SPECIAL $\alpha$-LIMIT POINTS FOR MAPS OF THE INTERVAL

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Abstract. The notion of a special $\alpha$-limit point is defined. For maps of the interval, it is shown that a point is a special $\alpha$-limit point if and only if it is an element of the attracting center.

1. Introduction

Our main result is that for continuous self-maps of an interval, the attracting center and the set of special $\alpha$-limit points coincide. Also, combining the work in this paper with that of Xiong in [9], we will obtain a necessary and sufficient condition for the existence of a periodic point with period not a power of two.

Throughout this paper, $f$ will be a continuous self-map of an interval $I$. For every positive integer $n$, $f^n$ will denote the function $f$ composed with itself $n$ times. Using the forward orbit of a point $x$, the $\omega$-limit set $\omega(x)$ of $x$ is defined to be the collection of all limit points of the sequence $(f^n(x))_{n=1}^{\infty}$. Elements of $\omega(x)$ are called $\omega$-limit points. A point $z$ is said to be recurrent if $z \in \omega(z)$.

In [9] the $\alpha$-limit set $\alpha(x)$ is defined for self-maps of the interval. In this paper we study the special $\alpha$-limit set of a point $x$ and denote it by $\sa(x)$. A point $y$ is an element of the set $\sa(x)$ provided there exists a sequence of positive integers $(n(i))_{i=1}^{\infty}$ and a sequence of points $(y_i)_{i=0}^{\infty}$ such that

\begin{enumerate}
  \item $x = y_0$,
  \item $f^{n(i)}(y_i) = y_{i-1}$,
  \item $\lim_{i \to \infty} y_i = y$.
\end{enumerate}

Note that if such a sequence exists then $y$ is an element of $\sa(y_i)$ for every $i$.

In order to state our theorem, we must define what is called the attracting center of a dynamical system. For a subset $Y$ of $I$, define $\Lambda(Y) = \bigcup_{x \in Y} \omega(x)$. Let $\Lambda^1 = \Lambda(I)$. For every $n > 1$, inductively define $\Lambda^n = \Lambda(\Lambda^{n-1})$. The attracting center $\Lambda^\infty$ is then the intersection of the sets $\Lambda^n$.

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Theorem. Suppose that \( f \) is a continuous self-map of an interval \( I \). Then the following are equivalent:

1. \( x \in s\alpha(y) \) for some \( y \),
2. \( x \in s\alpha(x) \),
3. \( x \in \Lambda^\infty \).

Combining this theorem with the work of Xiong [9], we obtain the following corollary. The period of a periodic point \( x \) is the least \( n \) such that \( f^n(x) = x \).

Corollary. The following are equivalent:

1. Some point \( y \) that is not recurrent is a special \( \alpha \)-limit point.
2. Some periodic point has period that is not a power of two.

A weaker form of recurrence than that of being an \( \omega \)-limit point is that of being a nonwandering point. Let \( y \in I \). Then \( y \) is a nonwandering point if for every open neighborhood \( V_y \) of \( y \), \( V_y \cap f^m(V_y) \neq \emptyset \) for some \( m \).

A pointwise definition of the set of nonwandering points can be obtained by considering the backwards orbit of a point. A point \( y \) is an element of \( \alpha(x) \) provided that a strictly increasing sequence of positive integers \( (n(i))_{i=1}^\infty \) and a sequence of points \( (y_i)_{i=1}^\infty \) can be found such that

1. \( f^{n(i)}(y_i) = x \),
2. \( \lim_{i \to \infty} y_i = y \).

Elements of \( \alpha(x) \) are called \( \alpha \)-limit points. The sets of all \( \alpha \)-limit points, in general, strictly contains the set of nonwandering points. A pointwise definition of the set \( \Omega \) of nonwandering points, however, can be obtained using

Proposition A [6]. If \( f \) is a self-map of an interval \( I \) then \( x \in \Omega \) if and only if \( x \in \alpha(x) \).

