RIGIDITY THEOREMS FOR NONPOSITIVE
EINSTEIN METRICS

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Abstract. In this paper we study the following problem: When must a complete Einstein metric \( g \) on an \( n \)-manifold with \( \text{Ric} = (n - 1)\lambda g \), \( \lambda \leq 0 \), be a metric of constant curvature \( \lambda \)?

1. Introduction and main results

Let \( g \) be a complete riemannian metric on an \( n \)-manifold \( M \). Denote by \( R \) the curvature tensor of \( g \). The Ricci curvature \( \text{Ric} \) is then defined as

\[
\text{Ric}(x, y) = \sum_{i=1}^{n} g(R(x, e_i)e_i, y), \quad x, y \in T_pM,
\]

where \( \{e_1, \ldots, e_n\} \) is an orthonormal basis for \( T_pM \). The metric \( g \) is said to be Einstein if the Ricci curvature is constant, i.e.,

\[
\text{Ric} = (n - 1)\lambda g
\]

for some constant \( \lambda \). \( \lambda \) is called the Einstein constant of \( g \). It is clear that in dimension three the metric \( g \) is Einstein with \( \text{Ric} = (n - 1)\lambda g \) if and only if it has constant curvature \( \lambda \), i.e.,

\[
R(x, y)z = \lambda(g(y, z)x - g(x, z)y), \quad x, y, z \in T_pM.
\]

In higher dimensions, this is not the case. One may ask if an Einstein metric has constant curvature whenever it has almost constant curvature in a certain sense. From now on we always assume that \( g \) is a complete Einstein metric with Einstein constant \( \lambda \). It is natural to consider the new tensor \( \hat{R} \), defined

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\[
\hat{R}(x, y)z = R(x, y)z - \lambda(g(y, z)x - g(x, z)y), \quad x, y, z \in T_p M.
\]

Denote by \( \sigma \) the pointwise norm of \( \hat{R} \), defined by

\[
\sigma = \sqrt{\sum_{ijkl} g(\hat{R}(e_i, e_j)e_k, e_l)^2},
\]

where \( \{e_1, \ldots, e_n\} \) is an orthonormal basis for \( T_p M \). By a formula in [H], one can easily show that \( \sigma \) satisfies

\[
(1) \quad \Delta \sigma + c_0(n)\sigma^2 - 2(n - 1)\lambda \sigma \geq 0
\]

in the sense of distribution, where \( c_0(n) \) is a positive constant depending only on \( n \) and \( \Delta \) denotes the Laplace-Beltrami operator (in \( \mathbb{R}^n \), \( \Delta = \sum_{i=1}^{n} \partial^2 / \partial x_i^2 \)).

In the case of \( \lambda > 0 \), Myers's Theorem (cf., e.g., [CE]) tells us that the manifold is closed. Thus by integrating (1), one obtains

**Theorem 1** (Berger [B]). Given \( n \), there is a small constant \( \varepsilon = \varepsilon(n) > 0 \) depending only on \( n \) such that if a complete Einstein metric \( g \), with \( \lambda > 0 \), on an n-manifold satisfies \( \sigma \leq \lambda \varepsilon \), then \( \sigma \equiv 0 \), i.e., \( g \) has constant curvature \( \lambda \).

In [S] the author gives an \( L^2 \)-version of Theorem 1, which says that if

\[
\int \sigma^2 \leq \lambda^2 \text{vol}(M)\varepsilon
\]

for some small \( \varepsilon = \varepsilon(n) > 0 \), depending only on \( n \), where \( \text{vol}(M) \) denotes the volume of \( (M, g) \), then \( \sigma \equiv 0 \).

In the case of \( \lambda \leq 0 \), the manifold can be compact or noncompact. First let us consider the case \( \lambda = 0 \). In this case, the following fact is known: There is a small constant \( \varepsilon = \varepsilon(n) > 0 \) depending only on \( n \), if a Ricci-flat metric \( g \) on a closed \( n \)-manifold satisfies

\[
(2) \quad \sigma \cdot \text{dia}(M)^2 \leq \varepsilon,
\]

