SPECIAL $\alpha$-LIMIT POINTS FOR MAPS OF THE INTERVAL

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ABSTRACT. The notion of a special $\alpha$-limit point is defined. For maps of the interval, it is shown that a point is a special $\alpha$-limit point if and only if it is an element of the attracting center.

1. Introduction

Our main result is that for continuous self-maps of an interval, the attracting center and the set of special $\alpha$-limit points coincide. Also, combining the work in this paper with that of Xiong in [9], we will obtain a necessary and sufficient condition for the existence of a periodic point with period not a power of two.

Throughout this paper, $f$ will be a continuous self-map of an interval $I$. For every positive integer $n$, $f^n$ will denote the function $f$ composed with itself $n$ times. Using the forward orbit of a point $x$, the $\omega$-limit set $\omega(x)$ of $x$ is defined to be the collection of all limit points of the sequence $(f^n(x))_{n=1}^{\infty}$. Elements of $\omega(x)$ are called $\omega$-limit points. A point $z$ is said to be recurrent if $z \in \omega(z)$.

In [9] the $\alpha$-limit set $\alpha(x)$ is defined for self-maps of the interval. In this paper we study the special $\alpha$-limit set of a point $x$ and denote it by $s\alpha(x)$. A point $y$ is an element of the set $s\alpha(x)$ provided there exists a sequence of positive integers $(n(i))_{i=1}^{\infty}$ and a sequence of points $(y_i)_{i=0}^{\infty}$ such that

1. $x = y_0$,
2. $f^{n(i)}(y_i) = y_{i-1}$,
3. $\lim_{i \to \infty} y_i = y$.

Note that if such a sequence exists then $y$ is an element of $s\alpha(y_i)$ for every $i$.

In order to state our theorem, we must define what is called the attracting center of a dynamical system. For a subset $Y$ of $I$, define $\Lambda(Y) = \bigcup_{x \in Y} \omega(x)$. Let $\Lambda^1 = \Lambda(I)$. For every $n > 1$, inductively define $\Lambda^n = \Lambda(\Lambda^{n-1})$. The attracting center $\Lambda^\infty$ is then the intersection of the sets $\Lambda^n$.

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Theorem. Suppose that $f$ is a continuous self-map of an interval $I$. Then the following are equivalent:

1. $x \in s\alpha(y)$ for some $y$,
2. $x \in s\alpha(x)$,
3. $x \in \Lambda^\infty$.

Combining this theorem with the work of Xiong [9], we obtain the following corollary. The period of a periodic point $x$ is the least $n$ such that $f^n(x) = x$.

Corollary. The following are equivalent:

1. Some point $y$ that is not recurrent is a special $\alpha$-limit point.
2. Some periodic point has period that is not a power of two.

A weaker form of recurrence than that of being an $\omega$-limit point is that of being a nonwandering point. Let $y \in I$. Then $y$ is a nonwandering point if for every open neighborhood $V_y$ of $y$, $V_y \cap f^m(V_y) \neq \emptyset$ for some $m$.

A pointwise definition of the set of nonwandering points can be obtained by considering the backwards orbit of a point. A point $y$ is an element of $\alpha(x)$ provided that a strictly increasing sequence of positive integers $(n(i))_{i=1}^\infty$ and a sequence of points $(y_i)_{i=1}^\infty$ can be found such that

1. $f^{n(i)}(y_i) = x$,
2. $\lim_{i \to \infty} y_i = y$.

Elements of $\alpha(x)$ are called $\alpha$-limit points. The sets of all $\alpha$-limit points, in general, strictly contains the set of nonwandering points. A pointwise definition of the set $\Omega$ of nonwandering points, however, can be obtained using

Proposition A [6]. If $f$ is a self-map of an interval $I$ then $x \in \Omega$ if and only if $x \in \alpha(x)$.