We now provide an example of a self-map of \([0, 4]\) for which \( \alpha(1) \neq s\alpha(1) \). A smooth version of this example is used in [7] to show that the point 1 is a nonwandering point but not an \( \omega \)-limit point. We will see later that if \( y \) is a nonwandering point that is not an \( \omega \)-limit point then \( y \in \alpha(y) - s\alpha(y) \).

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Example.

\[
\begin{align*}
f(x) &= \begin{cases} 
x + 2 & \text{if } x \in [0, 1), \\
3x & \text{if } x \in [1, \frac{4}{3}), \\
-\frac{8}{9}x + 10 & \text{if } x \in [\frac{4}{3}, 2), \\
2x - 3 & \text{if } x \in [2, \frac{7}{2}), \\
4 & \text{if } x \in [\frac{7}{2}, 4].
\end{cases}
\end{align*}
\]

It is clear from the graph of the function \( f \) that there is a strictly increasing sequence of points \( (w_i)_{i=1}^\infty \) with \( w_1 = 2 \), \( w_i \in (2, 3) \) and \( f(w_i) = w_{i-1} \) for \( i > 1 \). For every \( i \geq 1 \) there is a \( z_i \in [0, 1] \) with \( f(z_i) = w_i \). Thus \( f^{i+1}(z_i) = 1 \) and \( \lim_{i \to \infty} z_i = 1 \), hence \( 1 \in \alpha(1) \). The point 1, however, cannot be a special \( \alpha \)-limit point since \( f^n([1, \frac{14}{9}]) = [3, 4] \) for every \( n > 0 \) and \( f^{-1}([0, 1]) = \emptyset \). To see this, one needs only to realize that if 1 were a special \( \alpha \)-limit point, we could find points \( y_1, y_2, \) and \( y_3 \) all elements of \((0, \frac{14}{9})\) and integers \( n(3) \) and \( n(2) \) with \( f^{n(3)}(y_3) = y_2 \) and \( f^{n(2)}(y_2) = y_1 \).
2. Preliminaries

A point $y$ is said to be a $\gamma$-limit point of $x$ if $y \in \omega(x) \cap \alpha(x)$. Let $\gamma(x) = \omega(x) \cap \alpha(x)$ and $\Gamma = \bigcup_{x \in I} \gamma(x)$. It follows from Proposition A that the set of recurrent points $R$ is contained in the set $\Gamma$. In his investigation of the set $\Lambda^\infty$, Xiong [9] proves the following theorem.

**Theorem A.** Suppose that $f$ is a continuous self-map of an interval. Then

1. $\Gamma = \Lambda^2 = \Lambda^\infty \subseteq \overline{P}$, where $P$ is the set of periodic points;
2. $R = \Gamma$ if and only if the period of every periodic point is a power of 2, where $R$ is the set of all recurrent points.

A function $f$ displays a horseshoe if there are disjoint closed intervals $J$ and $K$ in the interval and a positive integer $n$ such that $f^n(J) \supset J$, $f^n(K) \supset K$, $f^n(K) \supset J$, and $f^n(J) \supset K$. The content of statement (2) of Theorem A is that $\Gamma$ strictly contains $R$ if and only if $f$ displays a horseshoe. In general, the following set relation holds for self-maps of the interval

$$P \subseteq R \subseteq \Gamma \subseteq \overline{P} \subseteq \Lambda^1 \subseteq \Omega.$$ 

In the above example, the set $\Lambda^1$ is strictly contained in $\Omega$. For an example of a function where $P$ is strictly contained in $\Lambda^1$ see [2]. In [3] an example of a function with $\Gamma \neq \overline{P}$ is constructed.

3. Proof of the theorem

We will prove several lemmas from which the theorem will follow immediately.

