THE $p$-PERIODICITY OF THE MAPPING CLASS GROUP
AND THE ESTIMATE OF ITS $p$-PERIOD

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Abstract. We determine completely the primes $p$ for which the Farrell-Tate cohomology of the mapping class group $\Gamma_g$ is $p$-periodic. We also estimate the $p$-period of a $p$-periodic $\Gamma_g$.

For $\Gamma$ any group of finite virtual cohomological dimension (vcd) and a prime $p$, we say that the group $\Gamma$ has $p$-periodic cohomology if there exists a positive integer $d$ such that the Farrell-Tate cohomology groups $\tilde{H}^i(\Gamma; M)$ and $\tilde{H}^{i+d}(\Gamma; M)$ have naturally isomorphic $p$-primary components for all $i \in \mathbb{Z}$ and $\mathbb{Z}\Gamma$-modules $M$. The $p$-period of $\Gamma$ is defined as the least value of $d$ [B1].

Recall that the mapping class group $\Gamma_g$ is defined to be the group of path components of orientation-preserving homeomorphisms of the orientable closed surface $S_g$ of genus $g$. We always assume $g > 1$. It is well known that the mapping class group $\Gamma_g$ is of finite vcd and the $\text{vcd}(\Gamma_g) = 4g - 5$ [H].

In this paper, we determine completely the primes $p$ for which $\Gamma_g$ is $p$-periodic. Furthermore, we estimate the $p$-period of a $p$-periodic $\Gamma_g$ by using the $p$-period of a metacyclic subgroup of $\Gamma_g$ as a lower bound and a homogeneous Chern class polynomial of the canonical homology representation $\Gamma_g \to \text{GL}(2g, \mathbb{Z})$ as an upper bound. The main results are as follows:

Theorem 1. (a) The mapping class group $\Gamma_g$ is never 2-periodic.

(b) The mapping class group $\Gamma_{kp+i}$ is always $p$-periodic when $i \not\equiv 1 \pmod{p}$ where $p$ is an odd prime and $k \geq 0$.

(c) The mapping class group $\Gamma_{kp+1}$ is $p$-periodic if and only if the interval $[(2k + 3)/p, (2k + 2)/(p - 1)]$ does not contain an integer and $k \not\equiv 0, -1 \pmod{p}$ where $p$ is an odd prime. In particular, $\Gamma_{kp+1}$ can be $p$-periodic only when $k \leq (p^2 - 5)/2$.

Theorem 2. If $k \not\equiv 0 \pmod{p}$ and $p > 2$, then $\Gamma_{(p-1)(kp-k-2)/2}$ is $p$-periodic and the $p$-period of $\Gamma_{(p-1)(kp-k-2)/2}$ is a multiple of $2(p-1)$. Moreover, if $k < (p-1)/2$, the $p$-period of $\Gamma_{(p-1)(kp-k-2)/2}$ equals $2(p-1)$.

Theorem 3. If $3 < d$ and $d$ divides $p - 1$, then $\Gamma_{(p-1)(d-2)/2}$ is $p$-periodic and the $p$-period of $\Gamma_{(p-1)(d-2)/2}$ is a multiple of $2d$.

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Theorem 4. Let $2g - 2 = mp - i$ where $0 \leq i \leq p - 1$, $p$ is an odd prime, and $p^{r-1} \leq m \leq p^r$. Assume that $\Gamma_g$ is $p$-periodic. Then

(a) If $[2g/(p-1)] < p^r$, then the $p$-period of $\Gamma_g$ divides $2p^{r-1}(p-1)$.
(b) If $[2g/(p-1)] \geq p^r$, then the $p$-period of $\Gamma_g$ divides $2p^r(p-1)^2$.

The remainder of this paper is organized as follows: In the first section we prove Theorem 1. In §2 we give a lower bound for the $p$-period of $\Gamma_g$ for $g = k(p - 1)$ and thus prove Theorem 3. In §3 we give an upper bound for the $p$-period of $\Gamma_g$ for $g = k(p - 1)$ and thus prove Theorem 2. In §4 we give an upper bound for the $p$-period of $\Gamma_g$ when $\Gamma_g$ is any $p$-periodic mapping class group and thus prove Theorem 4.

1. The $p$-Periodicity of the Mapping Class Group $\Gamma_g$

For a group $\Gamma$ of finite vcd, recall that $\Gamma$ is $p$-periodic if and only if $\Gamma$ does not contain $\mathbb{Z}/p \times \mathbb{Z}/p$ [B1]. Furthermore, the positive solution of the Nielsen conjecture by Kerckhoff [K] implies that the finite group $F$ is a subgroup of $\Gamma_g$ if and only if $F$ is isomorphic to a subgroup of Homeo$^+$($S_g$), the group of orientation preserving homeomorphisms of $S_g$.