where \( \text{dia}(M) \) denotes the diameter of \( g \), then \( \sigma \equiv 0 \), i.e., \( g \) is flat. The proof of this fact is trivial. By a theorem of Gromov [G], any almost flat manifold is aspherical, i.e., its universal cover is diffeomorphic to \( \mathbb{R}^n \). Thus for a sufficiently small \( \varepsilon = \varepsilon(n) \), (2) implies that the universal cover \( \tilde{M} \) is diffeomorphic to \( \mathbb{R}^n \). On the other hand, by the Cheeger-Gromoll's Splitting Theorem (cf. [CG]), \( \tilde{M} \) with the induced metric \( \tilde{g} \) is isometric to a riemannian product \( N \times \mathbb{R}^k \) for some closed riemannian manifold \( N \). Thus \( N \) must be a point and \( \tilde{g} \) is flat. Therefore \( g \) is flat. This argument in fact shows that all nonnegatively Ricci-curved aspherical manifolds are flat. By a theorem of Fukaya-Yamaguchi [FY], if \( \text{dia}(M) \leq D \), then condition (2) can be replaced by \( -1 \leq K_g \leq \varepsilon \) for a small number \( \varepsilon = \varepsilon(n, D) > 0 \), where \( K_g \) denotes the sectional curvature of \( g \).

In §2 we will prove an \( L^2 \)-version of the above fact, that is,
Theorem 2. Given $n$, there is a small constant $\varepsilon = \varepsilon(n) > 0$ depending only on $n$ such that if a Ricci-flat metric $g$ on a closed $n$-manifold satisfies

$$\int \sigma^\frac{n}{2} \leq \frac{\text{vol}(M)}{\text{dia}(M)^n} \varepsilon,$$

then $\sigma \equiv 0$, i.e., $g$ is flat.

For complete Ricci-flat metrics on noncompact $n$-manifolds, some rigidity phenomena have been discovered (cf. [A2, Ba, S], etc.). Roughly speaking, if a complete Ricci-flat metric $g$ has sufficiently small total curvature, i.e., there is a small $\varepsilon = \varepsilon(n) > 0$ such that if

$$\int M \left( \frac{1}{\sigma(x)} \right)^n \leq \nu_M^{n+1} \varepsilon,$$

where

$$\nu_M := \lim_{r \to +\infty} \frac{\text{vol}(B(p, r))}{\sigma_n r^n} > 0,$$

where $\sigma_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, then $\sigma \equiv 0$. It is worth mentioning that the result of Anderson [A2] does not require (3), but requires that $\nu_M \geq 1 - \varepsilon$ for a small $\varepsilon = \varepsilon(n) > 0$.

Now let us consider the case of $\lambda < 0$. The following theorem is first proved by Ye [Y, Theorem 2].

Theorem 3 ([Y]). Given $n$, $D > 0$, and $\lambda < 0$, there is a small constant $\varepsilon = \varepsilon(n, \sqrt{-\lambda} D) > 0$ such that if an Einstein metric $g$, with $\lambda < 0$, on a closed $n$-manifold satisfies $\text{dia}(M) \leq D$ and $\sigma \leq |\lambda| \varepsilon$, then $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda$.

In §3 we will prove the following $L^\frac{2}{3}$-version of Theorem 3.

Theorem 4. Given $n$, $D > 0$, and $\lambda < 0$, there is a small constant $\varepsilon = \varepsilon(n, \sqrt{-\lambda} D) > 0$ such that if an Einstein metric $g$, with $\lambda < 0$, on a closed $n$-manifold satisfies $\text{dia}(M) \leq D$ and

$$\int |\lambda|^\frac{2}{3} \text{vol}(M) \varepsilon,$$

then $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda$.

Complete Einstein metrics on noncompact $n$-manifolds with Einstein constant $\lambda < 0$ are still not completely understood. The Sobolev inequalities do not hold on such manifolds. Instead, the Poincaré inequalities hold, which will be used to prove the following

Theorem 5. Let $g$ be a complete Einstein metric on a noncompact simply connected $n$-manifold with $\lambda < 0$. Suppose $n \geq 10$. There is a small constant $\varepsilon = \varepsilon(n) > 0$ such that if

(i) $\sigma \leq |\lambda| \varepsilon$ and
(ii) for some $p \in M$,
\[
\lim_{r \to +\infty} e^{-\delta_n |\varphi|^2} \int_{B(p, r)} \sigma^2 = 0,
\]
where $\delta_n = \frac{1}{4} \sqrt{(n-1) (n-9)} > 0$ and $B(p, r)$ denotes the geodesic ball of radius $r$ around $p$, then $\sigma \equiv 0$, i.e., $g$ has constant curvature $\lambda < 0$.

The proof of Theorem 5 will be given in §4. The author does not know the case of $n \leq 9$.

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2. Closed Einstein manifolds with $\lambda = 0$

In this section we will prove Theorem 2. The argument given here is quite standard and similar to that given in Lemma 2.1 of [Al].