**Lemma 1.** If $x \in \sigma(y)$ then $f^n(x) \in \sigma(y)$ for every $n$ and $\omega(x) \subseteq \sigma(y)$.

**Proof.** Suppose that $x \in \sigma(y)$. Then we can find $y_i$ with $\lim_{i \to \infty} y_i = x$ and $n(i) > 0$ such that $f^{n(i)}(y_i) = y_{i-1}$ and $f^{n(1)}(y_1) = y$.

Consider the sequence

$$(y, f^{n(1)-1}(y_1), \ldots, y_1, f^{n(2)-1}(y_2), \ldots, y_2, f^{n(3)-1}(y_3), \ldots)$$

$$\equiv (y, z_1, z_2, z_3, \ldots).$$

Note that if $m < \sum_{i=1}^j n(i)$ then $f^m(y_j)$ is a term of the sequence $(z_i)_{i=1}^\infty$. One can see that any limit point of the sequence $(z_i)_{i=1}^\infty$ is an element of $\sigma(y)$.

Let $z \in \{f(x), f^2(x), \ldots\} \cup \omega(x)$ and let $\epsilon > 0$ be given. Then we can find an $m > 0$ with $f^m(x) \in (z-\epsilon, z+\epsilon)$. The function $f^m$ is continuous and $\lim_{i \to \infty} y_i = x$, thus for some $N > 0$ if $j > N$ then $f^m(y_j) \in (z-\epsilon, z+\epsilon)$. If we take $j > \max(N, m)$ then $m < j \leq \sum_{i=1}^j n(i)$. Thus $f^m(y_j) \in (z-\epsilon, z+\epsilon)$ and is a term of the sequence $(z_i)_{i=1}^\infty$. It follows that $z$ is a limit point of the sequence $(z_i)_{i=1}^\infty$. \qed

To state Lemma 2, we develop some notation. We will use the symbols $\alpha_R(x)$ (resp., $\alpha_L(x)$) to denote the set of all points $y$ such that there exists a sequences $(n(i))_{i=1}^\infty$ and $(x_i)_{i=1}^\infty$ with $\lim_{i \to \infty} n(i) = \infty$, $\lim_{i \to \infty} x_i = y$, $f^{n(i)}(x_i) = x$, and $y < x_i$ (resp., $x_i < y$) for every $i > 0$.

**Lemma 2.** (1) If for some $\epsilon > 0$, $x \in \alpha_L(y)$ for every $y \in (x-\epsilon, x)$, then for every $y \in (x-\epsilon, x)$, $x \in \sigma(y)$.
(2) If for some \( \varepsilon > 0 \), \( x \in \alpha_R(y) \) for every \( y \in (x, x + \varepsilon) \), then for every \( y \in (x, x + \varepsilon) \), \( x \in \sigma_a(y) \).

**Proof.** We will prove (1). Let \( y \in (x - \varepsilon, x) \). Since \( x \in \alpha_L(y) \), we can find \( y_1 \in (x - \varepsilon/2, x) \) and \( n(1) > 0 \), with \( f^{n(1)}(y_1) = y \). By hypothesis, \( x \in \alpha_L(y_1) \); thus we can find \( y_2 \in (x - \varepsilon/2^2, x) \) and \( n(2) > 0 \), with \( f^{n(2)}(y_2) = y_1 \). Continuing in this way, we obtain a sequence \( (y_i)_{i=1}^{\infty} \) and \( n(i) > 0 \), with \( y_i \in (x - \varepsilon/2^i, x) \) and \( f^{n(i)}(y_i) = y_{i-1} \).

Before proceeding any further, we need the definitions of almost periodic points and minimal sets. If given an open set \( V_y \) containing \( y \), one can find an integer \( n > 0 \) such that for any integer \( q > 0 \) there is an integer \( r, q \leq r \leq q + n \) with \( f^r(y) \in V_y \), then \( y \) is said to be almost periodic. If for every \( x \in \omega(y) \) we have that \( \omega(x) = \omega(y) \), then \( \omega(y) \) is said to be a minimal set. A well-known fact is that for compact metric spaces a point \( y \) is almost periodic if and only if \( y \in \omega(y) \) and \( \omega(y) \) is a minimal set.