**Proposition 1.1.** The finite group $F$ is isomorphic to a subgroup of Homeo$^+$($S_g$) with branching data $(h; n_1 \ldots n_b)$ if and only if $F$ satisfies the following conditions:

1. $F = \langle a_1, \ldots, a_h ; b_1, \ldots, b_h ; c_1, \ldots, c_b \rangle$;
2. $\prod_{1 \leq i \leq h}[a_i, b_i] \prod_{1 \leq j \leq b} c_j = 1$;
3. $\text{Order}(c_i) = n_i$;
4. Riemann-Hurwitz equation $2g - 2 = |F|(2h-2) + |F| \sum_{1 \leq i \leq b}(1 - 1/n_i)$.

**Proof.** See [T].

**Lemma 1.2.** Let $G$ be a finite subgroup of Homeo$^+$($S_g$). Then $G$ is also isomorphic to a subgroup of Homeo$^+($($S_g+k|G|$)$)$. Here $k$ is a nonnegative integer.

**Proof.** It follows immediately by Proposition 1.1.

**Proof of Theorem 1.** (a) We only need to show $\Gamma_i \supset Z/2 \times Z/2$ for $2 \leq i \leq 5$ by Lemma 1.2. In fact, write $Z/2 \times Z/2 = \langle x, y | x^2 = y^2 = 1, xy = yx \rangle$ in the following forms:

- (i) when $g = 2$, $Z/2 \times Z/2 = \langle x, x, x, y, xy \rangle$, $h = 0$, $b = 5$;
- (ii) when $g = 3$, $Z/2 \times Z/2 = \langle x; y; x, x \rangle$, $h = 1$, $b = 2$;
- (iii) when $g = 4$, $Z/2 \times Z/2 = \langle x; y; x, y, xy \rangle$, $h = 1$, $b = 3$;
- (iv) when $g = 5$, $Z/2 \times Z/2 = \langle x; y; x, y, x, y \rangle$, $h = 1$, $b = 4$.

It is straightforward to check by Proposition 1.1 that (i) $\Gamma_2 \supset Z/2 \times Z/2$ with branching data $(0; 2, 2, 2, 2, 2)$, (ii) $\Gamma_3 \supset Z/2 \times Z/2$ with branching data $(1; 2, 2)$, (iii) $\Gamma_4 \supset Z/2 \times Z/2$ with branching data $(1; 2, 2, 2)$, and (iv) $\Gamma_5 \supset Z/2 \times Z/2$ with branching data $(1; 2, 2, 2)$. We have proved that $\Gamma_2$ is never 2-periodic.

In the rest of this paper, $p$ is an odd prime.

(b) $\Gamma_{kp+i}$ is $p$-periodic for $i \neq 1$. Otherwise, we have $\Gamma_{kp+i} \supset Z/p \times Z/p$.

The Riemann-Hurwitz formula holds: $2(kp+i-2) = p^2(2h-2) + p^2(1-1/p)b$, i.e., $2k + (2i-2)/p = p(2h-2) + (p-1)b$ implies $2i-2 = 0 \mod(p)$, forcing $i = 1 \mod(p)$. This is a contradiction.
(c) Claim 1. If \( k = 0 \), \( -1 \mod(p) \) or the interval \([(2k + 3)/p, (2k + 2)/(p - 1)]\) contains an integer, then \( \Gamma_{kp+1} \supset Z/p \otimes Z/p \).

Case 1. \( k = 0 \mod(p) \). Suppose \( k = np \), where \( n \) is a nonnegative integer. We show a \( Z/p \times Z/p \) free action on \( S_{kp+1} \) by Proposition 1.1. In fact, write \( Z/p \times Z/p = (x, y|x^p = y^p = 1 \ xy = yx) = (x_1, x_2, \ldots, x_{n+1}; y_1, y_2, \ldots, y_{n+1}), h = 0, b = n+1 \), where \( x_i = x \) and \( y_i = y \), \( 1 \leq i \leq n+1 \). Notice that the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2n + 1) + p^2(1 - 1/p) \) holds.

Case 2. \( k = -1 \mod(p) \). We show that there is a \( Z/p \times Z/p \) action with two singular points on \( S_{kp+1} \) by Proposition 1.1. Write

\[
Z/p \times Z/p = (x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n; x, x^{-1}),
\]

\( h = n \), \( b = 2 \), where \( n = (k + 1)/p \geq 1 \), \( x_i = x \), \( y_i = y \), \( 1 \leq i \leq n \). Notice that the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2n - 1) + p^2(1 - 1/p) \) holds.

Case 3. The interval \([(2k + 3)/p, (2k + 2)/(p - 1)]\) contains an integer \( n \). We show that there is a \( Z/p \times Z/p \) action on \( S_{kp+1} \) with \( t = np - 2k \) singular points. Note \( t = np - 2k \geq 3 \).

