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TOPOLOGICAL TYPES OF QUASI-ORDINARY SINGULARITIES

KYUNGHO OH

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ABSTRACT. A germ (X, x) of a complex analytic hypersurface in $\mathbb{C}^{d+1}$ is quasi-ordinary if it can be represented as the image of an open neighborhood of 0 in $\mathbb{C}^d$ under the map $(s_1, \dots, s_d) \mapsto (s_1^n, \dots, s_d^n, \zeta(s_1, \dots, s_d))$, $n > 0$, where ζ is a convergent power series. It is shown that the topological type of the singularity $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ is determined by a certain set of fractional monomials, called the characteristic monomials, appearing in the fractional power series $\zeta(t_1^{1/n}, \dots, t_d^{1/n})$.

1. INTRODUCTION

Let (X, x) be a germ of an irreducible hypersurface of dimension d in $(\mathbb{C}^{d+1}, 0)$. By the Weierstrass preparation theorem, there is a finite map of analytic germs $\pi\colon (X, x) \to (\mathbb{C}^d, 0)$. If this map π has a normal crossing discriminant at $0 \in \mathbb{C}^d$ then the germ (X, x) is called a *quasi-ordinary* singularity. Such singularities are of the simplest type in terms of discriminants and arise naturally in the Jungian process of desingularization. For example, every plane curve singularity is quasi-ordinary $(d = 1)$. Quasi-ordinary surface singularities $(d = 2)$ were introduced by Jung [J] in 1908. Zariski [Z1] studied these singularities in the context of the problem of resolving singularities. More rigorous and comprehensive study began with Lipman's thesis [L3]. (For more details, see Lipman [L1, L2].)

An irreducible quasi-ordinary singularity (X, x) can be represented as the image of an open neighborhood of 0 in $\mathbb{C}^d$ by the map

$$(s_1, \dots, s_d) \mapsto (s_1^n, \dots, s_d^n, \zeta(s_1, \dots, s_d)), \qquad n > 0,$$

where ζ is a convergent power series [A, Theorem 3]. Among the fractional monomials appearing in the expansion $\zeta(t_1^{1/n}, \dots, t_d^{1/n}) = \sum c_{\alpha_1, \dots, \alpha_d} t_1^{\alpha_1} \cdots t_d^{\alpha_d}$, one can extract some monomials, called the characteristic monomials, whose exponents are classically called Puiseux pairs in the case of plane curve singularities.

In this paper, we will prove that these characteristic monomials determine the embedded topology of the pair $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ (Theorem 3.2). We show this by constructing a vector field on a $(2d + 1)$-dimensional sphere in $\mathbb{C}^{d+1}$

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whose integral curves give rise to a homeomorphism between the links of two quasi-ordinary singularities with the same characteristic monomials. Lipman [L2] indicated that this also follows from a saturation theorem of Zariski [Z1, Theorem 6.1], since the characteristic monomials determine the $\mathbb{C}\{t_1, \dots, t_d\}$-saturation of the local ring $\mathscr{O}_{X,x} = \mathbb{C}\{t_1, \dots, t_d\}[\zeta]$. The converse was proved by Gau [G] using Lipman's result on the topological invariance of the branching sequences [L1].

2. Quasi-ordinary singularities

We closely follow Lipman [L1, §5] for the basic definition and notations in this section.

Definition 2.1. An irreducible hypersurface analytic germ $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ is called a *quasi-ordinary* singularity if there exists a finite map of analytic germs $\pi\colon (X, x) \to (\mathbb{C}^d, 0)$ induced by a linear projection from $\mathbb{C}^{d+1}$ to $\mathbb{C}^d$ with a normal crossing discriminant Δ at 0; in other words, there is a local coordinate system $(t_1, \dots, t_d)$ of $\mathbb{C}^d$ at 0 such that the discriminant locus Δ is defined by $t_1^{e_1} \cdots t_d^{e_d} = 0$, $e_i \geq 0$. Moreover, the map π is called a quasi-ordinary projection.