**Lemma 3.** If \( x \) is almost periodic then \( x \in \sigma_a(x) \).

**Proof.** Since \( x \) is almost periodic, \( x \in \omega(x) \) and \( \omega(x) \) is a minimal set. Since \( f(\omega(x)) = \omega(x) \), we can find a sequence of points \( (z_i)_{i=1}^{\infty} \) with \( z_i \in \omega(x) \), \( f(z_i) = z_{i-1} \), and \( f(z_1) = x \). Let \( y \) be a limit point of this sequence. Then \( y \in \sigma_a(x) \) and, since \( \omega(x) \) is a closed set, \( y \in \omega(x) \). Since in this case \( \omega(x) \) is a minimal set, \( x \in \sigma_a(x) \). □

Let \( Y \) be a subset of \( I \). We will denote the closure of \( Y \) by \( \overline{Y} \). A point \( z \) is called a right-side (resp., left-side) accumulation point of \( Y \) if for every \( \varepsilon > 0 \), \( (z, z + \varepsilon) \cap Y \neq \emptyset \) (resp., \( (z - \varepsilon, z) \cap Y \neq \emptyset \)). The right-side closure \( \overline{Y}_R \) (resp., left-side closure \( \overline{Y}_L \)) is the union of \( Y \) and the set of right-side (resp., left-side) accumulation points of \( Y \).

In Lemma 4 bilateral limit points of periodic points are considered. We will see that such points are special \( \alpha \)-limit points. If in addition the point \( x \) in question is not almost periodic, then we show that \( x \) is the end point of interval \( J \) with \( x \in \sigma_a(y) \) for every \( y \in J \). In either case we will see that \( x \in \sigma_a(x) \).

**Lemma 4.** If \( x \in \overline{P}_L \cap \overline{P}_R \) then \( x \in \sigma_a(x) \). If in addition \( x \) is not almost periodic, then for some \( \varepsilon > 0 \), either \( x \in \sigma_a(y) \) for every \( y \in (x - \varepsilon, x) \), or \( x \in \sigma_a(y) \) for every \( y \in (x, x + \varepsilon) \).

**Proof.** According to Lemma 3, we may assume that \( x \) is not almost periodic. If for every \( \varepsilon > 0 \) we could find an \( n > 0 \) such that \( f^{nk}(x) \notin (x - \varepsilon, x + \varepsilon) \) for every \( k \), then \( x \) would be almost periodic. Thus we can find an \( \varepsilon > 0 \) such that if \( n > 0 \) is fixed then for some \( k > 0 \), \( f^{nk}(x) \notin (x - \varepsilon, x + \varepsilon) \).

Fix such an \( \varepsilon > 0 \). By our hypothesis, \( \lim_{i \to \infty} p_i = x \), \( \lim_{i \to \infty} q_i = x \) with \( p_i < x \) and \( x < q_i \) for two sequences of periodic points \( (p_i)_{i=1}^{\infty} \) and \( (q_i)_{i=1}^{\infty} \). Furthermore, we may assume that \( p_i \in (x - \varepsilon, x) \) and \( q_i \in (x, x + \varepsilon) \). Let \( n(i) \) be the period of \( p_i \) and \( m(i) \) the period of \( q_i \). Let \( l(i) = n(i)m(i) \). It follows from our choice of \( \varepsilon \) that we can find \( k(i) > 0 \) with \( f^{l(i)k(i)}(x) \notin (x - \varepsilon, x + \varepsilon) \).

