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ON THE JOINT SPECTRAL RADIUS. II

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ABSTRACT. In this paper we show that if $\mathbf{T} = (T_1, \ldots, T_n)$ is a commuting n-tuple of operators on a Hilbert space such that $\sigma(\mathbf{T}) = \prod_{i=1}^n \sigma(T_i)$, then the algebraic joint spectral radius is equal to the geometric one.

In [1] Bunce introduced the algebraic joint spectral radius formula $r_a(\mathbf{T})$ for a commuting n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on a Hilbert space. He conjectured the equality

$$(*) \qquad r_a(\mathbf{T}) = \sup\{|\mathbf{z}| : \mathbf{z} \in \sigma_\pi(\mathbf{T})\},$$

where the joint approximate point spectrum $\sigma_\pi(\mathbf{T})$ of $\mathbf{T}$ is defined as $\sigma_\pi(\mathbf{T}) = \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n :$ there exists a sequence $\{x_k\}$ of unit vectors such that $\lim_{k\to\infty} \|(T_i - z_i)x_k\| \to 0 \ (i = 1, \ldots, n)\}$. In [2] we proved that equality $(*)$ holds for commuting n-tuples of operators on finite-dimensional spaces. In this paper we show that equality $(*)$ also holds for some classes of commuting n-tuples of operators on Hilbert spaces.

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on a complex Hilbert space. We denote the Taylor joint spectrum of $\mathbf{T}$ by $\sigma(\mathbf{T})$. The joint operator norm $\|\mathbf{T}\|$ and the geometric spectral radius $r_g(\mathbf{T})$ of $\mathbf{T} = (T_1, \ldots, T_n)$ are given by

$$\|\mathbf{T}\| = \sup_{\|x\|=1} \left\{ \left(\sum_{i=1}^n \|T_i x\|^2 \right)^{1/2} \right\}$$

and

$$r_g(\mathbf{T}) = \sup\{|z| : z \in \sigma(\mathbf{T})\},$$

respectively. Also define $r_\pi(\mathbf{T}) = \sup\{|z| : z \in \sigma_\pi(\mathbf{T})\}$. Let $F(k, n)$ be the set of all functions from the set $\{1, 2, \ldots, k\}$ to the set $\{1, 2, \ldots, n\}$. For $f \in F(k, n)$, let

$$\mathbf{T}_f = T_{f(1)} \cdot T_{f(2)} \cdot \cdots \cdot T_{f(k)}.$$

The algebraic joint spectral radius $r_a(\mathbf{T})$ of $\mathbf{T}$ is given by

$$r_a(\mathbf{T}) = \inf_k \left\{ \left\| \sum_{f \in F(k, n)} \mathbf{T}_f^* \cdot \mathbf{T}_f \right\|^{1/(2k)} \right\}.$$

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For $n = 1$, $r_a(\mathbf{T})$ is the usual spectral radius formula of T_1. For an operator T, we denote the usual spectral radius of T by $r(T)$. By [1, Lemma 5]

$$r_a(\mathbf{T}) = \lim_{k\to\infty} \left\| \sum_{f\in F(k,n)} \mathbf{T}_f^* \cdot \mathbf{T}_f \right\|^{1/(2k)},$$

and by [1, Proposition 7] $r_\pi(\mathbf{T}) \le r_a(\mathbf{T})$. Letting $k = 1$, we have $r_a(\mathbf{T}) \le \|\mathbf{T}\|$ By [4, Theorem 1] or [5, Corollary 1.4]

$$\{\mathbf{z} \in \sigma(\mathbf{T}) : |\mathbf{z}| = r_g(\mathbf{T})\} \subset \sigma_\pi(\mathbf{T}).$$

Since $\sigma_\pi(\mathbf{T}) \subset \sigma(\mathbf{T})$, this implies that $r_\pi(\mathbf{T}) = r_g(\mathbf{T})$. Hence we have

Theorem 1. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting n-tuple of operators. Then* $r_g(\mathbf{T}) \le r_a(\mathbf{T}) \le \|\mathbf{T}\|$.

When $\mathbf{T} = (T_1, \dots, T_n)$ is a doubly commuting n-tuple of hyponormal operators, it holds that $r_g(\mathbf{T}) = \|\mathbf{T}\|$ (see [3, Theorem 3.4]). Hence equality $(*)$ holds for doubly commuting n-tuples of hyponormal operators.

Theorem 2. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting n-tuple of operators. Then* $r_a(\mathbf{T}) \le (r_a(\mathbf{S})^2 + r(T_n)^2)^{1/2}$, *where* $\mathbf{S} = (T_1, \dots, T_{n-1})$.

