is similar to the one in [5]. We omit the details here.

3. PROOFS OF THEOREMS

Let \( p \in (1, \infty) \), \( s \in (1, \infty) \), \( 0 < \lambda \leq \frac{1}{2}(n-1) \), and \( q > n \). Denote \( \alpha(p) = \frac{1}{p} - \frac{1}{2} \), \( p^1 = p(p-1)^{-1} \),

\[
\theta_1(p, s, \lambda) = \begin{cases} 
\frac{2\lambda}{n-1}, & \text{when } p = s = 2, \\
\alpha(p) \cdot \frac{p}{s}, & \text{when } 0 < \alpha(p) < \frac{2\lambda}{n-1}, \ p \neq 2, \\
0, & \text{otherwise}; 
\end{cases}
\]

\[
\theta_2(p, s, q) = \begin{cases} 
\left(1 + \frac{2n}{q-1}\right)^{-1}, & \text{when } p = s = 2, \\
\frac{\alpha(p)}{\alpha(s)} \cdot \frac{p}{s}, & \text{when } 0 < \alpha(p) \leq \left(1 + \frac{2n}{q-1}\right)^{-1}, \ p \neq 2, \\
\frac{1}{2} \left(1 - \frac{\alpha(p)}{\alpha(s)}\right) \left(1 - \frac{1}{q}\right) \frac{p}{s} \cdot \frac{q}{n}, & \text{when } \left(1 + \frac{2n}{q-1}\right)^{-1} < \frac{\alpha(p)}{\alpha(s)} < 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Then the following theorems are general versions of Theorems 1 and 3.

**Theorem 4.** Let \( 0 < \lambda \leq \frac{1}{2}(n-1) \), \( |\alpha(p)| < \lambda/(n-1) \), and \( \omega \in A_s \). Then

\[
\int |T_{\lambda}^* f(x)|^p \omega^{\theta_1(p, s, \lambda)}(x) \, dx \leq C \int |f(x)|^p \omega^{\theta_1(p, s, \lambda)}(x) \, dx.
\]

**Theorem 5.** Let \( \Omega \in L^q(S^{n-1}) \) for some \( q > n \) and \( \omega \in A_s \). Then

\[
\int |T^* f(x)|^p \omega^{\theta_2(p, s, q)}(x) \, dx \leq C \int |f(x)|^p \omega^{\theta_2(p, s, q)}(x) \, dx.
\]

By Lemma 3 and the fact \(|x|^a \in A_s \) for \(-n < a < n(s-1)\), we improve Theorem L.

**Proof of Theorem 4.** Let \( f, h \) be two nonnegative smooth functions with compact support and \( R(x) \) be any arbitrary positive measurable function bounded below and above, i.e., \( R(x)^{-1} \) and \( R(x) \) bounded. Suppose \( \omega \in A_s \) and \( \varepsilon_1 \) and \( \varepsilon_2 \) are sufficiently small positive constants chosen later, without loss of generality we assume \( \theta_1(p, s, \lambda) > 0 \). Let

\[
p^{-1} = \frac{1}{2} (1 - \theta) + s^{-1} \theta, \quad \lambda = (1 + \theta) \varepsilon_1 + \theta (\frac{1}{2} (n-1) + \varepsilon_2), \quad 0 < \theta < 1.
\]

Denote \( s(z)^{-1} = \frac{1}{2} (1 - z) + s^{-1} z \) for \( 0 < \text{Re } z < 1 \) and \( f_0(x) = \exp(-|x|^2) \).
Define
\[ f^s_{\delta_1, \delta_2}(x) = (f(x) + \delta_1 f_0(x))^{p(s)^{-1}-1}(w(x) + \delta_2)^{-s^{-1}}f(x), \]
\[ h^s_{\delta_3}(x) = (h(x) + \delta_3 f_0(x))^{p(1-s)^{-1}}h(x), \]
\[ w_N(x) = \begin{cases} w(x), & \text{when } w(x) \leq N, \\ N, & \text{when } w(x) > N, \end{cases} \]
and
\[ \widetilde{T}^{R(x)}_{z, \epsilon_1, \epsilon_2, \delta_3, N}(f)(x) = T^{R(x)}_{\epsilon_1 + z((n-1)/2 + \epsilon_2 - \epsilon_1)}f(x)(w_N(x) + \delta_2)^{s^{-1}}e^{-z^2} \]
where \( N^{-1}, \delta_i (i = 1, 2, 3) \) are small positive numbers.

Hence by Lemma 1 we can show easily
\[ g(z) = \int \widetilde{T}^{R(x)}_{z, \epsilon_1, \epsilon_2, \delta_3, N}(f)(x)h^s_{\delta_3}(x)\,dx \]
is analytic in the strip \( 0 < \Re z < 1 \) and continuous on the closed strip \( 0 \leq \Re z \leq 1 \). In addition \( g(z) \) is a bounded function, hence by Lemma 4 we have
\[ |g(\theta)| \leq C \left( \sup_{t \in \mathbb{R}} |g(it)| \right)^{1-\theta} \left( \sup_{t \in \mathbb{R}} |g(1+it)| \right)^\theta. \]