Suppose by choosing a subsequence, if needed, that \( f^{l(i)k(i)}(x) < x - \varepsilon \) for every \( i \). Since the period of \( p_i \) is \( n(i) \), \( f^{l(i)k(i)}(p_i) = p_i \); thus

\[
(*) \quad (x - \varepsilon, p_i) \subseteq f^{l(i)k(i)}([p_i, x]).
\]

Let \( y \in (x - \varepsilon, x) \). We will now show that \( x \in \sigma_a(y) \). Since \( \lim_{i \to \infty} p_i = x \), we can find a positive integer \( i(1) \) such that \( x - \varepsilon < y < p_{i(1)} < x \). It follows
from (*) that there exists $z_{i(1)} \in (p_i(1), x)$ with $f^{i(i(1))}k(i(1))(z_{i(1)}) = y$. Since $z_{i(1)} \in (p_i(1), x)$, we can find a positive integer $i(2)$ with

$$x - \varepsilon < y < p_i(1) < z_{i(1)} < p_i(2) < x.$$ 

It follows from (*) that there exists $z_{i(2)} \in (p_i(2), x)$ with $f^{i(i(2))}k(i(2))(z_{i(2)}) = z_{i(1)}$. Continuing in this way, it is possible to find $i(j), p_{i(j)},$ and $z_{i(j)}$ with

$$x - \varepsilon < y < p_i(1) < z_{i(1)} < p_i(2) < \cdots < p_i(j) < z_{i(j)} < x$$

such that

$$f^{i(i(j))}k(i(j))(z_{i(j)}) = z_{i(j-1)}.$$ 

Since $\lim_{j \to \infty} p_i = x$, we have that $\lim_{j \to \infty} z_{i(j)} = x$. Thus $x \in \mathcal{S}(y)$.

To finish the proof of the lemma it suffices to show that we can find a point $z \in (x - \varepsilon, x)$ and an $n > 0$ with $f^n(z) = x$. The argument that we will give here is taken from the proof of Proposition C in [9]. Let $g = f^{n(1)}$ and $L = [p_1, x]$. Then since $g(p_1) = p_1$, $K = L \cup g(L) \cup g^2(L) \cup \cdots$ is an interval. Let $l(j)$ denote the period of $p_j$ with respect to $g$. For $k = 1, 2, 3$ suppose that a subsequence of $g^{l(1)-k}(p_1)g^{l(2)-k}(p_2) \cdots g^{l(j)-k}(p_j) \cdots$ converges to $u_k \in K$. It is clear that $g^{k}(u_k) = x$. If $u_k' = u_k''$ for some $k' < k''$, then

$$g^{k''-k'}(x) = g^{k''-k'}(g^{k'}(u_k')) = g^{k''}(u_k') = g^{k''}(u_k'') = x.$$ 

However, $x$ is by assumption not periodic, thus we must have that $u_1, u_2, \text{and } u_3$ are distinct points in $K$. Since $K$ is an interval, $u_k \in K$ for some $k$. Hence for some point $z \in L$ and $n \geq 0$, $g^n(z) = u_k$. Thus $g^{n+k}(z) = x$. □

Note that the last argument in the proof of Lemma 4 shows that for points $x \in \overline{P}_L - P$ (resp., $\overline{P}_R - P$), we have $x \in \omega_{L}(x)$ (resp., $\omega_{R}(x)$). We will use the symbols $\omega_{L}(x)$ (resp., $\omega_{L}(x)$) to denote the set of right-side (resp., left-side) accumulation points of the set $\{f^n(x)\}_{n=1}^\infty$.

**Lemma 5.** If $x \in R$ then $x \in \mathcal{S}_{L}(x)$.

**Proof.** Let $x \in R$. If $\omega(x)$ is a minimal set, then according Lemma 3, $x \in \mathcal{S}_{L}(x)$.

We assume that $\omega(x)$ is not a minimal set. Applying Lemma 3 of [1] to the compact set $\omega(x)$, we have that the set $A = \{y|\omega(y) = \omega(x)\} \cap \omega(x)$ is an uncountable set. Let $\Delta$ denote the symmetric difference of sets. Since $\overline{P}_L \Delta \overline{P}_R$ and $\omega_{L}(x) \Delta \omega_{R}(x)$ are both countable, $\overline{P}_L \cap \overline{P}_R$ (resp., $\omega_{L}(x) \cap \omega_{R}(x)$) is the compliment of a countable set in $\overline{P}$ (resp., $\omega(x)$); since $\omega(x) \subseteq \overline{P}$, it follows that

