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CONVEXITY AND UNIQUENESS IN AN INVERSE PROBLEM OF POTENTIAL THEORY

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ABSTRACT. Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{R}^n$ whose intersection is convex. Suppose moreover that their volume potentials coincide in the complement of their union. Then $\Omega_1 = \Omega_2$.

In 1938, Novikov [4] proved that if two bounded, convex (or even starshaped) domains have the same volume potential in the complement of their union, then they are identical. In [5] Zalcman asked whether this is true if just one of the domains is assumed to be convex. Also, cf. [3, p. 86]. In our efforts to investigate this problem we were able to prove the following: If the intersection of two bounded domains, which have the same volume potential in the complement of their union, is convex, then they are identical.

Let us introduce some notation. For a bounded domain D we define U^D to be the volume potential of D, i.e.,

$$U^D(x) = c_n \int_D \frac{dy}{|x-y|^{n-2}}, \qquad x \in \mathbb{R}^n,$$

where c_n is a normalization factor so that $\Delta U^D = 1$ in D. We also define the *modified Schwarz potential* (MSP) of a bounded domain D with respect to μ to be the solution of $\Delta u = \chi_D - \mu$ (in the sense of distribution) and $u = 0$ on $\mathbb{R}^n \setminus D$, where μ is a distribution supported in $\overline{D}$ and χ_D is the characteristic function of D.

Theorem 1. *Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{R}^n$, whose intersection $\Omega_1 \cap \Omega_2$ is convex. Suppose moreover $U^{\Omega_1} = U^{\Omega_2}$ in $\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$. Then $\Omega_1 = \Omega_2$.*

Remark 1. In Theorem 1 we see, by putting $|x|^{n-2}U^{\Omega_1} = |x|^{n-2}U^{\Omega_2}$ and letting $|x| \to \infty$, that Ω_1 and Ω_2 have the same volume, therefore, $\Omega_1 \setminus \Omega_2$ and $\Omega_2 \setminus \Omega_1$ are nonempty unless they are identical. It also follows that $\Omega_1 \cap \Omega_2$ is not empty, else U^{Ω_1} can be continued harmonicly into $\mathbb{R}^n$ and violate Liouville's theorem.

The proof of Theorem 1 is based on the following lemmas.

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Lemma 2 (Gustafsson). *In Theorem* 1, *let* $\Omega_1 \not\equiv \Omega_2$. *Then each domain* Ω_j $(j = 1, 2)$ *admits a MSP* u_j *that is* $C(\mathbb{R}^n)$, *zero outside* Ω_j, *and satisfies* $\Delta u_j = \chi_{\Omega_j} - \mu$ *where* μ *is a distribution supported in* $\overline{\Omega}_1 \cap \overline{\Omega}_2$.

Proof. Define

$$U = \begin{cases} U^{\Omega_1} & \text{in } \mathbb{R}^n \setminus \Omega_1, \\ U^{\Omega_2} & \text{in } \mathbb{R}^n \setminus \Omega_2. \end{cases}$$

As $U^{\Omega_1} = U^{\Omega_2}$ in $\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$ this definition is consistent. Now extend U to $\mathbb{R}^n$ as a continuous function and set $\mu = \Delta U$. Then μ is a distribution supported in $\overline{\Omega}_1 \cap \overline{\Omega}_2$. Set $u_j = U^{\Omega_j} - U$ for $j = 1, 2$. Then u_j satisfies the conditions stated in the lemma. □

Remark 2. In Lemma 2 we can even assume the function u_1 (u_2) to be C^1 in a neighbourhood of any point $y \in \partial\Omega_2 \cap \Omega_1$ $(\partial\Omega_1 \cap \Omega_2)$. This will be needed for Lemma 4, where we will use the fact that the derivative of u_1 (u_2) at these points exists.

The following lemma is a modification of a result due to Caffarelli [1, Lemma 1].

Lemma 3. *For* Ω_j $(j = 1, 2)$ *in Theorem* 1, $\Omega_1 \not\equiv \Omega_2$, *and* u_j *in Lemma* 2 *the following hold*:

$$\sup_{\partial\Omega_2} u_1 \geq d_1^2/2, \qquad \sup_{\partial\Omega_1} u_2 \geq d_2^2/2,$$

where $d_1 = \sup_{x\in\Omega_1} \operatorname{dist}(x, \Omega_1 \cap \Omega_2)$ *and* $d_2 = \sup_{x\in\Omega_2} \operatorname{dist}(x, \Omega_1 \cap \Omega_2)$.