Proof. Let $\mathbf{T}(k) = \sum_{f\in F(k,n)} \mathbf{T}_f^* \cdot \mathbf{T}_f$ and $\mathbf{S}(k) = \sum_{f\in F(k,n-1)} \mathbf{S}_f^* \cdot \mathbf{S}_f$ for $k \in \mathbb{N}$. For every $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that $\|T_n^k\| < A^{k/2}$ and $\|\mathbf{S}(k)\| < B^k$ for $k \ge p$, where $A = (r(T_n) + \varepsilon)^2$ and $B = (r_a(\mathbf{S}) + \varepsilon)^2$.

Fix these ε and p. Since

$$\begin{aligned}\mathbf{T}(k) &= \mathbf{S}(k) + {}_kC_1 \cdot T_n^* \cdot \mathbf{S}(k-1) \cdot T_n \\ &\quad + {}_kC_2 \cdot T_n^{*2} \cdot \mathbf{S}(k-2) \cdot T_n^2 + \cdots + {}_kC_k \cdot T_n^{*k} \cdot T_n^k,\end{aligned}$$

we have, for $k > 2p$,

$$\begin{aligned}\|\mathbf{T}(k)\| &\le \|S(k)\| + {}_kC_1 \cdot \|T_n\|^2 \cdot \|\mathbf{S}(k-1)\| \\ &\quad + \cdots + {}_kC_{p-1} \cdot \|T_n\|^{2(p-1)} \cdot \|\mathbf{S}(k-p+1)\| \\ &\quad + {}_kC_p \cdot \|T_n^p\|^2 \cdot \|\mathbf{S}(k-p)\| + \cdots + {}_kC_{k-p} \cdot \|T_n^{k-p}\|^2 \cdot \|\mathbf{S}(p)\| \\ &\quad + {}_kC_{k-p+1} \cdot \|T_n^{k-p+1}\|^2 \cdot \|\mathbf{S}(p-1)\| + \cdots + \|T_n^k\|^2 \\ &\le {}_kC_{p-1} \cdot M_1 \cdot B^k + (A+B)^k + {}_kC_{p-1} \cdot M_2 \cdot A^{k-p+1} \\ &\le M \cdot {}_kC_{p-1} \cdot (A+B)^k,\end{aligned}$$

where

$$\begin{aligned}M_1 &= 1 + \frac{a}{B} + \left(\frac{a}{B}\right)^2 + \cdots + \left(\frac{a}{B}\right)^{p-1} \qquad (a = \|T_n\|^2), \\ M_2 &= \|\mathbf{S}(p-1)\| + A \cdot \|\mathbf{S}(p-2)\| + \cdots + A^{p-2}\|\mathbf{S}(1)\| + A^{p-1}, \\ M &= M_1 + 1 + M_2 A^{1-p}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\|\mathbf{T}(k)\|^{1/(2k)} &\le M^{1/(2k)} \cdot ({}_kC_{p-1})^{1/(2k)} \cdot (A+B)^{1/2} \\ &\to (A+B)^{1/2} \qquad (k \to \infty).\end{aligned}$$

Since ε is arbitrary, it follows that

$$\lim_{k\to\infty} \|\mathbf{T}(k)\|^{1/(2k)} \le (r_a(\mathbf{S})^2 + r(T_n)^2)^{1/2}. \quad \text{Q.E.D.}$$

Immediately we have the following corollaries.

Corollary 1. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting* n*-tuple of operators. If there exists* $\mathbf{z} = (z_1, \dots, z_n) \in \sigma(\mathbf{T})$ *such that* $|z_i| = r(T_i)$ *for every* i $(i = 1, \dots, n)$, *then equality* $(*)$ *holds for* $\mathbf{T}$ *and*

$$r_g(\mathbf{T}) = r_a(\mathbf{T}) = (r(T_1)^2 + \cdots + r(T_n)^2)^{1/2}.$$

In particular, if $\sigma(\mathbf{T}) = \prod_{i=1}^n \sigma(T_i)$, *then the above equality holds.*

Corollary 2. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting* n*-tuple of operators. If* $\sigma(T_i) = \{0\}$ *for every* i $(i = 1, \dots, n)$, *then equality* $(*)$ *holds for* $\mathbf{T}$ *and* $r_g(\mathbf{T}) = r_a(\mathbf{T}) = 0$.

Corollary 3. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting* n*-tuple of operators. If* $\sigma(T_i) = \{0\}$ *for* $i = 1, \dots, n-1$, *then equality* $(*)$ *holds for* $\mathbf{T}$ *and* $r_g(\mathbf{T}) = r_a(\mathbf{T}) = r(T_n)$.

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