$$\omega_{L}(x) \cap \omega_{R}(x) \subseteq \overline{P}_L \cap \overline{P}_R$$

is the compliment of a countable set in $\omega(x)$. But then $A$, being an uncountable subset of $\omega(x)$, must intersect the set $\omega_{L}(x) \cap \omega_{R}(x) \cap \overline{P}_L \cap \overline{P}_R$. Let $y$ be a point from $A$ in this intersection.

Since $\omega(x) = \omega(y)$ and $\omega(x)$ is not a minimal set, it must be that $y$ is not almost periodic. Without loss of generality, by Lemma 4 we may choose an $\varepsilon > 0$ such that if $z \in (y - \varepsilon, y]$ then $y \in \mathcal{S}(z)$. Now $y \in \omega_{L}(x)$; hence for some positive integer $n$, $f^n(x) \in (y - \varepsilon, y)$. Since the function $f^n$ is continuous, for some $\delta > 0$

$$f^n((x - \delta, x + \delta)) \subseteq (y - \varepsilon, y).$$
Now \( R \subseteq \Omega \), so by Proposition A, \( x \in \alpha(x) \). Thus for some \( z \in (x - \delta, x + \delta) \) and \( m > n \), \( f^m(z) = x \). By \((**)\), \( f^n(z) \in (y - \varepsilon, y) \), hence \( y \in s\alpha(f^n(z)) \) and \( f^{m-n}(f^n(z)) = x \). It follows that \( y \in s\alpha(x) \). Since \( x \in \omega(y) \), by Lemma 1, \( x \in s\alpha(x) \). □

We consolidate our proof that (1) implies (2) implies (3) in our theorem into the proof of Proposition 1. Before we prove Proposition 1, two well-known results must be stated. The first, Lemma B is a description of the dynamical behavior that can occur on subintervals of \( I \) containing no periodic points. The second, Proposition C, is a description of a situation under which one can conclude that a point is an element of \( \omega_R(x) \) (resp., \( \omega_L(x) \)) for some \( x \). For a proof of Lemma B see [5] and for Proposition C see [4]. Proposition C is due to Sharkovskii in [8].

**Lemma B.** Let \( J \) be a subinterval of \( I \) that does not contain any periodic points. If \( x \) and \( y \) are elements of \( J \) and if for some \( n \) and \( m \) both \( f^n(x) \) and \( f^m(y) \) are elements of \( J \), then either \( f^n(x) < x \) and \( f^m(y) < y \), or \( f^n(x) > x \) and \( f^m(y) > y \).

**Proposition C.** If for every \( \varepsilon > 0 \), an \( n > 0 \) can be found with
\[
(c, c + \varepsilon) \cap f^n((c, c + \varepsilon)) \neq \emptyset
\]
(resp., \( (c - \varepsilon, c) \cap f^n((c - \varepsilon, c)) \neq \emptyset \)), then \( c \in \omega_R(y) \cup P \) (resp., \( c \in \omega_L(y) \cup P \)) for some \( y \in I \).

**Proposition 1.** If \( x \in s\alpha(y) \) for some \( y \) then \( x \in s\alpha(x) \) and \( x \in \Gamma \).

**Proof.** Let \( x \in s\alpha(y) \). Since \( R \subseteq \Gamma \), by Lemma 5 we may assume that \( x \) is not recurrent.

Fix an \( \varepsilon > 0 \) such that \( f^n(x) \notin (x - \varepsilon, x + \varepsilon) \) for every \( n > 0 \). As \( x \in s\alpha(y) \), we can find a sequence \( (y_i)_{i=1}^{\infty} \) and \( n(i) > 0 \) with \( \lim_{i \to \infty} y_i = x \) and \( f^{n(i)}(y_i) = y_{i-1} \). By choosing a subsequence of the sequence \( (y_i)_{i=1}^{\infty} \), we may assume that \( y_i \in (x - \varepsilon, x) \) for every \( i > 0 \).