We now provide an example of a self-map of $[0, 4]$ for which $\alpha(1) \neq s\alpha(1)$. A smooth version of this example is used in [7] to show that the point 1 is a nonwandering point but not an $\omega$-limit point. We will see later that if $y$ is a nonwandering point that is not an $\omega$-limit point then $y \in \alpha(y) - s\alpha(y)$.

Example.

$$f(x) = \begin{cases} x + 2 & \text{if } x \in [0, 1), \\ 3x & \text{if } x \in [1, \frac{4}{3}), \\ -\frac{9}{8}x + 10 & \text{if } x \in [\frac{4}{3}, 2), \\ 2x - 3 & \text{if } x \in [2, \frac{7}{2}), \\ 4 & \text{if } x \in [\frac{7}{2}, 4]. \end{cases}$$

It is clear from the graph of the function $f$ that there is a strictly increasing sequence of points $(w_i)_{i=1}^\infty$ with $w_1 = 2$, $w_i \in (2, 3)$ and $f(w_i) = w_{i-1}$ for $i > 1$. For every $i \geq 1$ there is a $z_i \in [0, 1]$ with $f(z_i) = w_i$. Thus $f^i(z_i) = 1$ and $\lim_{i \to \infty} z_i = 1$, hence $1 \in \alpha(1)$. The point 1, however, cannot be a special $\alpha$-limit point since $f^n([1, \frac{14}{9}]) = [3, 4]$ for every $n > 0$ and $f^{-1}([0, 1]) = \emptyset$. To see this, one needs only to realize that if 1 were a special $\alpha$-limit point, we could find points $y_1$, $y_2$, and $y_3$ all elements of $(0, \frac{14}{9})$ and integers $n(3)$ and $n(2)$ with $f^n(y_3) = y_2$ and $f^n(y_2) = y_1$.

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2. Preliminaries

A point \( y \) is said to be a \( \gamma \)-limit point of \( x \) if \( y \in \omega(x) \cap \alpha(x) \). Let \( \gamma(x) = \omega(x) \cap \alpha(x) \) and \( \Gamma = \bigcup_{x \in I} \gamma(x) \). It follows from Proposition A that the set of recurrent points \( R \) is contained in the set \( \Gamma \). In his investigation of the set \( \Lambda^\infty \), Xiong [9] proves the following theorem.

**Theorem A.** Suppose that \( f \) is a continuous self-map of an interval. Then

1. \( \Gamma = \Lambda^2 = \Lambda^\infty \subseteq \overline{P} \), where \( P \) is the set of periodic points;
2. \( R = \Gamma \) if and only if the period of every periodic point is a power of 2, where \( R \) is the set of all recurrent points.

A function \( f \) displays a horseshoe if there are disjoint closed intervals \( J \) and \( K \) in the interval and a positive integer \( n \) such that \( f^n(J) \supset J \), \( f^n(K) \supset K \), \( f^n(J) \supset J \), and \( f^n(K) \supset K \). The content of statement (2) of Theorem A is that \( \Gamma \) strictly contains \( R \) if and only if \( f \) displays a horseshoe. In general, the following set relation holds for self-maps of the interval

\[ P \subseteq R \subseteq \Gamma \subseteq \overline{P} \subseteq \Lambda^1 \subseteq \Omega. \]

In the above example, the set \( \Lambda^1 \) is strictly contained in \( \Omega \). For an example of a function where \( P \) is strictly contained in \( \Lambda^1 \) see [2]. In [3] an example of a function with \( \Gamma \neq \overline{P} \) is constructed.

3. Proof of the theorem

We will prove several lemmas from which the theorem will follow immediately.

**Lemma 1.** If \( x \in s\alpha(y) \) then \( f^n(x) \in s\alpha(y) \) for every \( n \) and \( \omega(x) \subseteq s\alpha(y) \).

**Proof.** Suppose that \( x \in s\alpha(y) \). Then we can find \( y_i \) with \( \lim_{i \to \infty} y_i = x \) and \( n(i) > 0 \) such that \( f^{n(i)}(y_i) = y_i \) and \( f^{n(1)}(y_1) = y \).