Let \( h = k+1-n(p-1)/2 \), then \( h \geq k+1-(2k+2)/2 = 0 \). Write \( Z/p \times Z/p = (x_1, x_2, \ldots, x_{h}; y_1, y_2, \ldots, y_{h}; y, x_1, x_2, \ldots, x_{h-2}, (\Pi_{i=1<i<h}x_i^{-1})^{-1}y^{-1}) \). Here \( x_i = x_1 = x \) and \( y_i = y \), \( 1 \leq i \leq h \), \( 1 \leq j \leq h-2 \). Note that the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2h - 2) + p^2(1 - 1/p) \) holds, i.e., \( 2kp = p^2(2k - n(p - 1)) + p(p - 1)(np - 2k) = 2kp^2 - np^2(p - 1) + np^2(p - 1) - 2kp(p - 1) \).

Claim 2. Conversely, if \( \Gamma_{kp+1} \) is not \( p \)-periodic, then \( k = 0 \mod(p) \), \( k = -1 \mod(p) \), or the interval \([(2k + 3)/p, (2k + 2)/(p - 1)]\) contains an integer.

Let \( \Gamma_{kp+1} \supset Z/p \times Z/p \), i.e., there exists a \( Z/p \times Z/p \) action on \( S_{kp+1} \).

Case 1. The \( Z/p \times Z/p \) acts freely on \( S_{kp+1} \). Then the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2h - 2) \) implies \( k = p(h-1) \), i.e., \( k = 0 \mod(p) \).

Case 2. The \( Z/p \times Z/p \) acts on \( S_{kp+1} \) with two singular points. Then the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2h - 2) + p^2(1 - 1/p)2 \) implies \( k = p(h-1) + p - 1 \), i.e., \( k = -1 \mod(p) \).

Case 3. The \( Z/p \times Z/p \) acts on \( S_{kp+1} \) with more than three singular points. Then the Riemann-Hurwitz formula \( 2(kp + 1) - 2 = p^2(2h - 2) + p^2(1 - 1/p)t \) implies \( (2k + t)p = (2h - 2 + t)p^2 \). Suppose \( n = 2h - 2 + t \) and \( np = (2h - 2 + t)p = 2kp + t \geq 2k + 3 \) because of \( t \geq 3 \), i.e., \( n \geq (2k + 3)/p \). Also, \( 2kp - (2h - 2)p^2 = p(p - 1)t \), or \( t = (2k - 2kp + 2p)/(p - 1) \) implies

\[
n = 2h - 2 + t = [(2h - 2)(p - 1) + 2k - 2hp + 2p]/(p - 1)
= (2k + 2 - 2h)/(p - 1) \leq (2k + 2)/(p - 1)
\]

since \( h \geq 0 \). So \( (2k + 3)/p \leq n \leq (2k + 2)/(p - 1) \) where \( n \) is an integer.

Claim 3. \( \Gamma_{kp+1} \) is not \( p \)-periodic if \( k \geq (p^2 - 3)/2 \). Since \( p \geq 3 \), it follows that

\[
(2k + 2)/(p - 1) - (2k + 3)/p = (2kp + 2p - 2kp + 2k - 3p + 3)/[p(p - 1)]
\geq (p^2 - p)/[p(p - 1)] = 1
\]

implies that there exists at least an integer \( n \in [(2k + 3)/p, (2k + 2)/(p - 1)] \).

We have proved Theorem 1.
2. THE METACYCLIC SUBGROUPS OF THE MAPPING CLASS GROUP

Let $M_{p,k} = \langle a, b | a^k a^{-1} = b^k = 1, b^{-1} a b^{-1} = b^k \rangle$ and $N_{p,d} = \langle a, b | a^d = 1, b^d = 1, \text{ord}_p(\text{ord}_p(s)) = p-1, \text{ord}_p(\text{ord}_p(t)) = d \rangle$ denote metacyclic groups, where $p$ is an odd prime, $k$ is a positive integer, $d \geq 3$ and divides $p-1$, $s$ and $t$ are mod $(p)$ integers such that the order of $s$ in the multiple group $(\mathbb{Z}/p\mathbb{Z})^*$ is equal to $p-1$, and the order of $t$ in the multiple group $(\mathbb{Z}/p\mathbb{Z})^*$ is equal to $d$. Then the order $|M_{p,k}| = kp(p-1)$ and $|N_{p,d}| = pd$.

Lemma 2.1. (a) $\Gamma_{(p-1)(kp-2-k)/2} \supset M_{p,k}$ except for $k = 1, p = 3$.
(b) $\Gamma_{(p-1)(d-2)/2} \supset N_{p,d}$.