**Claim 1.** The point \( x \) is an element of \( \bar{P}_L - P \) and hence, as noted above, an element of \( \alpha_L(x) \).

**Proof of Claim 1.** Let \( \delta > 0 \) be given. It is possible to find a \( j > 0 \) with \( y_j \) and \( y_{j-1} \) both elements of \( (x - \delta, x) \) and hence, \( (x - \delta, x) \cap f^{n(j)}((x - \delta, x)) = \emptyset \). Since \( x \) is not periodic by Proposition C, \( x \in \omega_L(y) \) for some \( y \). Now if, on the contrary, \( x \notin \bar{P}_L \) then we can find an \( \varepsilon > 0 \) such that \( (x - \varepsilon, x) \cap P = \emptyset \). Since \( x \in \omega_L(y) \), we can assume that \( y \in (x - \varepsilon, x) \). Thus for some \( n > 0 \), \( x - \varepsilon < y < f^n(y) < x \). Since \( \lim_{i \to \infty} y_i = x \), we can find an integer \( j \) with \( x - \varepsilon < y_{j-1} < y_j < x \). This is, however, a contradiction to Lemma B as \( f^{n(j)}(y_j) = y_{j-1} \). Thus \( x \in \bar{P}_L - P \) and hence \( x \in \alpha_L(x) \).

**Claim 2.** For some \( \delta > 0 \), \( x \in s\alpha(y) \) for every \( y \in (x - \delta, x) \).

**Proof of Claim 2.** To prove this claim we first define the integers \( M(k, j) \) for every \( k > 0 \) and \( j > 0 \), according to the equation \( M(k, j) = \sum_{i=1}^{j} n(k + i) \). Since \( f^{n(i)}(y_i) = y_{i-1} \), we have that \( f^{M(k, j)}(y_{k+j}) = y_k \). We consider two cases.
Case 1. For some \(k > 0\), \(x + \varepsilon < f^{M(k,j)}(x)\) for an infinite number of \(j\). Fix such a \(k > 0\). Let \((h(i))_{i=1}^{\infty}\) be an increasing subsequence of positive integers such that \(x < f^{M(k,h(i))}(x)\) for every \(i > 0\). In this case we will show that for every \(y \in (y_k, x)\), \(x \in s\alpha(y)\). Since \(x \in \alpha_L(x)\), by Lemma 2 it suffices to show that \(x \in \alpha_L(y)\) for every \(y \in (y_k, x)\). So let \(y \in (y_k, x)\). For every \(i > 0\),
\[
[y_k, f^{M(k,h(i))}(x)] \subseteq f^{M(k,h(i))}([y_{k+h(i)}, x])
\]
Hence for every \(i > 0\) we can find a point \(z_i \in [y_{k+h(i)}, x]\) with \(f^{M(k,h(i))}(z_i) = y\). Since \(\lim_{i \to \infty} y_{k+h(i)} = x\), it follows that \(x \in \alpha_L(y)\).

Case 2. For every \(k > 0\), we can find a positive integer \(M(k)\) so that if \(j > M(k)\) then \(f^{M(k,j)}(x) < x - \varepsilon\). In this case we will prove that for \(y \in (x - \varepsilon, x), x \in \alpha_L(y)\). Let \(y \in (x - \varepsilon, x)\). We can find a \(k > 0\) with \(y < y_k\). By our assumption for \(j > M(k)\), we have
\[
f^{M(k,j)}(x) < x - \varepsilon < y < y_k.
\]
It follows that for every \(j > M(k)\) we can find a \(z_j \in [y_{k+j}, x]\) with \(f^{M(k,j)}(z_j) = y\). As in Case 1, since \(\lim_{j \to \infty} y_{k+j} = x\), \(x \in \alpha_L(y)\). By Lemma 2 and the fact that \(x \in \alpha_L(x)\), we see that for every \(y \in (x - \varepsilon, x), x \in \alpha_L(y)\). This completes the proof of the claim.

