NORMS ON UNITIZATIONS OF BANACH ALGEBRAS

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Abstract. Equivalence of various norms on the unitization of a nonunital Banach algebra is established, with bounds (1 and 6 exp(1)) uniform over the class of such algebras. A tighter bound, 3, is obtained in C*-algebras for elements with Hermitian nonunital parts.

The algebra norm $\| \cdot \|$ on a nonunital Banach algebra $A$ can be extended to an algebra norm on the unitization $A^+$ in many ways. Proposition 4.3 in [3] states that among these extensions, the $l_1$-norm

$$\|\lambda e + a\|_1 = |\lambda| + \|a\|$$

is maximal and the operator norm

$$\|\lambda e + a\|_{op} = \sup\{\|\lambda x + ax\| : \|x\| \leq 1\}$$

is minimal, provided that it does extend $\| \cdot \|$, i.e., that $\| \cdot \|$ is a regular (= operator) norm.

In the latter case, $A^+$ is complete under both $\| \cdot \|_1$ and $\| \cdot \|_{op}$, so by the "two-norm lemma" [2, II.2.5] these two norms are equivalent; the pure existence nature of the lemma does not yield an explicit bound $M$ in $\| \cdot \|_1 \leq M \| \cdot \|_{op}$ and such a bound seems to depend on the algebra $A$.

The present theorem establishes uniform equivalence of the two unitization norms over the class of nonunital Banach algebras with regular norms.

Theorem. For every nonunital Banach algebra $A$ with unitization $A^+$ and with regular norm, and for every $\lambda \in \mathbb{C}$ and $a \in A$, we have

$$\|\lambda e + a\|_{op} \leq \|\lambda e + a\|_1 \leq (6 \exp 1)\|\lambda e + a\|_{op}.$$  

If $A$ is a C*-algebra, $a \in A$ is hermitian, and $\lambda$ is complex then

$$\|\lambda e + a\|_1 \leq 3\|\lambda e + a\|_{op}$$

and the constant 3 is best (minimal) possible.

Proof. In a general algebra $A$ with a regular norm, we have an extension of the classical inequality for the numerical radius $v(a)$ [1, Theorem 4.1]:

$$v(a) \leq \|a\| \leq (\exp 1)v(a).$$

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Without loss of generality, assume that \( a \neq 0 \). We know that the closure \( K \) of the numerical range of \( a \) in a nonunital algebra contains 0; our first task is to estimate \( v(\lambda + a) \) from below: From the geometry of the complex plane we see that the diameter \( d \) of the compact \( K \) is realized as the distance \( d = |\alpha - \beta| \) with \( \alpha, \beta \in K \), and comparison with the special case \( \lambda_0 = -(\alpha + \beta)/2 \) leads to

\[
v(\lambda e + a) = \max\{|\lambda + \xi| : \xi \in K\} \geq \frac{1}{2}d.
\]

Also, since \( 0 \in K \), we have \( d \geq v(a) \). Altogether,

\[
v(\lambda e + a) \geq \frac{1}{2}v(a).
\]

Now we split estimates into cases \( |\lambda| \leq 2\|a\| \) and \( |\lambda| > 2\|a\| \). The former case gives

\[
\frac{\|\lambda e + a\|_{\text{op}}}{|\lambda| + \|a\|} \geq \frac{v(a)/2 - \|a\| (\exp 1)v(a)}{2\|a\| (\exp 1)v(a)} \geq \frac{v(a)/2 - \|a\| (\exp 1)v(a)}{2\|a\| (\exp 1)v(a)} = \frac{1}{6\exp 1};
\]

the latter case \( |\lambda| > 2\|a\| \) gives, using the triangle inequality and the fact that the fraction in the middle increases with \( |\lambda| \),

\[
\frac{\|\lambda e + a\|_{\text{op}}}{|\lambda| + \|a\|} \geq \frac{|\lambda| - \|a\|}{|\lambda| + \|a\|} \geq \frac{1}{3}.
\]

We conclude that for all complex \( \lambda \),

\[
\|\lambda + a\|_{1} \leq (6\exp 1)\|\lambda + a\|_{\text{op}}.
\]

