THE WEYL TRANSFORM AND $L^p$ FUNCTIONS ON PHASE SPACE

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ABSTRACT. This is primarily a negative paper showing that a bound of the form
$$\|W(f)\|_{\text{operator norm}} \leq c\|f\|_p$$
fails for the Weyl transform if $p > 2$. $L^p$ properties of Wigner distribution functions are discussed as well as Cwikel's theorem.

Trace ideal properties of operators of the form $f(x)g(-i\nabla)$ on $L^2(\mathbb{R}^n)$ have been an important element in the study of Schrödinger operators (both scattering theory and, via the Birman-Schwinger principle, bound state problems) and Yukawa quantum field theories (see [4, Chapter 4]). The main results here are

**Theorem 1.** If $f, g \in L^p(\mathbb{R}^n)$, $2 \leq p < \infty$, then $f(x)g(-i\nabla) \in \mathcal{S}_p$ and $$\|f(x)g(-i\nabla)\|_p \leq \|f\|_p\|g\|_p.$$  

**Theorem 2 (Cwikel).** If $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, $2 < p < \infty$, then $f(x)g(-i\nabla) \in \mathcal{S}_p'$ and $$\|f(x)g(-i\nabla)\|_{p',w} \leq c_p\|f\|_p\|g\|_{p',w}.$$  

It is natural to try to extend this to some nonproduct functions. Define for $F \in \mathcal{S}(\mathbb{R}^{2n})$
$$W(F) = (2\pi)^{-n} \int \hat{F}(k,y)e^{i(kx+y\nabla)}dkdy,$$
the Weyl quantization, and its asymmetrical form
$$A(F) = (2\pi)^{-n} \int \hat{F}(k,y)e^{ikx}e^{iy\nabla}dkdy.$$  

Then $A(f(x)g(p)) = f(x)g(-i\nabla)$, so one might expect that Theorem A extends to a result of the form

$$\|A(F)\|_p \leq c\|F\|_p \quad \text{or} \quad \|W(F)\|_p \leq c\|F\|_p.$$  

At first sight this might seem incompatible with the fact that $f, g \in L^p_w$ does not imply that $f(x)g(-i\nabla)$ is even compact (consider $f(x) = g(x) = |x|^{-n/p}$); but in fact, it is consistent, for $f \in L^p_w$ and $g \in L^p_w$ does not imply that $f(x)g(y)$ on $\mathbb{R}^{2n}$ is in weak $L^p$, e.g., $f(x) = g(x) = |x|^{-n/p}$ where $|[x,y]|f(x)g(y) \geq 1| = \infty$. But $f \in L^p$ and $g \in L^p_w$ imply that $f(x)g(y) \in L^p_w(\mathbb{R}^{2n})$ by a simple argument. Indeed, for $f$ fixed, $f(x)g(y) \in L^p_w(\mathbb{R}^{2n})$
for all \( g \in L^p_w(\mathbb{R}^n) \) if and only if \( f \in L^p(\mathbb{R}^n) \). Thus if (1) holds, a simple consequence would be Cwikel's theorem via a standard application of weak interpolation theorems.

On the other hand, since it is natural to view the map \( F \mapsto A(F) \) as an operator-valued Fourier transform \([2]\), one would not expect Theorem A to extend to a result of the form (1). This is confirmed in the following theorem.

**Theorem 1.** Let \( p \geq 2 \). Then (1) holds if and only if \( p = 2 \).

That (1) holds if \( p = 2 \) follows formally from

\[
\text{Tr}(e^{i(kx+yP)}) = (2\pi)^{n/2}\delta(k)\delta(y) = \text{Tr}(e^{ikx}e^{iyP}) ,
\]

which yields

\[
\text{Tr}(W^*(F)W(F)) = \text{Tr}(A^*(F)A(F)) = (2\pi)^{-n} \int |F(x, k)|^2 \, dx \, dk.
\]

A proof of this well-known fact \([1, 3]\) follows from the Plancherel theorem and the explicit form of the integral kernels for \( W \) and \( A \):

\[
W(F)(x, z) = (2\pi)^{-n} \int \hat{F}(k, z-x)e^{ikx}e^{ik(z-x)/2} \, dk,
\]

\[
A(F)(x, z) = (2\pi)^{-n} \int \hat{F}(k, z-x)e^{ikx} \, dk.
\]

So we turn to proving that (1) fails for \( p > 2 \). Indeed, we will even prove that

\[
\|A(F)\|_\infty \leq c\|F\|_p , \quad \|W(F)\|_\infty \leq c\|F\|_p
\]

both fail where \( \| \cdot \|_\infty \) is the operator norm.