Consider the sequence

\[ (y, f^{n(1)-1}(y_1), \ldots, y_1, f^{n(2)-1}(y_2), \ldots, y_2, f^{n(3)-1}(y_3), \ldots) \]

\[ \equiv (y, z_1, z_2, z_3, \ldots). \]

Note that if \( m < \sum_{i=1}^{j} n(i) \) then \( f^m(y_j) \) is a term of the sequence \( (z_i)_{i=1}^{\infty} \). One can see that any limit point of the sequence \( (z_i)_{i=1}^{\infty} \) is an element of \( s\alpha(y) \).

Let \( z \in \{ f(x), f^2(x), \ldots \} \cup \omega(x) \) and let \( \varepsilon > 0 \) be given. Then we can find an \( m > 0 \) with \( f^m(x) \in (z - \varepsilon, z + \varepsilon) \). The function \( f^m \) is continuous and \( \lim_{i \to \infty} y_i = x \). Thus for some \( N > 0 \) if \( j > N \) then \( f^m(y_j) \in (z - \varepsilon, z + \varepsilon) \). If we take \( j > \max(N, m) \) then \( m < j \leq \sum_{i=1}^{j} n(i) \). Thus \( f^m(y_j) \in (z - \varepsilon, z + \varepsilon) \) and is a term of the sequence \( (z_i)_{i=1}^{\infty} \). It follows that \( z \) is a limit point of the sequence \( (z_i)_{i=1}^{\infty} \). \( \square \)

To state Lemma 2, we develop some notation. We will use the symbols \( \alpha_R(x) \) (resp., \( \alpha_L(x) \)) to denote the set of all points \( y \) such that there exists a sequences \( (n(i))_{i=1}^{\infty} \) and \( (x_i)_{i=1}^{\infty} \) with \( \lim_{i \to \infty} n(i) = \infty \), \( \lim_{i \to \infty} x_i = y \), \( f^{n(i)}(x_i) = x \), and \( y < x_i \) (resp., \( x_i < y \)) for every \( i > 0 \).

**Lemma 2.** (1) If for some \( \varepsilon > 0 \), \( x \in \alpha_L(y) \) for every \( y \in (x - \varepsilon, x) \), then for every \( y \in (x - \varepsilon, x) \), \( x \in \alpha_R(y) \).
(2) If for some $\varepsilon > 0$, $x \in \alpha_R(y)$ for every $y \in (x, x + \varepsilon)$, then for every $y \in (x, x + \varepsilon)$, $x \in s\alpha(y)$.

Proof. We will prove (1). Let $y \in (x - \varepsilon, x)$. Since $x \in \alpha_L(y)$, we can find $y_1 \in (x - \varepsilon/2, x)$ and $n(1) > 0$, with $f^{n(1)}(y_1) = y$. By hypothesis, $x \in \alpha_L(y_1)$; thus we can find $y_2 \in (x - \varepsilon/2^2, x)$ and $n(2) > 0$, with $f^{n(2)}(y_2) = y_1$. Continuing in this way, we obtain a sequence $(y_i)_{i=1}^\infty$ and $n(i) > 0$, with $y_i \in (x - \varepsilon/2^i, x)$ and $f^{n(i)}(y_i) = y_{i-1}$. □

Before proceeding any further, we need the definitions of almost periodic points and minimal sets. If given an open set $V_y$ containing $y$, one can find an integer $n > 0$ such that for any integer $q > 0$ there is an integer $r$, $q \leq r \leq q + n$ with $f^r(y) \in V_y$, then $y$ is said to be almost periodic. If for every $x \in \omega(y)$ we have that $\omega(x) = \omega(y)$, then $\omega(y)$ is said to be a minimal set. A well-known fact is that for compact metric spaces a point $y$ is almost periodic if and only if $y \in \omega(y)$ and $\omega(y)$ is a minimal set.