In §1 we have shown that every Ricci-flat metric satisfying (2) for some small $\varepsilon = \varepsilon(n)$ is flat. Throughout this section $M = (M, g)$ always denotes a closed Ricci-flat manifold of dimension $n \geq 4$ and $c_i(n)$’s denote constants depending only on $n$. In the case of $\lambda = 0$, (1) is equivalent to
\[
\Delta \sigma + c_0(n) \sigma^2 \geq 0
\]
in the sense of distribution. Recall that the following Sobolev inequality holds in $M$ (cf. [Be] for references):
\[
\|f\|_{L^{2n/n-2}} \leq c_1(n) \text{vol}(M)^{-\frac{1}{2}} [\text{vol}(M)^{\frac{1}{2}}] \|\nabla f\|_2 + \|f\|_2
\]
for every $f \in C^\infty(M)$.

Multiply (4) by $\sigma^{\alpha}$ for $\alpha \geq 1$. Integration by parts gives
\[
c_0(n) \int \sigma^{\alpha+2} \geq \frac{4\alpha}{(\alpha + 1)^2} \int |\nabla \sigma^{\frac{\alpha+1}{2}}|^2 \geq \frac{1}{\alpha} \int |\nabla \sigma^{\frac{\alpha+1}{2}}|^2.
\]
Taking $f = \sigma^{\frac{\alpha+1}{2}}$ in (5), we obtain by (6)
\[
\|\sigma^{\frac{\alpha+1}{2}}\|_{L^{2n/n-2}} \leq c_2(n) \text{vol}(M)^{-\frac{1}{2}} [\text{vol}(M)^{\frac{1}{2}}] \|\sigma \cdot \sigma^{\alpha+1}\|_{L^1} + \|\sigma^{\alpha+1}\|_2].
\]
Taking $\alpha + 1 = \frac{q}{2}$ in (7) and applying Hölder’s inequality to $\|\sigma \cdot \sigma^{n/2}\|_1$, we have
\[
\|\sigma^{\frac{q}{2}}\|_{L^{2n/n-2}} \leq c_3(n) \text{vol}(M)^{-\frac{1}{2}} [\text{vol}(M)^{\frac{1}{2}}] \|\sigma\|_{L^1} \|\sigma^{\frac{q}{2}}\|_{L^{2n/n-2}} + \|\sigma\|_{L^1}^{\frac{q}{2}}.
\]
It follows from (8) that there is a small constant $\varepsilon(n) > 0$, such that for some $\varepsilon \leq \varepsilon(n)$
\[
\|\sigma\|_{L^1}^{\frac{q}{2}} \leq \frac{\text{vol}(M)^{\frac{1}{2}}}{\text{vol}(M)^{\frac{1}{2}}} \varepsilon,
\]
then
\[
\|\sigma\|_{L^1}^{\frac{q}{2}} = \|\sigma^{\frac{q}{2}}\|_{L^{2n/n-2}} \leq c_4(n) \text{vol}(M)^{-\frac{q}{2n}} \|\sigma\|_{L^1}^{\frac{q}{2}} \leq c_5(n) \text{vol}(M)^{\frac{1}{2} - \frac{4}{n} \text{vol}(M)^{-1}}.
\]
where $q = \frac{n}{2} \cdot \frac{2n}{n-2}$. For general $\alpha \geq 1$, by Hölder's inequality, the interpolation inequality, and (10), we have that for all $\theta > 0$

(11) $\|\sigma \cdot \sigma^{\alpha+1}\|_1 \leq \|\sigma\|_\frac{q}{2} \|\sigma^{\frac{n+1}{2}}\|_2^2$

\[ \leq c_5(n)\text{vol}(M)^{\frac{2}{n-2}}\text{dia}(M)^{-2}(\theta\|\sigma^{\alpha+1}\|_\frac{2n}{n-2} + \theta^{-\frac{2}{n-2}}\|\sigma^{\alpha+1}\|_2)^2. \]

Thus it follows from (7) and (11) that

(12) $\|\sigma^{\frac{n+1}{2}}\|_\frac{2n}{n-2} \leq c_6(n)[\alpha^\frac{1}{2}\text{vol}(M)^{-\frac{3}{2}}\theta\|\sigma^{\alpha+1}\|_2^\frac{2n}{n-2} + (\alpha^\frac{1}{2}\text{vol}(M)^{-\frac{3}{2}}\theta^{-\frac{2}{n-2}} + \text{vol}(M)^{-\frac{1}{2}})\|\sigma^{\alpha+1}\|_2].$