On the other hand by Lemma 1 we get
\[ \sup_{t \in \mathbb{R}} |g(it)| \leq C_{\epsilon_1, \epsilon_2} e^{-t^2} e^{\epsilon_1|t|} \left( \int |f(x)|^p \,dx \right)^{1/2} \left( \int |h(x)|^{p'} \,dx \right)^{1/2} \]
\[ \leq C_{\epsilon_1, \epsilon_2} \|f\|_p^{1/2} \|h\|_{p'}^{1/2} \]
and by Lemma 2
\[ \sup_{t \in \mathbb{R}} |g(1+it)| \leq C_{\epsilon_1, \epsilon_2} e^{-t^2} \left( \int |T^{R(x)}_{1+it, \epsilon_1, \epsilon_2, \delta_3, N}(f)^{1+it}(x)|^s \,dx \right)^{1/s} \]
\[ \times \left( \int |h^s_{\delta_3}(x)|^{s(s-1)^{-1}} \right)^{s-1/3} \leq C_{\epsilon_1, \epsilon_2} \|f\|_p^{p-1} \|h\|_{p'}^{p'-1(s-1)}. \]

Therefore
\[ (1) \quad |g(\theta)| \leq C_{\epsilon_1, \epsilon_2} \|f\|_p \|h\|_{p'} \]
where \( C_{\epsilon_1, \epsilon_2} \) is independent of \( \delta_1, \delta_2, \delta_3, \) and \( N \).

Write \( g(\theta) \) as
\[ g(\theta) = \int \widetilde{T}^{R(x)}_{\theta, \epsilon_1, \epsilon_2, \delta, N}(f(w + \delta_2)^{-\theta s^{-1}})(x)h(x)\,dx \]
\[ = \int T^{R(x)}_{\lambda}(f(w + \delta_2)^{-\theta s^{-1}})(x)(w_N(x) + \delta_2)^{\theta s^{-1}}h(x)\,dx. \]

Notice that for every \( f \in L^{p'} \) there exist two nonnegative smooth function sequences \( \{f^1_n\} \) and \( \{f^2_n\} \) with compact supports such that
\[ \|f^i_n\|_{p'} \leq C \|f\|_{p'}, \quad i = 1, 2, \quad n \in N, \]
and 
\[ \|f_n^1 - f_n^2 - f\|_{p'} \to 0, \quad as \ n \to \infty. \]
Hence by (1) and (2) we get 
\[ \int |T_R^x(f(w + \delta_2)^{-\theta s^{-1}}(x))|^p \, dw \leq C \int |f(x)|^p \, dx. \]
Let \( N \) tend to infinity in the inequality above. Then we have 
\[ \int |T_R^x(f(w + \delta_2)^{-\theta s^{-1}}(x))|^p \, dw \leq C \int |f(x)|^p \, dx. \]
Notice that \( C \) is independent of \( \delta_2 \) and \( R(x) \) being a measurable function bounded below and above. Therefore 
\[ \int |T_f^x s^{-1}(x)|^p w^{\theta s^{-1}}(x) \, dx \leq C \int |f(x)|^p w^{\theta s^{-1}}(x) \, dx \]
holds for all \( \theta < \theta_1(p, s, \lambda) \neq 0 \) by choosing appropriate \( \varepsilon_1 \) and \( \varepsilon_2 \). By Lemma 3 we can replace \( w \) in (3) by \( w^{1+\delta} \). Hence Theorem 4 holds by choosing \( \theta = \theta_1(p, s, x)/(1 + \delta) \neq 0 \).

**Proof of Theorem 2.** By Lemmas 7 and 8, \( T_z \) is bounded on \( L^p(w) \) provided \( w \in A_p(1-1/q) \) and \( \text{Re} z > 0 \). In addition \( T_z \) is bounded on \( L^2(R^n) \) when \( |\text{Re} z| < \frac{1}{2}(1 - 1/q) \). By the complex interpolation theorem (see [10]) and Lemma 3, \( T_0 \) is bounded on \( L^p(w) \) provided \( p > q(q - 1)^{-1} \) and \( w \in A_p(1-1/q) \). Therefore Theorem 2 holds.

**Proof of Theorem 5.** Define 
\[ T_z^*f(x) = \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} K_z(y) f(x - y) \, dy \right|. \]
Then by Lemmas 9 and 10 we get 
\[ \int |T_z^* f(x)|^p w(x) \, dx \leq C(z) \int |f(x)|^p w(x) \, dx \]
provided \( w \in A_p \) and \( n/q < \text{Re} z < 1 \). In addition the pointwise estimate 
\[ \sup_{\varepsilon > 0} |T_z, \varepsilon f(x)| \leq C(z) \mu f(x) + T_z^* f(x) \]
holds for all \( x \) in \( R^n \), \( n/q < \text{Re} z < 1 \), and 
\[ C(z) \leq C(|\text{Re} z - 1|^{-\epsilon} + |\text{Re} z - n/q|)^{-\epsilon} e^{\epsilon |\text{Im} z|}. \]
Hence we have 
\[ \int \sup_{\varepsilon > 0} |T_z, \varepsilon f(x)|^p w(x) \, dx \leq C(z) \int |f(x)|^p w(x) \, dx \]
for \( n/q < \text{Re} z < 1 \) and \( w \in A_p \). By (4), Lemmas 3, 5, and 6 we can prove the following in such a way as we prove Theorem 4, 
\[ \int \sup_{\varepsilon > 0} |T_z, \varepsilon f(x)|^p w^{(\text{Re} z)n/q}(x) \, dx \leq C(z) \int |f(x)|^p w^{(\text{Re} z)n/q}(x) \, dx \]
provided $0 < \Re z < n/q$, $w \in A_s$, and $C(z) \leq C|\Re z|^{-C} \exp(2|\Im z|^2)$.

Hence
\[
\sup_{\varepsilon > 0} |T_z, \varepsilon f|^p(x) w^\varepsilon(x) dx \leq C \int |f(x)|^p w^\varepsilon(x) dx
\]
holds for $1 < p < \infty$, $q > n$, and $w \in A_s$ and Theorem 5 holds.

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REFERENCES


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