Lemma 3. If $x$ is almost periodic then $x \in s\alpha(x)$.

Proof. Since $x$ is almost periodic, $x \in \omega(x)$ and $\omega(x)$ is a minimal set. Since $f(\omega(x)) = \omega(x)$, we can find a sequence of points $(z_i)_{i=1}^\infty$ with $z_i \in \omega(x)$, $f(z_i) = z_{i-1}$, and $f(z_1) = x$. Let $y$ be a limit point of this sequence. Then $y \in s\alpha(x)$ and, since $\omega(x)$ is a closed set, $y \in \omega(x)$. Since in this case $\omega(x)$ is a minimal set, $x \in \omega(y)$. It follows from Lemma 1 that $x \in s\alpha(x)$. □

Let $Y$ be a subset of $I$. We will denote the closure of $Y$ by $\overline{Y}$. A point $z$ is called a right-side (resp., left-side) accumulation point of $Y$ if for every $\varepsilon > 0$, $(z, z + \varepsilon) \cap Y \neq \emptyset$ (resp., $(z - \varepsilon, z) \cap Y \neq \emptyset$). The right-side closure $\overline{Y}_R$ (resp., left-side closure $\overline{Y}_L$) is the union of $Y$ and the set of right-side (resp., left-side) accumulation points of $Y$.

In Lemma 4 bilateral limit points of periodic points are considered. We will see that such points are special $\alpha$-limit points. If in addition the point $x$ in question is not almost periodic, then we show that $x$ is the end point of interval $J$ with $x \in s\alpha(y)$ for every $y \in J$. In either case we will see that $x \in s\alpha(x)$.

Lemma 4. If $x \in \overline{P}_L \cap \overline{P}_R$ then $x \in s\alpha(x)$. If in addition $x$ is not almost periodic, then for some $\varepsilon > 0$, either $x \in s\alpha(y)$ for every $y \in (x - \varepsilon, x]$, or $x \in s\alpha(y)$ for every $y \in [x, x + \varepsilon)$.

Proof. According to Lemma 3, we may assume that $x$ is not almost periodic. If for every $\varepsilon > 0$ we could find an $n > 0$ such that $f^{nk}(x) \in (x - \varepsilon, x + \varepsilon)$ for every $k$, then $x$ would be almost periodic. Thus we can find an $\varepsilon > 0$ such that if $n > 0$ is fixed then for some $k > 0$, $f^{nk}(x) \notin (x - \varepsilon, x + \varepsilon)$.

Fix such an $\varepsilon > 0$. By our hypothesis, $\lim_{i \to \infty} p_i = x$, $\lim_{i \to \infty} q_i = x$ with $p_i < x$ and $x < q_i$ for two sequences of periodic points $(p_i)_{i=1}^\infty$ and $(q_i)_{i=1}^\infty$. Furthermore, we may assume that $p_i \in (x - \varepsilon, x)$ and $q_i \in (x, x + \varepsilon)$. Let $n(i)$ be the period of $p_i$ and $m(i)$ the period of $q_i$. Let $l(i) = n(i)m(i)$. It follows from our choice of $\varepsilon$ that we can find $k(i) > 0$ with $f^{l(i)k(i)}(x) \notin (x - \varepsilon, x + \varepsilon)$.

Suppose by choosing a subsequence, if needed, that $f^{l(i)k(i)}(x) < x - \varepsilon$ for every $i$. Since the period of $p_i$ is $n(i)$, $f^{l(i)k(i)}(p_i) = p_i$; thus

\[(x - \varepsilon, p_i] \subseteq f^{l(i)k(i)}([p_i, x)).\]