Proof. We apply Lemma 1.1 to show $\text{Homeo}^+(S_{(p-1)(kp-2-k)/2}) \supset M_{p,k}$ and $\text{Homeo}^+(S_{(p-1)(d-2)/2}) \supset N_{p,d}$. Write the groups $M_{p,k} = \langle b^{-1}, a, a^{-1} b \rangle (h = 0, b = 3)$ and $N_{p,d} = \langle b^{-1}, a, a^{-1} b \rangle (h = 0, b = 3)$. Note the order $\text{ord}_p(a^{-1} b) = k(p-1)$ in $M_{p,k}$ and the order $\text{ord}_p(a^{-1} b) = d$ in $N_{p,d}$. It is direct to check that the Riemann-Hurwitz formula holds for both groups $M_{p,k}$ and $N_{p,d}$:

\[2(p-1)(kp-2-k)/2 - 2 = kp(p-1)(2(0)-2) + kp(p-1)(1-1/p)\]
\[+ kp(p-1)(1-1/(k(p-1)))/2,\]

\[2(p-1)(d-1)/2 - 2 = pd(2(0)-2) + pd(1-1/p) + pd(1-1/d)/2.\]

Lemma 2.2. The finite group $M_{p,k}$ (resp. $N_{p,d}$) is $p$-periodic with the $p$-period $2(p-1)$ (resp. $2d$) for $k \neq 0 \mod (p)$.

Proof. The order $|M_{p,k}| = kp(p-1)$ (resp. $|N_{p,d}| = pd$) implies that the group $M_{p,k}$ (resp. $N_{p,d}$) does not contain $\mathbb{Z}/p \times \mathbb{Z}/p$ except $k = 0 \mod (p)$, i.e., $M_{p,k}$ (resp. $N_{p,d}$) is $p$-periodic. We will compute the $p$-period of $M_{p,k}$ (resp. $N_{p,d}$) by using a result of Swan [S] that states that the $p$-period of a $p$-periodic finite group equals $2|N(S_p)/C(S_p)|$ for an odd prime $p$, where $N(-)$ and $C(-)$ denote the normalizer and centralizer and $S_p$ is a $p$-Sylow subgroup of $G$. In fact, the order $|N(S_p)/C(S_p)| = |N(\mathbb{Z}/p)/C(\mathbb{Z}/p)| = kp(p-1)/kp = p-1$ in $M_{p,k}$ and the order $|N(\mathbb{Z}/p)/C(\mathbb{Z}/p)| = pd/p = d$ in $N_{p,d}$. This completes the proof of Lemma 2.2.

The $p$-periodicity of $\Gamma_{(p-1)(kp-2-k)/2}$ and $\Gamma_{(p-1)(d-2)/2}$ are clear by Theorem 1(b). So a lower bound of the $p$-period of certain mapping class group is obtained by combining Lemmas 2.1 and 2.2.

We have thus proved Theorem 3 and the following lemma.

Lemma 2.3. The $p$-period of $\Gamma_{(p-1)(kp-2-k)/2}$ is a multiple of $2(p-1)$.

3. THE CHERN CLASSES OF THE CANONICAL HOMOLOGY REPRESENTATION OF THE MAPPING CLASS GROUP

Recall that for a complex representation $f: \Gamma \to \text{GL}(k, C)$ of the discrete group $\Gamma$ the Chern classes $c_i(f) \in H^{2i}(\Gamma; \mathbb{Z})$ are defined as Chern classes of the flat $C^\infty$-bundle over $K(\Gamma, 1)$ classified by $Bf: K(\Gamma, 1) \to B\text{GL}(k, C)$. Let $Q$ be a subring in $C$. If $f: \Gamma \to \text{GL}(k, Q)$ is a representation over $Q$, we will write $c_i(f)$ for the $i$th Chern class of the associated complex representation $\Gamma \to \text{GL}(k, Q) \to \text{GL}(k, C)$.
It is well known that over $Q$ the group $Z/nZ$ has a unique faithful irreducible representation $\sigma_n : Z/nZ \rightarrow GL(\varphi(n), Q)$, where $\varphi(n)$ is the Euler function. Glover and Mislin showed a result in [GM].

**Proposition 3.1** (Glover and Mislin). Let $r : Z/p^\alpha \rightarrow GL(k, Q)$ be a $Q$ representation. Suppose that in the decomposition of $r$ into $Q$ irreducible representation $\sigma_{p^\alpha}$ occurs with multiplicity $m$, where $m$ is not divisible by $p$. Then for every $j > 0$, $(C_{\varphi(p^j)}(r))^j \in H^{2j}(\pi,(Q)(Z/p^\alpha ; Z)$ has order $p^\alpha$.