Now the C*-algebra case: The closure of the numerical range of a Hermitian \( a \) is the smallest real interval \([\alpha, \beta]\) containing the spectrum of \( a \), and for all complex \( \lambda \) we have

\[
\|\lambda e + a\|_{1} = |\lambda| + \max(|\alpha|, |\beta|),
\]

\[
\|\lambda e + a\|_{\text{op}} = \max(|\lambda + \alpha|, |\lambda + \beta|).
\]

The expression to minimize is

\[
q(\lambda) = \frac{\max(|\lambda + \alpha|, |\lambda + \beta|)}{|\lambda| + \max(|\alpha|, |\beta|)}.
\]

Without loss of generality, we assume that \( \alpha \leq 0 < \beta \) and \( \gamma = (\alpha + \beta)/2 \geq 0 \) (recall that 0 is in the spectrum of \( a \)); otherwise we replace \( a \) with \(-a\).

From now on, this is a problem about complex numbers. We split it into four cases:

(C1) \( \lambda \) real,

(C2) \( \lambda \) not real, \( \Re \lambda > -\gamma \),

(C3) \( \lambda \) is not real, \( \Re \lambda < -\gamma \),

(C4) \( \lambda \) not real, \( \Re \lambda = -\gamma \).

In (C1) \( q \) is continuous, piecewise monotone with breakpoints \(-\beta, 0, -\gamma, -\alpha\), and respective values,

\[
\frac{\beta + |\alpha|}{2\beta} \geq \frac{1}{2}, \quad \frac{\beta}{\beta} = 1, \quad \frac{\beta + |\alpha|}{3\beta - |\alpha|} \geq \frac{1}{3}, \quad \frac{\beta + |\alpha|}{\beta + |\alpha|} = 1,
\]

and \( q \) approaches 1 as \( |\lambda| \to \infty \). The best we can say about \( q \), therefore, is \( q \geq \frac{1}{3} \), attained when \( \alpha = 0 \).
Case (C4). Write, for symmetry, \( \alpha = \gamma - \rho \leq 0 \), \( \beta = \gamma + \rho > 0 \), so that \( \rho = (\beta - \alpha)/2 \). Also, we substitute \( p = -\gamma + \sqrt{\gamma^2 + \nu^2} \) (note \( p \geq 0 \)), so that \( \nu^2 = p^2 + 2p\gamma \). To prove that \( Q(\nu) = q(-\gamma + i\nu) \geq \frac{1}{9} \), write

\[
Q(\nu) = \frac{|\lambda + \alpha|}{|\lambda| + \max(|\alpha|, |\beta|)} = \frac{\sqrt{p^2 + \nu^2}}{\sqrt{\gamma^2 + \nu^2} + \gamma + \rho},
\]

\[
Q^2(\nu) - \frac{1}{9} = \frac{9(p^2 + p^2 + 2p\gamma) - (p + \rho + 2\gamma)^2}{9(p + \rho + 2\gamma)^2}
\]

\[
= \frac{2(\rho - p - \gamma)^2 + 6(\rho + \gamma)(\rho - \gamma + p)}{9(p + \rho + 2\gamma)^2} \geq 0
\]

since both \( \rho + \gamma > 0 \) and \( \rho - \gamma + p = |\alpha| + p \geq 0 \).

Cases (C2) and (C3). Except on the set \( \{ \lambda | \Re \lambda = -\gamma \text{ or } \lambda = 0 \} \), \( q \) has a gradient

\[
\nabla q(\lambda) = \frac{\lambda + \beta)(|\lambda| + \beta)/|\lambda + \beta| - |\lambda + \beta|\lambda/|\lambda|}{(|\lambda| + \beta)^2} \quad \text{for } \Re \lambda > -\gamma,
\]

\[
= \frac{(\lambda + \alpha)(|\lambda| + \beta)/|\lambda + \alpha| - |\lambda + \alpha|\lambda/|\lambda|}{(|\lambda| + \beta)^2}.
\]

Remark. The bound \( 6 \exp 1 \) is not the best; by splitting at \( (1 + 1/(2 \exp 1))||a|| \) instead of at \( 2||a|| \) in the proof, we could reduce the bound \( 6 \exp 1 \) to \( 1 + 4 \exp 1 \), but we suspect that even this can be improved.

References