Let $y \in (x - \varepsilon, x)$. We will now show that $x \in s\alpha(y)$. Since $\lim_{i \to \infty} p_i = x$, we can find a positive integer $i(1)$ such that $x - \varepsilon < y < p_{i(1)} < x$. It follows
from (*) that there exists \( z_{i(1)} \in (p_{i(1)}, x) \) with \( f^{l(i(1))k(i(1))}(z_{i(1)}) = y \). Since \( z_{i(1)} \in (p_{i(1)}, x) \), we can find a positive integer \( i(2) \) with

\[
x - \varepsilon < y < p_{i(1)} < z_{i(1)} < p_{i(2)} < x.
\]

It follows from (*) that there exists \( z_{i(2)} \in (p_{i(2)}, x) \) with \( f^{l(i(2))k(i(2))}(z_{i(2)}) = z_{i(1)} \). Continuing in this way, it is possible to find \( i(j), p_{i(j)}, \) and \( z_{i(j)} \) with

\[
x - \varepsilon < y < p_{i(1)} < z_{i(1)} < p_{i(2)} < \cdots < p_{i(j)} < z_{i(j)} \cdots < x
\]

such that

\[
f^{l(i(j))k(i(j))}(z_{i(j)}) = z_{i(j-1)}.
\]

Since \( \lim_{j \to \infty} p_i = x \), we have that \( \lim_{j \to \infty} z_{i(j)} = x \). Thus \( x \in sa(y) \).

To finish the proof of the lemma it suffices to show that we can find a point \( z \in (x - \varepsilon, x) \) and an \( n > 0 \) with \( f^n(z) = x \). The argument that we will give here is taken from the proof of Proposition C in [9]. Let \( g = f^{n(1)} \) and \( L = [p_1, x] \). Then since \( g(p_1) = p_1 \), \( K = L \cup g(L) \cup g^2(L) \cup \cdots \) is an interval. Let \( l(j) \) denote the period of \( p_j \) with respect to \( g \). For \( k = 1, 2, \) or 3 suppose that a subsequence of \( g^{l(1) - k}(p_1)g^{l(2) - k}(p_2) \cdots g^{l(j) - k}(p_j) \cdots \) converges to \( u_k \in K \). It is clear that \( g^{k'}(u_k) = x \). If \( u_{k'} = u_k^{k''} \) for some \( k' < k'' \), then

\[
g^{k'' - k'}(x) = g^{k'' - k'}(g^{k'}(u_{k'})) = g^{k''}(u_{k'}) = g^{k''}(u_{k''}) = x.
\]

However, \( x \) is by assumption not periodic, thus we must have that \( u_1, u_2, \) and \( u_3 \) are distinct points in \( K \). Since \( K \) is an interval, \( u_k \in K \) for some \( k \). Hence for some point \( z \in L \) and \( n \geq 0, g^n(z) = u_k \). Thus \( g^{n+k}(z) = x \). \( \square \)

Note that the last argument in the proof of Lemma 4 shows that for points \( x \in \overline{P}_L - P \) (resp., \( \overline{P}_R - P \)), we have \( x \in \alpha_L(x) \) (resp., \( \alpha_R(x) \)). We will use the symbols \( \omega_L(x) \) (resp., \( \omega_L(x) \)) to denote the set of right-side (resp., left-side) accumulation points of the set \( \{f^n(x)\}_{n=1}^{\infty} \).

Lemma 5. If \( x \in R \) then \( x \in sa(x) \).

Proof. Let \( x \in R \). If \( \omega(x) \) is a minimal set, then according Lemma 3, \( x \in sa(x) \).

We assume that \( \omega(x) \) is not a minimal set. Applying Lemma 3 of [1] to the compact set \( \omega(x) \), we have that the set \( A = \{ y | \omega(y) = \omega(x) \} \cap \omega(x) \) is an uncountable set. Let \( \Delta \) denote the symmetric difference of sets. Since \( \overline{P}_L \Delta \overline{P}_R \) and \( \overline{\omega}_L(x) \Delta \omega_R(x) \) are both countable, \( \overline{P}_L \cap \overline{P}_R \) (resp., \( \omega_L(x) \cap \omega_R(x) \)) is the complement of a countable set in \( \overline{P} \) (resp., \( \omega(x) \)); since \( \omega(x) \subseteq \overline{P} \), it follows that

\[
\omega_L(x) \cap \omega_R(x) \subseteq \overline{P}_L \cap \overline{P}_R
\]

is the compliment of a countable set in \( \omega(x) \). But then \( A \), being an uncountable subset of \( \omega(x) \), must intersect the set \( \omega_L(x) \cap \omega_R(x) \cap \overline{P}_L \cap \overline{P}_R \). Let \( y \) be a point from \( A \) in this intersection.