Let $\mu : \Gamma \rightarrow Sp(2g, Z)$ be the map obtained by allowing a homeomorphism $h$ of $S_g$ to act on $H_1(S_g ; Z)$ and let $i : Sp(2g, Z) \rightarrow GL(2g, Q)$ be the canonical inclusion. Then $\epsilon = i\mu : \Gamma \rightarrow GL(2g, Q)$ is a representation over $Q$. If $p : Z/p \rightarrow \Gamma \rightarrow GL(2g, Q)$ is the composite of inclusion and $\epsilon$, and since $\rho$ is faithful, $\chi_{\rho} = m_{\rho} \chi_{\text{tr}} + n_{\rho} \chi_{\sigma}$, where $\chi$ stands for the character of the representation, $\tau$ denotes the trivial representation, $\sigma$ is the unique irreducible representation of $Z/p$, and the integers $m_{\rho}$ and $n_{\rho}$ depend on $\rho$ [Se].

**Proposition 3.2.** Suppose $\Gamma$ is $p$-periodic for an odd prime $p$, $p = \epsilon i : Z/p \rightarrow \Gamma \rightarrow GL(2g, Q)$ is a representation of $Z/p$ over $Q$ for any inclusion $i : Z/p \rightarrow \Gamma$, and $\chi_{\rho} = m_{\rho} \chi_{\text{tr}} + n_{\rho} \chi_{\sigma}$. If $n_{\rho}$ is not divisible by $p$ where $i$ ranges over all inclusions, then $\Gamma$ has $p$-period $m_{\rho}$, dividing $2\varphi(p) = 2(p-1)$.

**Proof.** It is well known that $\Gamma$ is $p$-periodic ($p > 2$) if and only if every $p$-Sylow subgroup $S_p$ of $\Gamma$ is cyclic. We need to show that there exists an element $a \in \hat{H}^{2\varphi(p)}(\Gamma ; Z)$ such that $\text{Res}(a) \in \hat{H}^{2\varphi(p)}(Z/p ; Z)$ is nontrivial for every $Z/p$ inclusion by the Brown-Venkov theorem.

Let $g^* : H^{2\varphi(p)}(\Gamma ; Z) \rightarrow \hat{H}^{2\varphi(p)}(\Gamma ; Z)$ be the canonical map from the ordinary cohomology to the Farrell-Tate cohomology. The following diagram is commutative:

\[
\begin{array}{ccc}
H^{2\varphi(p)}(\Gamma ; Z) & \xrightarrow{\text{Res}} & \hat{H}^{2\varphi(p)}(Z/p ; Z) \\
\uparrow g^* & & \uparrow g^* \\
\hat{H}^{2\varphi(p)}(\Gamma ; Z) & \xrightarrow{\text{Res}} & H^{2\varphi(p)}(Z/p ; Z)
\end{array}
\]

Let $a = g^* [c_{p-1}(\epsilon)] \in \hat{H}^{2\varphi(p)}(\Gamma ; Z)$. Then $\text{Res}(a) = \text{Res} g^* [c_{p-1}(\epsilon)] = g^* \text{Res}[c_{p-1}(\epsilon)] = g^* [c_{p-1}(\rho)]$ has nontrivial order in $\hat{H}^{2\varphi(p)}(Z/p ; Z)$ for every $Z/p$ inclusion since the irreducible representation $\sigma$ occurs with multiplicity $n$ that is not divisible by $p$, i.e., the cup-product with $g^* [c_{p-1}(\epsilon)]$ gives an isomorphism for all integers $i$ and a $\Gamma$-module $Z : \hat{H}^i(\Gamma ; Z)_{(p)} \rightarrow \hat{H}^{2\varphi(p)+i}(\Gamma ; Z)_{(p)}$.

Next, we give an upper bound of the $p$-period of $\Gamma_{(p-1)(kp-2-k)/2}$ when $k < (p-1)/2$. Consider any inclusion $i : Z/p \rightarrow \Gamma_{(p-1)(kp-2-k)/2}$. The Riemann-Hurwitz formula $2(p-1)(kp-2) - 2 - 2p(2k-2) + (p-1)t$ implies $t = kp-2k-2ph/(p-1) = k(p-1)-sp$, where $s = 2h/(p-1)$ must be an integer. The Lefschetz-Hopf trace formula also tells us $\chi_{\rho}(T) = 2-t = 2-k(p-1)+sp$, where $T$ is a generator of $Z/p$ and $\rho = \epsilon i : Z/p \rightarrow GL((p-1)(kp-2-k), Q)$.

Since $\chi_{tr}(1) = 1$, $\chi_{tr}(T) = 1$, $\chi_{\sigma}(1) = p-1$, and $\chi_{\sigma}(T) = -1$, we have $\chi_{\rho}(1) = m+n(p-1) = (p-1)(kp-2-k)$ and $\chi_{\rho}(T) = m-n = 2-k(p-1)+sp$. 