To complete the proof of the lemma we must show that \(x \in \Gamma\). Let \(\delta > 0\) be such that for every \(y \in (x - \delta, x), x \in \alpha_L(y)\). According to the proof of Claim 1, \(x \in \omega_L(y)\) for some \(y \in (x - \delta, x)\), and hence \(x \in \omega_L(y) \cap \alpha(y)\). Since \(\alpha_L(y) \subseteq \alpha_L(y), x \in \Gamma\).

The following lemma will complete the proof of the theorem.

**Lemma 6.** If \(x \in \Gamma\) then \(x \in \alpha_L(x)\).

**Proof.** If \(x \in \Gamma\) then, by Lemma 5, \(x \in \alpha_L(x)\). According to Xiong in [9],
\[
\Gamma = \left( \bigcup_{y \in I} \omega_L(y) \cap \alpha_L(y) \right) \cup \left( \bigcup_{y \in I} \omega_R(y) \cap \alpha_R(y) \right) \cup P.
\]
Thus we assume that \(x \notin \Gamma\) and that \(y\) has been chosen with \(x \in \omega_L(y) \cap \alpha_L(y)\). Let \(\varepsilon > 0\) be chosen so that if \(n > 0\) then \(f^n(x) \notin (x - \varepsilon, x + \varepsilon)\).

Let \((y_i)_{i=1}^{\infty}\) and \((z_i)_{i=1}^{\infty}\) be sequences with \(y_i \in (x - \varepsilon, x), z_i \in (x - \varepsilon, x), f^{n(i)}(y) = y_i, f^{m(i)}(z_i) = y, \lim_{i \to \infty} n(i) = \infty, \) and \(\lim_{i \to \infty} m(i) = \infty\). Let \(L(i, j) = m(i) + n(j)\). Then \(f^{L(i,j)}(z_i) = y_j\) for all \(i > 0\) and \(j > 0\).

**Case 1.** There exists \(j > 0\) such that \(f^{L(i,j)}(x) > x + \varepsilon\) for an infinite number of \(i\). Let \((h(i))_{i=1}^{\infty}\) be an increasing subsequence of positive integers such that \(f^{L(h(i),j)}(x) > x + \varepsilon\). Since \(f^{L(h(i),j)}(z_{h(i)}) = y_j\), it follows that
\[(***) \quad [y_j, x + \varepsilon) \subseteq f^{L(h(i),j)}([z_{h(i)}, x]).\]
To finish the proof by Lemmas 2 and Proposition 1, it suffices to show that for every \(y \in (y_j, x), x \in \alpha_L(y)\).

Let \(y \in (y_j, x)\). For every \(i > 0\), there exists \(w_i \in (z_{h(i)}, x)\) with \(f^{L(h(i),j)}(w_i) = y\). It follows that \(x \in \alpha_L(y)\).

**Case 2.** If \(j > 0\) then there exists an integer \(M(j)\) such that for \(i > M(j), f^{L(i,j)}(x) < x - \varepsilon\).
To finish the proof in this case, we will show that if \( y \in (x - \varepsilon, x) \) then, \( x \in \alpha_L(y) \). Let \( y \in (x - \varepsilon, x) \). Since \( \lim_{i \to \infty} y_i = x \), we can find a \( j > 0 \) with \( x - \varepsilon > y < y_j \). For every \( i > M(j) \) we have
\[
(x - \varepsilon, y_j] \subseteq f^{L(i,j)}([z_i, x]).
\]
Hence for every \( i > M(j) \), we can find \( w_i \in (z_i, x) \) with \( f^{L(i,j)}(w_i) = y \). It follows that \( x \in \alpha_L(y) \).

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References


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