Since \( \omega(x) = \omega(y) \) and \( \omega(x) \) is not a minimal set, it must be that \( y \) is not almost periodic. Without loss of generality, by Lemma 4 we may choose an \( \varepsilon > 0 \) such that if \( z \in (y - \varepsilon, y] \) then \( y \in sa(z) \). Now \( y \in \omega_L(x) \); hence for some positive integer \( n \), \( f^n(x) \in (y - \varepsilon, y) \). Since the function \( f^n \) is continuous, for some \( \delta > 0 \)

\[
(\ast\ast) \quad f^n((x - \delta, x + \delta)) \subseteq (y - \varepsilon, y).
\]
Now \( R \subseteq \Omega \), so by Proposition A, \( x \in \alpha(x) \). Thus for some \( z \in (x - \delta, x + \delta) \) and \( m > n \), \( f^m(z) = x \). By (**), \( f^n(z) \in (y - \varepsilon, y) \), hence \( y \in \sigma(f^n(z)) \) and \( f^{m-n}(f^n(z)) = x \). It follows that \( y \in \sigma(x) \). Since \( x \in \omega(y) \), by Lemma 1, \( x \in \sigma(x) \). □

We consolidate our proof that (1) implies (2) implies (3) in our theorem into the proof of Proposition 1. Before we prove Proposition 1, two well-known results must be stated. The first, Lemma B is a description of the dynamical behavior that can occur on subintervals of \( I \) containing no periodic points. The second, Proposition C, is a description of a situation under which one can conclude that a point is an element of \( \omega_R(x) \) (resp., \( \omega_L(x) \)) for some \( x \). For a proof of Lemma B see [5] and for Proposition C see [4]. Proposition C is due to Sharkovskii in [8].

**Lemma B.** Let \( J \) be a subinterval of \( I \) that does not contain any periodic points. If \( x \) and \( y \) are elements of \( J \) and if for some \( n \) and \( m \) both \( f^n(x) \) and \( f^m(y) \) are elements of \( J \), then either \( f^n(x) < x \) and \( f^m(y) < y \), or \( f^n(x) > x \) and \( f^m(y) > y \).

**Proposition C.** If for every \( \varepsilon > 0 \), an \( n > 0 \) can be found with

\[
(c, c + \varepsilon) \cap f^n((c, c + \varepsilon)) \neq \emptyset
\]

(resp., \( (c - \varepsilon, c) \cap f^n((c - \varepsilon, c)) \neq \emptyset \)), then \( c \in \omega_R(y) \cup P \) (resp., \( c \in \omega_L(y) \cup P \)) for some \( y \in I \).

**Proposition 1.** If \( x \in \sigma_\alpha(y) \) for some \( y \) then \( x \in \sigma(x) \) and \( x \in T \).

**Proof.** Let \( x \in \sigma_\alpha(y) \). Since \( R \subseteq \Gamma \), by Lemma 5 we may assume that \( x \) is not recurrent. Fix an \( \varepsilon > 0 \) such that \( f^n(x) \notin (x - \varepsilon, x + \varepsilon) \) for every \( n > 0 \). As \( x \in \sigma_\alpha(y) \), we can find a sequence \((y_i)_{i=1}^\infty\) and \( n(i) > 0 \) with \( \lim_{i \to \infty} y_i = x \) and \( f^{n(i)}(y_i) = y_{i-1} \). By choosing a subsequence of the sequence \((y_i)_{i=1}^\infty\), we may assume that \( y_i \in (x - \varepsilon, x) \) for every \( i > 0 \).