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Then \( np = (p - 1)(kp - k - 2) - 2 + k(p - 1) - sp = (kp - k - 2 - s)p \) implies \( n = kp - k - 2 - s \). Notice \( t \geq 0 \) implies \( 0 \leq s \leq k - 1 \). If \( k < (p - 1)/2 \), then \((k - 1)p < kp - k - 1 - 2 \leq kp - k - s - 2 = n < kp \). So \( n \) is not divisible by \( p \). By Proposition 3.2 we conclude that the \( p \)-period of \( \Gamma_{(p-1)(kp-2-k)/2} \) divides \( 2(p-1) \) if \( 0 < k < (p-1)/2 \).

We have now proved Theorem 2 by combining the results of these two sections.

4. AN UPPER BOUND OF THE \( p \)-PERIOD OF A \( p \)-PERIODIC MAPPING CLASS GROUP \( \Gamma_g \)

In this section, we construct a homogeneous Chern class polynomial of the canonical homology representation \( \Gamma_g \to \text{GL}(2g, \mathbb{Q}) \), which depends only upon the genus \( g \) and the prime \( p \) so that the restrictions of the homogeneous Chern class polynomial to all \( \mathbb{Z}/p \) inclusions in \( \Gamma_g \) are nontrivial. Therefore an upper bound of the \( p \)-period of a \( p \)-periodic \( \Gamma_g \) is obtained.

We still suppose \( g > 1 \) and \( p \) an odd prime. Let \( 2g - 2 = mp - i \), where \( 0 \leq i \leq p - 1 \). Define a fixed point number set \( B_{g,p} \) as follows:

\[
\begin{align*}
B_{g,p} &= \{i, i + p, i + 2p, \ldots, i + \lfloor 2g/(p - 1) \rfloor - m p \} \quad \text{if } i \neq 1, \\
B_{g,p} &= \{1 + p, 1 + 2p, \ldots, 1 + \lfloor 2g/(p - 1) \rfloor - m p \} \quad \text{if } i = 1.
\end{align*}
\]

Remark. Define \( B_{g,p} = \emptyset \) if in the cases (a) \( i \neq 1 \) and \( 2p/(p - 1) < m \) and (b) \( i = 1 \) and \( 2g/(p - 1) < m + 1 \).

Lemma 4.1. If \( \mathbb{Z}/p = \langle x \rangle \) acts on the surface \( S_g \) and \( 2g - 2 = mp - i \), where \( 0 \leq i \leq p - 1 \), then the number \( t \) of fixed points of \( x \) belongs to \( B_{g,p} \). Conversely, any number \( t \in B_{g,p} \) can occur as the number of fixed points of an order \( p \) homeomorphism \( x \) on the surface \( S_g \).

Proof. We again use Proposition 1.1. If \( \mathbb{Z}/p = \langle x \rangle \) acts on the surface, the Riemann-Hurwitz formula \( 2g - 2 = (2h - 2)p + tp(1 - 1/p) \) implies \( t = 2g(h - 1)/(p - 1) - 2h - 2 = n - (2g - n(p - 1))/2 = np - 2g + 2 \). Here \( g - h = n(p - 1)/2 \), \( n \) is an integer, and \( n \leq [2g/(p - 1)] \) since \( h \geq 0 \). Therefore, \( t = -2g + 2 = i \mod(p) \) and \( 0 \leq t \leq 2g/(p - 1) - m \), i.e., \( t \in B_{g,p} \). Notice, if \( i = 1 \) then \( t \neq 1 \), since the number of fixed points of \( \mathbb{Z}/p \) action cannot be 1.

Conversely, if \( t \in B_{g,p} \), i.e., \( t = i + kp \), where \( 0 \leq k \leq [2g/(p - 1)] - m \) if \( i \neq 1 \); \( 0 < k \leq [2g/(p - 1)] - m \) if \( i = 1 \). Let \( n = k + m \leq [2g/(p - 1)] \). Then \( h = g - n(p - 1) = g - (k + m)(p - 1)/2 \geq 0 \). We show that there exists a \( \mathbb{Z}/p = \langle x \rangle \) action on the surface \( S_g \) with \( t \) fixed points numbered by Proposition 1.1. In fact, write

\[
Z/p = \langle x \rangle = \langle x_1 \ldots x_h; x_1^{-1} \ldots x_h^{-1}; x_{h+1}, \ldots, x_{h+t-1}, x^{-(t-1)} \rangle
\]

for \( t \neq 1 \mod(p) \);

\[
Z/p = \langle x \rangle = \langle x_1 \ldots x_h; x_1^{-1} \ldots x_h^{-1}; x_{h+1}^2, \ldots, x_{h+t-1}, x^{-t} \rangle
\]