This will follow from the simple duality argument. If we define for \( \psi \in L^2(\mathbb{R}^n) \)

\[
\rho^A(\psi)(x, p) = (2\pi)^{-2n} \int e^{-ixk}e^{-ipy}(\psi|e^{ikx}e^{iyP}|\psi) \, dk \, dy,
\]

\[
\rho^W(\psi)(x, p) = (2\pi)^{-2n} \int e^{-ixk}e^{-ipy}(\psi|e^{ik(x+yP)}|\psi) \, dx \, dy
\]

(with \( \langle \psi|B|\psi \rangle = (\psi, B\psi) \)); then

\[
\langle \psi|A(F)|\psi \rangle = \int F(x, p)\rho^A(\psi)(x, p) \, dx \, dp,
\]

\[
\langle \psi|W(F)|\psi \rangle = \int F(x, p)\rho^W(\psi)(x, p) \, dx \, dp.
\]

From this we conclude:

**Proposition.** If \( \|A(F)\|_\infty \leq c\|F\|_p \) (resp., \( \|W(F)\|_\infty \leq c\|F\|_p \)), then for \( p' = p/p - 1 \) we have that \( \|\rho^A(\psi)\|_{p'} \leq c \) (resp., \( \|\rho^W(\psi)\|_{p'} \leq c \)) for all \( \psi \in L^2(\mathbb{R}^n) \).

By straightforward calculation,

\[
\rho^A(\psi)(x, p) = (2\pi)^{-n/2}e^{ipx}|\psi|^*(x)\psi(p),
\]

\[
\rho^W(\psi)(x, p) = (2\pi)^{-n} \int e^{ipy}|\psi|^*(x - \frac{1}{2}y)\psi(x + \frac{1}{2}y) \, dy.
\]
From (2) we see that \( \rho^A(\psi) \in L^q \) if and only if both \( \psi \) and \( \hat{\psi} \) lie in \( L^q \). Since \( \psi \) is arbitrary in \( L^2 \), there are \( \psi \) with \( \rho^A(\psi) \in L^q \) if and only if \( q = 2 \); so by the proposition, \( \|A(f)\|_\infty \leq c\|f\|_p \) only if \( p = 2 \).

The Weyl case is a little more subtle. Note first that if \( \psi \) is supported in \( \{x| |x| \leq 1\} \), then by (3), \( \rho^W(\psi)(x, p) \neq 0 \) only if there is a \( y \) with \( |x \pm \frac{1}{2}y| \leq 1 \); so \( |x| \leq \frac{1}{2}(|x + \frac{1}{2}y| + |x - \frac{1}{2}y|) \leq 1 \), i.e., \( \rho^W(\psi)(x, p) = 0 \) if \( |x| \geq 1 \). Suppose that \( \int |\rho^W|^q dxdp < \infty \). Then it follows, since the characteristic function of the unit ball is in \( L^q/q-1 \), that we have for any \( \theta(x,p) \)

\[
\int \rho^W(x,p)e^{i\theta(x,p)} \, dx \in L^q(d\rho).
\]

We will find \( \psi \) and \( \theta \) in (4) false if \( 1 < q < 2 \). Pick \( \theta(x,p) = 2px \). Then by (3)

\[
\int e^{i\theta(x,p)} \rho^W(x,p) \, dx = (2\pi)^{-n} \int e^{2ip(x-y/2)} \psi^*(x - \frac{1}{2}y) \psi(x + \frac{1}{2}y) \, dy \, dx
\]

\[
= (2\pi)^{-n} \int e^{2ipu} \psi(u) \psi^*(z) \, dy \, dz
\]

\[
= (2\pi)^{-n/2} \psi(0) \psi(2p).
\]

Now take

\[
\psi(x) = \begin{cases} 
|x|^{-\alpha}, & |x| < 1, \\
0, & |x| \geq 1
\end{cases}
\]

with \( 2\alpha < 9 \). Then, \( \psi(0) \neq 0 \) and \( \psi(p) \sim |p|^{-(n-\alpha)} \) for \( p \) large; so \( \rho^W \notin L^q \) if \( q(n-\alpha) < n \). Since \( \alpha \) can be arbitrarily closer to \( n/2 \), \( q \) can be chosen anywhere in \([1,2)\). This concludes the proof of Theorem 1.

Along the way, we proved the following of independent interest:

**Theorem 2.** In general, \( \rho^A(\psi) \) may not lie in any \( L^p \), \( p \neq 2 \). In general, \( \rho^W(\psi) \) may not lie in \( L^p \), \( 1 \leq p < 2 \).

It is easy to show \( \rho^W \in L^\infty \) so in \( L^p \), \( 2 \leq p \leq \infty \). Since \( \rho^W \) is a “density”, \( \int |\rho^W|^1dxdp = \infty \) is notable!

**References**


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