**Claim 1.** The point \( x \) is an element of \( \overline{P_L} - P \) and hence, as noted above, an element of \( \alpha_L(x) \).

**Proof of Claim 1.** Let \( \delta > 0 \) be given. It is possible to find a \( j > 0 \) with \( y_j \) and \( y_{j-1} \) both elements of \( (x - \delta, x) \) and hence, \( (x - \delta, x) \cap f^{n(j)}((x - \delta, x)) \neq \emptyset \). Since \( x \) is not periodic by Proposition C, \( x \in \omega_L(y) \) for some \( y \). Now if, on the contrary, \( x \notin \overline{P_L} \), then we can find an \( \varepsilon > 0 \) such that \( (x - \varepsilon, x) \cap P = \emptyset \). Since \( x \in \omega_L(y) \), we can assume that \( y \in (x - \varepsilon, x) \). Thus for some \( n > 0 \), \( x - \varepsilon < y < f^n(y) < x \). Since \( \lim_{i \to \infty} y_i = x \), we can find an integer \( j \) with \( x - \varepsilon < y_{j-1} < y_j < x \). This is, however, a contradiction to Lemma B as \( f^{n(j)}(y_j) = y_{j-1} \). Thus \( x \in \overline{P_L} - P \) and hence \( x \in \alpha_L(x) \).

**Claim 2.** For some \( \delta > 0 \), \( x \in \sigma_\alpha(y) \) for every \( y \in (x - \delta, x) \).

**Proof of Claim 2.** To prove this claim we first define the integers \( M(k, j) \) for every \( k > 0 \) and \( j > 0 \), according to the equation \( M(k, j) = \sum_{i=1}^{j} n(k + i) \). Since \( f^{n(i)}(y_i) = y_{i-1} \), we have that \( f^{M(k, j)}(y_{k+j}) = y_k \). We consider two cases.
Case 1. For some \( k > 0 \), \( x + \varepsilon < f^{M(k,j)}(x) \) for an infinite number of \( j \). Fix such a \( k > 0 \). Let \((h(i))_{i=1}^{\infty}\) be an increasing subsequence of positive integers such that \( x < f^{M(k,h(i))}(x) \) for every \( i > 0 \). In this case we will show that for every \( y \in (y_k, x) \), \( x \in s\alpha(y) \). Since \( x \in \alpha_L(x) \), by Lemma 2 it suffices to show that \( x \in \alpha_L(y) \) for every \( y \in (y_k, x) \). So let \( y \in (y_k, x) \). For every \( i > 0 \),

\[
[y_k, f^{M(k,h(i))}(x)] \subseteq f^{M(k,h(i))}([y_k+h(i), x]).
\]

Hence for every \( i > 0 \) we can find a point \( z_i \in [y_k+h(i), x] \) with \( f^{M(k,h(i))}(z_i) = y \). Since \( \lim_{i \to \infty} y_{k+h(i)} = x \), it follows that \( x \in \alpha_L(y) \).

Case 2. For every \( k > 0 \), we can find a positive integer \( M(k) \) so that if \( j > M(k) \) then \( f^{M(k,j)}(x) < x - \varepsilon \). In this case we will prove that for \( y \in (x - \varepsilon, x) \), \( x \in \alpha_L(y) \). Let \( y \in (x - \varepsilon, x) \). We can find a \( k > 0 \) with \( y < y_k \). By our assumption for \( j > M(k) \), we have

\[
f^{M(k,j)}(x) < x - \varepsilon < y < y_k.
\]

It follows that for every \( j > M(k) \) we can find a \( z_j \in [y_{k+j}, x] \) with \( f^{M(k,j)}(z_j) = y \). As in Case 1, since \( \lim_{j \to \infty} y_{k+j} = x \), \( x \in \alpha_L(y) \). By Lemma 2 and the fact that \( x \in \alpha_L(x) \), we see that for every \( y \in (x - \varepsilon, x) \), \( x \in s\alpha(y) \). This completes the proof of the claim.