for \( t = 1 \mod(p) \), where \( x_j = x, 0 < j < h + t \); it is easy to check the Riemann-Hurwitz formula \((2h - 2)p + (p - 1)t = (2g - n(p - 1) - 2)p + (p - 1)(i + kp) = mp - i = 2g - 2 \) since \( n = k + m \).
Lemma 4.2. Let \( Z/p = \langle x \rangle \) act on the surface \( S_g \) and \( \rho = \varepsilon i : Z_p \to \Gamma_g \to \text{GL}(2g, Q) \) be a representation for an inclusion \( i: Z_p \to \Gamma_g \). Then \( \rho \) is equivalent to one of the following representations as a complex representation: 

\[
\rho_k = (m+k)\sigma_p \oplus nTr, \text{ where } 2g-2 = mp-i, \ 0 \leq i \leq p-1. \text{ If } i \neq 1, \ 0 \leq k \leq \lfloor 2g/(p-1) \rfloor - m, \text{ if } i = 1, \ 1 \leq k \leq \lfloor 2g/(p-1) \rfloor - m, \ n = 2g - (m+k)(p-1). 
\]

Conversely, any representation of \( \rho_k = (m+k)\sigma_p \oplus nTr \) as above can be equivalent to \( \rho = \varepsilon i \) for some inclusion \( i: Z/p \to \Gamma_g \).

Proof. It is clear that the character \( \chi_{\rho_k}(Id) = (m+k)(p-1) + n = 2g \) and \( \chi_{\rho_k}(x) = (m+k) + n = 2g - (m+k)p = 2 - i - kp \). On the other hand, by using the Lefschetz fixed point theorem, for any \( \rho = \varepsilon i \), \( \chi_{\rho}(Id) = 2g \), \( \chi_{\rho}(x) = 2-t \in 2-B_g,p = \{2-i-kp\} \). Lemma 4.1 implies \( \chi_{\rho} = \chi_{\rho_k} \) for some \( k \). Conversely, by Lemma 4.1, there exists a \( Z/p \) representation \( \rho \) such that \( \chi_{\rho} = \chi_{\rho_k} \) for every \( k \). This completes the proof of Lemma 4.2.

Lemma 4.3. Let \( 2g-2 = mp-i \), where \( p \) is an odd prime and \( 0 \leq i \leq p-1 \). If \( m < p^r \) then \( \lfloor 2g/(p-1) \rfloor < p^{r+1} - p^r \).

Proof. We have \( 2g-2+i = mp < p^{r+1}, \) i.e., \( 2g/(p-1) < (p^{r+1} + 2)/(p-1) \). Since \( 3p^{r+1} + 2 < p^{r+2} + p^r \) implies \( p^{r+1} + 2 < p^{r+2} - 2p^{r+1} + p^r \), we obtain \( (p^{r+1} + 2)/(p-1) \leq p^{r+1} - p^r \).

Lemma 4.4. If \( 1 \leq k \leq p-1 \) and \( kp^r \leq n < (k+1)p^r \), then \( n!/(p^r)(n-p^r)! = k \mod(p) \).

Proof. Write the integer \( n!/(p^r)(n-p^r)! = [(p^r+1)/(1)][(p^r+2)/(2)] \times[(p^r+3)/(3)] \cdots [(p^r+p)/(p)][(p^r+p+1)/(p+1)] \cdots [(p^r+p^2)/(p^2)] \cdots \times[(p^r+p^r)/(p^r)][(p^r+p^r+1)/(p^r+1)] \cdots [(p^r+2p^r)/(2p^r)][(p^r+2p^r+1)/(2p^r+1)] \cdots [(p^r+3p^r)/(3p^r)] \cdots [(kp^r)/(k-1)p^r] \cdots [(n)/(n-p^r)] \).

If \( i \neq 0 \mod(p) \) then \( [(p^r+i)/(i)] = 1 \) in the field \( F_p \).

If \( i = 0 \mod(p^{s-1}) \) and \( i \neq 0 \mod(p^s) \) (\( s \leq r \)), then \( [(p^r+i)/(i)] = 1 \) in the field \( F_p \).

If \( i = mp^r \), \( m = 0, 1, 2, \ldots, k-1 \), then \( [(p^r+i)/(i)] = [(m+1)/(m)] \) in \( F_p \).

So \( n!/(p^r)(n-p^r)! = [(1)][(1)][(2)] \cdots [(k)/(k-1)] = [k] \) in \( F_p \), i.e., the integer \( n!/(p^r)(n-p^r)! = k \mod(p) \) when \( kp^r \leq n < (k+1)p^r \).

