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A NOTE ON THE LARGEST REGULAR SUBALGEBRA OF A BANACH ALGEBRA

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ABSTRACT. A simple proof of Albrecht's result on the existence of the largest closed regular subalgebra of a semisimple commutative Banach algebra is given.

Albrecht [1] proved that any semisimple commutative Banach algebra with identity has the largest closed regular subalgebra, using the theory of decomposable operators. In this short note, we give a simple proof of Albrecht's result without the assumption of semisimplicity. Our proof is quite different from Albrecht's, and based on the consideration of the hull-kernel topology of the carrier space of the given Banach algebra.

Theorem. *If A is a commutative Banach algebra with identity, then there exists a closed regular subalgebra of A which contains all closed regular subalgebras of A.*

Our proof is essentially based on the following

Lemma. *Let X be a commutative Banach algebra with identity and B a Banach subalgebra of X. If B is regular, then for any $b \in B$ the Gelfand transform of b as an element of X is continuous on the carrier space Φ_X of X in the hull-kernel topology.*

Proof. We can assume without loss of generality that B contains the identity of X. Then it is sufficient to show that the restriction map $\theta\colon \Phi_X \to \Phi_B$; $\varphi \to \varphi|B$ is continuous in the hull-kernel topology. To do this let F be a closed subset of Φ_B in the hull-kernel topology. Then $\{\varphi \in \Phi_X\colon \varphi|\ker F = 0\} = \theta^{-1}(F)$. Also since $\ker F \subset \ker \theta^{-1}(F)$, it follows that $\operatorname{hul}(\ker \theta^{-1}(F)) \subset \{\varphi \in \Phi_X\colon \varphi|\ker F = 0\}$. Therefore $\theta^{-1}(F)$ is closed in the hull-kernel topology. In other words, θ is continuous in this topology. □

Proof of Theorem. Let $\operatorname{reg}(A)$ be the closed subalgebra of A generated by the class of all closed regular subalgebras of A. It is sufficient to show that $\operatorname{reg}(A)$ is regular. Note that $\operatorname{reg}(A)$ is a commutative Banach algebra with identity. If

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b is an element of A which belongs to a certain regular Banach subalgebra B of A, then $B \subset \mathrm{reg}(A)$. Hence by the preceding lemma the Gelfand transform of b as an element of $\mathrm{reg}(A)$ is continuous on $\Phi_{\mathrm{reg}(A)}$ in the hull-kernel topology. Since $\mathrm{reg}(A)$ is generated by such b's, the Gelfand transform of an arbitrary element of $\mathrm{reg}(A)$ is also continuous on $\Phi_{\mathrm{reg}(A)}$ in the hull-kernel topology. Consequently $\mathrm{reg}(A)$ is regular. □

References

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