Choosing $\theta = \frac{1}{2}c_6(n)^{-\alpha^\frac{1}{2}}\text{vol}(M)^{\frac{3}{2}}$, we obtain by (12)

(13) $\|\sigma^{\frac{n+1}{2}}\|_\frac{2n}{n-2} \leq c_7(n)\alpha^\frac{3}{2}\text{vol}(M)^{-\frac{3}{2}}\|\sigma^{\alpha+1}\|_2.$

Let $\chi = \frac{n}{n-2}$ and $\alpha + 1 = \frac{n}{2}\chi^i$, $i \geq 0$. It follows from (13) that

\[ \|\sigma\|_\frac{\chi^i+1}{\chi^i} \leq c_8(n)\chi^\frac{1}{\chi^i}\text{vol}(M)^{-\frac{3}{2}}\chi^\frac{1}{\chi^i}\|\sigma\|_\frac{\chi^i}{\chi^i} \]

\[ \leq c_8(n)\chi^{\frac{1}{\chi^i}+\frac{1}{2}}\text{vol}(M)^{-\frac{3}{2}}(\chi^{\frac{1}{\chi^i}+\frac{1}{2}}\|\sigma\|_\frac{1}{\chi^i}). \]

Letting $i \to +\infty$, we obtain

\[ \sigma \leq c_9(n)\text{vol}(M)^{-\frac{3}{2}}\|\sigma\|_\frac{1}{\chi^i} \leq c_9(n)\text{dia}(M)^{-2}\varepsilon, \]

i.e.,

\[ \sigma \cdot \text{dia}(M)^2 \leq c_9(n)\varepsilon. \]

The last inequality follows from (9). Choosing a smaller $\varepsilon$ in (9) if necessary, by the argument in §1, we conclude that $\sigma \equiv 0$, i.e., $g$ is flat. This completes the proof of Theorem 2.

3. Closed Einstein manifolds with $\lambda < 0$

In this section we will only give a sketch of the proof of Theorem 4. The method applied here is very standard and similar to that given in §2. Let $M = (M, g)$ be a closed Einstein $n$-manifold with Einstein constant $\lambda < 0$ and $\text{dia}(M) \leq D$. Throughout this section $c_i(n)$'s always denote positive constants depending only on $n$.

First one has the following Sobolev inequality in $M$ (cf., e.g., [Be] for references): for every $f \in C^\infty(M)$

(14) $\|f\|_\frac{2n}{n-2} \leq c_1(n)C(\sqrt{|\lambda|}D)^{-\frac{n}{2}}(\sqrt{|\lambda|}^{-\frac{1}{2}}\|\nabla f\|_2 + \|f\|_2),$

where $C(x)$, $x > 0$, is the unique positive root of the equation

\[ y \int_0^x (\cosh t + y \sinh t)^{n-1} dt = \int_0^x \sin^{n-1} t dt. \]
Similarly, by (1) and (14) we obtain that there is a constant \( \varepsilon(n) > 0 \) if for some \( \varepsilon < \varepsilon(n) \)
\[ \| \sigma \|_{\frac{q}{2}} \leq |\lambda| \text{vol}(M)^{\frac{q}{2}} \varepsilon \]
then for \( q = \frac{n}{2} \cdot \frac{2n}{n-2} \)
\[ \| \sigma \|_{\frac{q}{2}} \leq c_2(n) C(\sqrt{|\lambda| D})^{-\frac{q}{4}} \text{vol}(M)^{-\frac{q}{4}} \| \sigma \|_{\frac{q}{2}} \]
\[ \leq c_3(n) C(\sqrt{|\lambda| D})^{2-\frac{q}{4}} |\lambda| \text{vol}(M)^{\frac{q}{2} - \frac{q}{4}} \]
and for \( \alpha \geq 1 \),
\[ \| \sigma^{\alpha \frac{q}{2}} \|_{\frac{q}{2} \alpha} \leq c_4(n) C(\sqrt{|\lambda| D})^{-\frac{q}{4}} \alpha \| \sigma \|^{\alpha \frac{q}{2}} \]
Then the last argument in \( \S 2 \) carries over to give
\[ \sigma \leq c_5(n) C(\sqrt{|\lambda| D})^{-\frac{q}{2}} |\lambda| \varepsilon. \]
Choosing a smaller \( \varepsilon \) in (15) if necessary, by Theorem 3 (Ye), one concludes that \( \sigma = 0 \), i.e., \( g \) has constant curvature \( \lambda < 0 \).