To complete the proof of the lemma we must show that \( x \in \Gamma \). Let \( \delta > 0 \) be such that for every \( y \in (x - \delta, x) \), \( x \in \alpha_L(y) \). According to the proof of Claim 1, \( x \in \omega_L(y) \) for some \( y \in (x - \delta, x) \), and hence \( x \in \omega_L(y) \cap s\alpha(y) \). Since \( s\alpha(y) \subseteq \alpha(y) \), \( x \in \Gamma \).

The following lemma will complete the proof of the theorem.

**Lemma 6.** If \( x \in \Gamma \) then \( x \in s\alpha(x) \).

**Proof.** If \( x \in R \) then, by Lemma 5, \( x \in s\alpha(x) \). According to Xiong in [9],

\[
\Gamma = \left( \bigcup_{y \in I} \omega_L(y) \cap \alpha_L(y) \right) \cup \left( \bigcup_{y \in I} \omega_R(y) \cap \alpha_R(y) \right) \cup P.
\]

Thus we assume that \( x \notin R \) and that \( y \) has been chosen with \( x \in \omega_L(y) \cap \alpha_L(y) \). Let \( \varepsilon > 0 \) be chosen so that if \( n > 0 \) then \( f^n(x) \notin (x - \varepsilon, x + \varepsilon) \).

Let \((y_i)_{i=1}^{\infty}\) and \((z_i)_{i=1}^{\infty}\) be sequences with \( y_i \in (x - \varepsilon, x) \), \( z_i \in (x - \varepsilon, x) \), \( f^n(y_i) = y_i \), \( f^n(z_i) = y_i \), \( \lim_{i \to \infty} n(i) = \infty \), and \( \lim_{i \to \infty} m(i) = \infty \). Let \( L(i, j) = m(i) + n(j) \). Then \( f^{L(i,j)}(z_i) = y_j \) for all \( i > 0 \) and \( j > 0 \).

**Case 1.** There exists \( j > 0 \) such that \( f^{L(i,j)}(x) > x + \varepsilon \) for an infinite number of \( i \). Let \((h(i))_{i=1}^{\infty}\) be an increasing subsequence of positive integers such that \( f^{L(h(i),j)}(x) > x + \varepsilon \). Since \( f^{L(h(i),j)}(z_{h(i)}) = y_j \), it follows that

\[
(y_j, x + \varepsilon) \subseteq f^{L(h(i),j)}([z_{h(i)}, x]).
\]

To finish the proof by Lemmas 2 and Proposition 1, it suffices to show that for every \( y \in (y_j, x) \), \( x \in \alpha_L(y) \).

Let \( y \in (y_j, x) \). For every \( i > 0 \), there exists \( w_i \in (z_{h(i)}, x) \) with \( f^{L(h(i),j)}(w_i) = y \). It follows that \( x \in \alpha_L(y) \).

**Case 2.** If \( j > 0 \) then there exists an integer \( M(j) \) such that for \( i > M(j) \), \( f^{L(i,j)}(x) < x - \varepsilon \).
To finish the proof in this case, we will show that if $y \in (x - \varepsilon, x)$ then, $x \in \alpha_L(y)$. Let $y \in (x - \varepsilon, x)$. Since $\lim_{i \to \infty} y_i = x$, we can find a $j > 0$ with $x - \varepsilon > y < y_j$. For every $i > M(j)$ we have
\[(x - \varepsilon, y_j] \subseteq f^{L(i,j)}([z_i, x]).\]
Hence for every $i > M(j)$, we can find $w_i \in (z_i, x)$ with $f^{L(i,j)}(w_i) = y$. It follows that $x \in \alpha_L(y)$. □

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