ON THE ALMOST SPLIT SEQUENCES
FOR RELATIVELY PROJECTIVE MODULES
OVER A FINITE GROUP

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Abstract. Let $G$ be a finite group with a subgroup $H$. Given a field $k$ of characteristic $p$ dividing the order of $G$, denote by mod $kG$ the category of finite-dimensional over $k$ left $G$-modules, and let $?$ be the full subcategory of mod $kG$ determined by the relatively projective modules. Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence in mod $kG$ with $L, M, N \in ?$. It is proved that the sequence is an almost split sequence in $?$ if and only if it is an almost split sequence in mod $kG$. This implies, together with a recent result of Carlson and Happel, that $?$ has almost split sequences if and only if it is closed under extensions, i.e., if and only if $p$ is coprime to either the order of $H$ or the index of $H$ in $G$.

Throughout the paper, we fix an artin algebra $\Lambda$ [1] with Jacobson radical $r$ and denote by mod $\Lambda$ the category of finitely generated left $\Lambda$-modules. We also fix $?$, a full subcategory of mod $\Lambda$ closed under isomorphisms, direct sums, and direct summands. Denote by ind $?$ the set of pairwise nonisomorphic indecomposable modules in $?$, and write ind $\Lambda$ for ind(mod $\Lambda$). We say that $?$ is closed under $DTr$ and $TrD$ if for each $C \in$ ind $?$, $DTrC$ and $TrDC$ [2] are in $?$. We freely use the notions of a functorially finite subcategory and of a left or right almost split morphism in $?$, introduced in [5, 6] as a generalization of the corresponding notions in [2].

The only modification is that we replace Ext-projective and Ext-injective modules in $?$ [5] by extension projective and extension injective modules, respectively, which are defined as follows. A module $N \in ?$ is called extension projective in $?$ if every exact sequence $0 \rightarrow X \rightarrow Y \rightarrow N \rightarrow 0$ in $?$ splits. Recall that a module $N \in ?$ is called Ext-projective in $?$ if Ext$^1_\Lambda (N, X) = 0$ for all $X \in ?$. Clearly, every Ext-projective module is extension projective. On the other hand, in the context of [5], $?$ is closed under extensions, i.e., in every exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ with $X, Z \in ?$, we have $Y \in ?$ so that a module is Ext-projective if and only if it is extension projective. Extension injective modules in $?$ are introduced dually.

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Recall that an exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{C}$ is called *almost split* in $\mathcal{C}$ if $f$ is a left almost split morphism in $\mathcal{C}$ and $g$ is a right almost split morphism in $\mathcal{C}$. $\mathcal{C}$ is said to have almost split sequences if it satisfies the following conditions:

(a) $\mathcal{C}$ has almost split morphisms.

(b) If $N \in \text{ind } \mathcal{C}$ is not extension projective, then there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathcal{C}$.

(c) Dual to (b).

It is obvious that if an exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{C}$ is almost split in $\text{mod } \Lambda$, then it is almost split in $\mathcal{C}$. The converse is true only under certain conditions, as was pointed out by K. W. Roggenkamp, whose short proof replaces here an argument of the author.

**Proposition 1.** (a) Let $0 \to L \to M \to N \to 0$ be an almost split sequence in $\mathcal{C}$, and let $0 \to U \to V \to N \to 0$ be the almost split sequence in $\text{mod } \Lambda$ with right end $N$, where $U \in \mathcal{C}$. Then the sequences are isomorphic.

(b) If $\mathcal{C}$ is closed under $DTr$ and $TrD$, then a short exact sequence in $\mathcal{C}$ is almost split in $\mathcal{C}$ if and only if it is almost split in $\text{mod } \Lambda$.

**Proof.** (a) Since $g$ is not a splittable epimorphism, we have the following exact commutative diagram:

\[
\begin{array}{cccccccc}
0 & \to & L & \to & M & \to & N & \to & 0 \\
& & \downarrow{h} & & \downarrow{j} & & \| & \\
0 & \to & U & \to & V & \to & N & \to & 0 \\
\end{array}
\]

If $h$ is not a splittable monomorphism, it can be extended to $M$ because $U \in \mathcal{C}$ and $f$ is a left almost split morphism in $\mathcal{C}$. Then the bottom row splits, a contradiction. Thus, $h$ is a splittable monomorphism and must be an isomorphism because $U \in \text{ind } \Lambda$.