4. PROOF OF THEOREM 5

Let \( M = (M, g) \) be a complete \( n \)-manifold. Denote by \( \lambda_1(M, g) \) the first eigenvalue of \( M \), defined as
\[ \lambda_1(M, g) = \inf f \frac{|\nabla f|^2}{f^2}, \]
where the infimum is taken over all \( f \in C^\infty_0(M) \) with compact support in \( M \). It is proved in [M] that if \( M \) is simply connected with sectional curvature \( K_g \leq -\Lambda^2 \) (\( \Lambda > 0 \)),
\[ \lambda_1(M, g) \geq \frac{(n-1)^2}{4} \Lambda^2, \]
i.e., for every \( f \in C^\infty_0(M) \),
\[ \frac{1}{4} (n-1)^2 \Lambda^2 \int f^2 \leq \int |\nabla f|^2. \]
From now on \( (M, g) \) always denotes a complete Einstein \( n \)-manifold with Einstein constant \( \lambda < 0 \) and \( c_1(n) \)'s denote positive constants depending only on \( n \). Clearly, there is a small constant \( \varepsilon(n) > 0 \) such that if for some \( \varepsilon \leq \varepsilon(n) \), \( \sigma \leq |\lambda| \varepsilon \), then the sectional curvature satisfies
\[ K_g \leq -(1 - c_1(n)\varepsilon)|\lambda| < 0. \]
By (1) and (17) we have
\[ \Delta \sigma + (c_0(n)\varepsilon + 2(n-1))|\lambda| \sigma \geq 0. \]
Multiply (18) by \( \sigma \eta^2 \), where \( \eta \) is a cut off function of compact support in \( M \).
Integration by parts gives
\[ (c_0(n)\varepsilon + 2(n-1))|\lambda| \int (\sigma \eta)^2 \geq \int |\nabla (\sigma \eta)|^2 - \int |\nabla \eta|^2 \sigma^2. \]
Taking $f = \eta \sigma$ in (16), by (17) we have
\[
\frac{1}{4}(n-1)^2(1 - c_1(n)\varepsilon)|\lambda| \int (\sigma \eta)^2 \leq \int |\nabla (\sigma \eta)|^2;
\]
so that by (19)
\[
\int |\nabla \eta|^2 \sigma^2 \geq \left[ \frac{1}{4}(n-1)(n-9) - c_2(n)\varepsilon \right] |\lambda| \int (\sigma \eta)^2.
\]

Now suppose $n \geq 10$. Take $\delta(n) = \frac{1}{3}\sqrt{(n-1)(n-9)}$. One can choose a smaller $\varepsilon$ in (17) if necessary, such that
\[
\frac{1}{4}(n-1)(n-9) - c_2(n)\varepsilon \geq \frac{1}{4}e^2\delta(n)^2.
\]
Choosing $\eta(x) = \eta(d(p, x))$, where
\[
\eta(t) = \begin{cases} 
1 & \text{if } t \leq r, \\
\frac{R - t}{R - r} & \text{if } r \leq t \leq R, \\
0 & \text{if } t \geq R,
\end{cases}
\]
we obtain by (20)
\[
\frac{1}{(R-r)^2} \int_{B(p, r)} \sigma^2 \geq \frac{1}{4} e^2 \delta(n)^2 |\lambda| \int_{B(p, r)} \sigma^2.
\]

For any $r_0 > 0$, take $r_j = 2\delta(n)^{-1}|\lambda|^{-\frac{1}{2}}j + r_0$, $j \geq 0$. It then follows from (21) that
\[
\int_{B(p, r_j)} \sigma^2 \geq e^2 \int_{B(p, r_{j-1})} \sigma^2 \geq e^{2j} \int_{B(p, r_0)} \sigma^2 = e^{\delta(n)|\lambda|^{-\frac{1}{2}}(r_j - r_0)} \int_{B(p, r_0)} \sigma^2.
\]

Thus it is easy to see that
\[
\int_{B(p, r_0)} \sigma^2 \leq e^{-\delta(n)|\lambda|^{-\frac{1}{2}}(r_j - r_0)} \int_{B(p, r_j)} \sigma^2.
\]
Letting $r_j \to +\infty$, by Theorem 5(ii), one obtains $\sigma \equiv 0$ on $B(p, r_0)$. Since $r_0$ is arbitrary, one concludes that $\sigma \equiv 0$ on $M$, i.e., $g$ has constant curvature $\lambda < 0$. This completes the proof of Theorem 5.

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