Proof. Let $y \in \partial(\Omega_1 \cap \Omega_2)$ and $z \in \partial\Omega_1$ be two points such that $d_1 = |z - y|$. We may, by rotation and translation, assume y is the origin and $z = (z_1, 0')$. Now set $\Omega_1' = \{x \in \Omega_1 : x_1 > 0\}$ and let $\{z^j\}$ be a sequence in Ω_1' converging to z and satisfying $u_1(z^j) \geq 0$ (as $\Delta u_1 \geq 0$ near z the existence of such points is obvious). Define now

$$w(x) = u_1(x) - u_1(z^j) - \tfrac{1}{2}(x_1 - z_1^j)^2 \quad \text{in } \Omega_1',$$

and observe that $w(z^j) = 0$. Then w, being harmonic in Ω_1', attains its positive maximum on the boundary of Ω_1'. Since $w < 0$ on $\partial\Omega_1$, the maximum value is attained at $x_1 = 0$ and it is positive. Now letting $z^j \to z$ we obtain

$$\sup_{\partial\Omega_2} u_1 \geq \sup_{x_1=0} u_1 \geq z_1^2/2 = d_1^2/2,$$

where the first inequality is a consequence of the maximum principle, applied to u_1 in $\Omega_1 \setminus \Omega_2$. One, similarly, proves the second statement. Thus the lemma is proved. □

Lemma 4. *Let* Ω_1 *and* Ω_2 *be as in Theorem* 1, $\Omega_1 \not\equiv \Omega_2$ *and* u_1, u_2 *as in Lemma* 2. *Then at least one of the following holds*:

(1) *There is a point* $y^1 \in \Omega_1 \setminus \Omega_2$ *such that* $d_1^2/2 < -u_1(y^1)$ *and* $|\nabla u_1|(y^1) = 0$.

(2) *There is a point* $y^2 \in \Omega_2 \setminus \Omega_1$ *such that* $d_2^2/2 < -u_2(y^2)$ *and* $|\nabla u_2|(y^2) = 0$.

Proof. Let u_j $(j = 1, 2)$ be the MSP of Ω_j with respect to μ (see Lemma 2). Set

$$W_1 = \{x \in \Omega_1 \setminus \Omega_2 : u_1(x) < 0\}, \qquad W_2 = \{x \in \Omega_2 \setminus \Omega_1 : u_2(x) < 0\},$$

and define $v = (u_1 - u_2)^2$. Then v is subharmonic in $\mathbb{R}^n \setminus (\overline{W_1 \cup W_2})$ and it tends to zero at infinity. Hence, by the maximum principle, v attains its maximum value at $y^1 \in \partial W_1 \cup \partial W_2$. If $W_1 \cup W_2$ is empty then $u_1 = u_2$, which implies $\Omega_1 = \Omega_2$, and the proof will be completed. So assume it is not empty. Then as $u_1 = u_2 = 0$ on $(\partial W_1 \cup \partial W_2) \setminus (\partial\Omega_1 \cup \partial\Omega_2)$, we conclude that $y^1 \in (\partial\Omega_1 \cup \partial\Omega_2) \cap (\partial W_1 \cup \partial W_2)$, i.e., $y^1 \in (\partial\Omega_1 \cap \partial W_2) \cup (\partial\Omega_2 \cap \partial W_1)$. Without loss of generality, let $y^1 \in \partial\Omega_2 \cap \partial W_1$. Then

$$-u_1(y^1) = \sup_{\partial\Omega_2} \sqrt{v} > \sup_{\partial\Omega_2} u_1 \geq d_1^2/2;$$

the last inequality follows from Lemma 3. By Remark 2 we may assume v is C^1 at y^1. Now either $|\nabla v|(y^1) = 0$, which (since u_2 has vanishing Cauchy-data) will complete the proof, or $\nu.\nabla v(y^1) > 0$ for some vector ν pointing outward Ω_2. Since u_2 has vanishing Cauchy-data at y^1, we obtain $\nu.\nabla u_1^2(y^1) > 0$. As $u_1(y^1) < 0$ we arrive at $\nu.\nabla u_1(y^1) < 0$. Thus u_1 decreases strictly in direction ν. Therefore there is another point in $\Omega_1 \setminus \overline{\Omega_2}$, which we (again) label y^1, at which u_1 attains its minimum value and consequently $|\nabla u_1|(y^1) = 0$. This completes the proof. □

Proof of Theorem 1. Suppose the conclusion in the theorem does not hold. Then, by Lemmas 2 and 4, we may assume that the MSP u_1 of Ω_1 exists and satisfies:

(1) $\Delta u_1 = 1$ in $\Omega_1 \setminus \Omega_2$ (which, by Remark 1, is not empty),
(2) there is a point $y^1 \in \Omega_1 \setminus \Omega_2$ such that $d_1^2/2 < -u_1(y^1)$ and $|\nabla u_1|(y^1) = 0$.

We may obviously assume that $u_1(y^1) = \inf_{\Omega_1 \setminus \Omega_2} u_1$. Now by rotation and translation we may assume y^1 is the origin and $\Omega_1 \cap \Omega_2$ is in the half-space $\{x : x_1 < 0\}$. Set $v = \frac{1}{2}x_1^2 - u_1$ in $\Omega_1' = \{x \in \Omega_1 : x_1 > 0\}$. Then v, being harmonic in Ω_1', attains its maximum value on Ω_1' at the boundary $\partial\Omega_1'$. By (2) and $u_1 = 0$ on $\partial\Omega_1$ we conclude that this maximum value is attained on $\partial\Omega_1' \setminus \partial\Omega_1$, hence at the origin. Therefore, by Hopf's maximum principle [2],

$$0 > \frac{\partial v}{\partial x_1}(0) = 0,$$

since both $\frac{1}{2}x_1$ and u_1 have vanishing gradient at the origin. Thus a contradiction is obtained and the theorem is proved. □

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