(b) Follows from (a) and its dual. □

**Proposition 2.** Let $\Lambda$ be connected. Suppose that

(i) $\mathcal{C}$ has almost split sequences and is closed under $DTr$ and $TrD$,

(ii) all extension projectives in $\mathcal{C}$ are projective in $\text{mod } \Lambda$,

(iii) all extension injectives in $\mathcal{C}$ are injective in $\text{mod } \Lambda$,

(iv) $\mathcal{C}$ is functorially finite.

Then either $\mathcal{C} = \text{mod } \Lambda$ or $\mathcal{C}$ consists of projective-injective modules in $\text{mod } \Lambda$; in particular, $\mathcal{C}$ is closed under extensions.

**Proof.** Let $\mathcal{C}$ satisfy (i) and (ii), and suppose $L \in \text{ind } \mathcal{C}$ is not extension injective. Show that if $X \to L$ and $L \to Y$ are irreducible maps with $X, Y \in \text{ind } \Lambda$, then $X, Y \in \mathcal{C}$. Really, by assumption, there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathcal{C}$ that is almost split in $\text{mod } \Lambda$ by Proposition 1(b). Then $Y \in \mathcal{C}$ for $Y$ is a direct summand of $M$. If $L$ is not extension projective in $\mathcal{C}$, there is an almost split sequence $0 \to U \to V \to L \to 0$ in $\mathcal{C}$ and $X \in \mathcal{C}$ for $X$ is a direct summand of $V$. If $L$ is extension projective, it is projective and $X$ is not injective in $\text{mod } \Lambda$. Then there exists an almost split
sequence $0 \to X \to L \oplus L' \to Z \to 0$ in $\text{mod } \Lambda$ with $Z$ a direct summand of $M$, whence $Z \in \mathcal{C}$ and $X \in \mathcal{C}$ by (i).

From the preceding argument and its dual, we conclude that if $L \in \text{ind } \mathcal{C}$ is not projective-injective in $\text{mod } \Lambda$, then for all irreducible maps $X \to L$ and $L \to Y$, we have $X, Y \in \mathcal{C}$. Let $\mathcal{D} = \{P_i|i = 1, \ldots, s\}$ be the set of all those pairwise nonisomorphic projective-injective modules in $\text{ind } \Lambda$ for which the ends of the almost split sequence $0 \to rP_i \to rP_i \otimes P_i \to P_i \otimes P_i \to 0$ [3, Proposition 4.11] are not in $\mathcal{C}$. Then $\mathcal{C}_\mathcal{D}$, the full subcategory of $\mathcal{C}$ determined by all modules having no direct summands in $\mathcal{D}$, has the following property. If $X \to Y$ is an irreducible map with $X, Y \in \text{ind } \Lambda$, then $X \in \mathcal{C}_\mathcal{D}$ if and only if $Y \in \mathcal{C}_\mathcal{D}$.

Now suppose $\mathcal{C} \neq \text{mod } \Lambda$ is functorially finite. Then $\mathcal{C}_\mathcal{D}$ is functorially finite by [6, Proposition 3.13], and, according to [8, Corollary 2.2] and its dual, we have $\text{Hom}_\Lambda(C, W) = 0 = \text{Hom}_\Lambda(W, C)$ for all $C \in \text{ind } \mathcal{C}_\mathcal{D}$ and $W \in (\text{ind } \Lambda) - (\text{ind } \mathcal{C}_\mathcal{D})$. Since $\Lambda$ is connected, indecomposable projective $\Lambda$-modules must either all be in $\text{ind } \mathcal{C}_\mathcal{D}$, or all belong to $(\text{ind } \Lambda) - (\text{ind } \mathcal{C}_\mathcal{D})$. Clearly, the latter holds, i.e., $\text{ind } \mathcal{C} = \mathcal{D}$. □

**Remark 3.** Proposition 2 is false without assumptions (ii) and (iii). Really, if $\Lambda$ is the group algebra of a finite group of order 2 over a field of characteristic 2 and $\mathcal{C}$ is the additive subcategory of $\text{mod } \Lambda$ generated by the trivial module, then $\mathcal{C}$ satisfies (i) and (iv) but is not closed under extensions.