Proof of Theorem 4. (a) If \( 2g/(p-1) < p^r \), let \( \rho_k : Z/p \to \Gamma_g \to \text{GL}(2g, Q) \) be a representation equivalent to the complex \( \rho_k = (m+k)\sigma_p \oplus nTr \). The total Chern class \( C \) satisfies

\[
C(\rho_k) = C(\sigma_p)^{m+k} = [1 + c_{p-1}(\sigma_p)]^{m+k} = \sum_{0 \leq t \leq m+k} (m+k)!/t!(m+k-t)! [c_{p-1}(\sigma_p)]^t.
\]

So

\[
i^*c_{\varphi(p^r)}(\varepsilon) = c_{\varphi(p^r)}(\rho_k) = (m+k)!/p^{r-1}(m+k-p^r-1)! [c_{p-1}(\sigma_p)]^{p^r-1} \not\equiv 0 \mod(p)
\]

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in $H^{2p'^{-1}(p-1)}(Z/p; Z)$ for every inclusion $i: Z/p \to \Gamma_g$ since $p'^{-1} \leq m + k \leq [2g/(p-1)] < p'$ by Lemma 4.4. Note that $\Gamma_g$ is $p$-periodic by assumption. Using an argument similar to the proof of Proposition 3.2, we obtain an upper bound $2p'^{-1}(p-1)$ of the $p$-period of $\Gamma_g$ in this case.

(b) If $[2g/(p-1)] \geq p'$, consider the restriction of a homogeneous Chern class polynomial to every $Z/p$ inclusion. We have

$$i^*\left\{[c_\varphi(p')(e)]^{p(p-1)}[c_\varphi(p'+1)(e)]^{p-1}\right\}
\begin{align*}
= & \left\{[c_\varphi(p')(\rho_k)]^{p(p-1)} + [c_\varphi(p'+1)(\rho_k)]^{p-1}\right\}
= \left\{(m + k)!/p'^{-1}!(m + k - p'^{-1})!\right\}^{p(p-1)}[c_{p-1}(\sigma_p)]^{p'(p-1)}
\end{align*}
$$

in $H^{2p'^{-1}(p-1)^2}(Z/p; Z)$.

If $m + k < p'$, the second term above vanishes and the first term is nontrivial by Lemma 4.4 since $p'^{-1} \leq m + k < p'$. If $m + k \geq p'$, then $m + k \leq [2g/(p-1)] < p'^{-1} - p'$ by Lemma 4.3. It implies that the second term above always equals 1 mod $(p)$ and the first term above is 0 or 1 mod $(p)$. So, the element $i^*\left\{[c_\varphi(p')(e)]^{p(p-1)} + [c_\varphi(p'+1)(e)]^{p-1}\right\} = [c_\varphi(p')(\rho_k)]^{p(p-1)} + [c_\varphi(p'+1)(\rho_k)]^{p-1}$ has order $p$ in $H^{2p'^{-1}(p-1)}(Z/p; Z)$ for every $Z/p$ inclusion. It follows that $2p'(p-1)^2$ is an upper bound of the $p$-period of a periodic $\Gamma_g$, and we have completed the proof of Theorem 4.

**Remark.** The upper bound of the $p$-period of $\Gamma_g$ in Theorem 4 is a little bit rough. It can be improved by individually computing the Chern classes of the homology representation of $\Gamma_g$ with the same method.

**Example.** Consider the 3-periodic group $\Gamma_3$, $i: Z/3 \to \Gamma_3$ is an inclusion, the number of possible fixed points are 2 or 5; i.e., the associated representations are $\rho_1 = 2\sigma_3 \oplus 2Tr$ or $\rho_2 = 2\sigma_3$. But $i_1 * c_2(e) = c_2(\rho_1) = c_2(2\sigma_3 \oplus 2Tr) = 2c_2(\sigma_3)$, $i_1 * c_6(e) = c_6(\rho_1) = c_6(2\sigma_3 \oplus 2Tr) = 0$. Therefore, $i_1 * \{[c_2(e)]^3 + c_6(e)\} = 2[c_2(\sigma_3)]^3$ is nontrivial. Similarly, $i_2 * c_2(e) = c_2(\rho_2) = c_2(3\sigma_3) = 3c_2(\sigma_3) = 0 \mod(3)$, $i_2 * c_6(e) = c_6(\rho_2) = c_6(3\sigma_3) = [c_2(\sigma_3)]^3$. So, $i_2 * \{[c_2(e)]^3 + c_6(e)\} = c_2(\sigma_3)$ is nontrivial, i.e., the element $[c_2(e)]^3 + c_6(e)$ is nontrivial when restricted to every $Z/3$ subgroup. Thus, we obtain an upper bound 12 of the 3-period of $\Gamma_3$, which is better than the upper bound 24 given by Theorem 4.

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**References**


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