Let $G$ be a finite group of order $|G|$ with a subgroup $H$ of index $[G:H]$. Let $k$ be a field of characteristic $p$ dividing $|G|$. From now on, put $\Lambda = kG, \Gamma = kH$, and denote by $\mathcal{C}$ the full subcategory of relatively projective modules [9, 10] in $\text{mod } \Lambda$, i.e., of all modules isomorphic to a direct summand of $\Lambda \otimes_\Gamma X$ for some $X \in \text{mod } \Gamma$. It is well known that $\mathcal{C}$ is functorially finite, and the extension projective modules, as well as the extension injective modules in $\mathcal{C}$, are projective in $\text{mod } \Lambda$. In addition, $\mathcal{C}$ is closed under $DTr$ and $TrD$ by [4, p. 550]. Thus, Proposition 1 and 2 hold for every block of $\Lambda$. Since the necessary and sufficient conditions for the closure of $\mathcal{C}$ under extensions are well known, we can restate Proposition 2 as follows.

**Corollary 4.** $\mathcal{C}$ has almost split sequences if and only if $p$ is coprime to either $|H|$ or $[G:H]$.

Now suppose that $\mathcal{C}$ does not have almost split sequences. Since $\mathcal{C}$ is self-dual, it follows from Proposition 1 and from [7, Theorem 1.2], where the term extension projective should have been used instead of Ext-projective, that

(i) for some nonprojective $N \in \text{ind } \mathcal{C}$, the middle term of the almost split sequence $0 \to L \to M \to N \to 0$ in $\text{mod } \Lambda$ is not in $\mathcal{C}$.

(ii) For some nonprojective $N' \in \text{ind } \mathcal{C}$, the kernel of one (and every!) right almost split morphism $V \to N'$ in $\mathcal{C}$ is not in $\mathcal{C}$. Our next statement shows that for a given nonprojective module in $\text{ind } \mathcal{C}$, (i) and (ii) are equivalent.

**Proposition 5.** Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence in $\text{mod } \Lambda$ with $L \in \mathcal{C}$. Then

(a) In the exact sequence $0 \to K \to \Lambda \otimes_\Gamma M \xrightarrow{m} N \to 0$, we have $K \simeq L \oplus X(M)$, where $m : \Lambda \otimes_\Gamma M \to M$ is the multiplication map and $X(M) = \text{Ker } m$.  

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(b) $M \in \mathcal{C}$ if and only if $K \in \mathcal{C}$.

(c) If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an almost split sequence in $\text{mod}\Lambda$, then $gm$ is a right almost split morphism in $\mathcal{C}$.

Proof. (a) We have the following exact commutative diagram of $\Lambda$-modules:

$$
\begin{array}{ccccccccc}
0 & 0 & 0 \\
\downarrow & & \downarrow \\
X(M) & = & X(M) \\
\downarrow & & \downarrow \\
0 & \to & K & \xrightarrow{h} & \Lambda \otimes_{\Gamma} M & \xrightarrow{gm} & N & \to & 0 \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \to & 0 \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
$$

The middle column splits as a sequence of $\Gamma$-modules. Since $L$ is relatively projective, $f$ can be lifted to $\Lambda \otimes_{\Gamma} M$, and the left column splits.

(b) By the Krull-Schmidt theorem, it follows from (a) that $K \in \mathcal{C}$ if and only if $X(M) \in \mathcal{C}$, because $L \in \mathcal{C}$. Since every module in $\mathcal{C}$ is relatively injective and the middle column of the diagram splits over $\Gamma$, $X(M) \in \mathcal{C}$ if and only if $m$ is a splittable epimorphism over $\Lambda$, i.e., if and only if $M \in \mathcal{C}$.

(c) Clearly, $N \in \mathcal{C}$, and $gm$ is a right almost split morphism in $\mathcal{C}$ by the proof of [6, Proposition 3.10]. $\Box$

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