After a suitable coordinate change, we may assume that the map π is the restriction of the projection map $\mathbb{C}^{d+1} \to \mathbb{C}^d$ sending $(t_1, \dots, t_d, z) \mapsto (t_1, \dots, t_d)$ and that its discriminant locus Δ is contained in the coordinate hyperplanes of $\mathbb{C}^d$. For such a coordinate system, the Weierstrass preparation theorem implies that the germ $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ is defined by an equation

$$F(t_1, \dots, t_d, z) = z^m + f_1(t_1, \dots, t_d)z^{m-1} + \cdots + f_m(t_1, \dots, t_d) = 0$$

where $(t_1, \dots, t_d, z)$ is a local coordinate system of $\mathbb{C}^{d+1}$ and the f_i are nonunit convergent power series. From Riemann Extension Theorem and the fact that the fundamental group of the complement of the discriminant locus is a free abelian group, it follows that the roots ζ_i of F as a polynomial in z belong to a fractional convergent power series ring $\mathbb{C}\{t_1^{1/n}, \dots, t_d^{1/n}\}$ for some $n > 0$ (cf. [A, Theorem 3] for a purely algebraic proof). Thus the germ (X, x) can be represented as the image of an open neighborhood of 0 in $\mathbb{C}^d$ under the map $(s_1, \dots, s_d) \mapsto (s_1^n, \dots, s_d^n, \zeta(s_1, \dots, s_d))$ $n > 0$ where ζ is a convergent power series. In this sense, ζ is called a *parametrization* of the germ (X, x). Since the discriminant Δ of F has a normal crossing, Δ is of the form

$$\prod_{i \neq j} (\zeta_i - \zeta_j) = t_1^{e_1} \cdots t_d^{e_d} \cdot \varepsilon(t_1, \dots, t_d), \qquad \varepsilon(0, \dots, 0) \neq 0.$$

Since $t_1^{1/n}, \dots, t_d^{1/n}$ are irreducible elements in the unique factorization domain $\mathbb{C}\{t_1^{1/n}, \dots, t_d^{1/n}\}$, we have

$$\zeta_i - \zeta_j = M_{ij}\varepsilon_{ij}(t_1^{1/n}, \dots, t_d^{1/n}), \qquad \varepsilon_{ij}(0, \dots, 0) \neq 0, \tag{1}$$

where

$$M_{ij} = t_1^{a_1/n} t_2^{a_2/n} \cdots t_d^{a_d/n} \tag{2}$$

with integers $a_l \geq 0$ depending on (i, j).

In general, a fractional convergent power series $\zeta \in \mathbb{C}\{t_1^{1/n}, \dots, t_d^{1/n}\}$ is called a *quasi-ordinary branch* if any two conjugates $\zeta_i \neq \zeta_j$ of ζ over $\mathbb{C}\{t_1, \dots, t_d\}$ satisfy equation (1). The fractional monomials M_{ij} are called the *characteristic monomials* of ζ. Note that the characteristic monomials M_{ij} are not units since ζ_i and ζ_j are nonunits in $\mathbb{C}\{t_1^{1/n}, \dots, t_d^{1/n}\}$. Now, we shall give some elementary properties of the characteristic monomials of a given quasi-ordinary branch ζ. Let

$$\mathscr{L} \subset \mathscr{L}(\zeta) \subset \mathscr{L}_n$$

be the respective fraction fields of

$$\mathbb{C}\{t_1, \dots, t_d\} \subset \mathbb{C}\{t_1, \dots, t_d\}[\zeta] \subset \mathbb{C}\{t_1^{1/n}, \dots, t_d^{1/n}\}.$$

Then $\mathscr{L}(\zeta)$ is a Galois extension of $\mathscr{L}$ since $\mathscr{L}_n$ (and hence $\mathscr{L}(\zeta)$) is an abelian extension of $\mathscr{L}$. Hence the Galois group $G = \mathrm{Aut}(\mathscr{L}(\zeta)/\mathscr{L})$ consists of $m = [\mathscr{L}(\zeta) : \mathscr{L}]$ elements and the set of all conjugates $\{\zeta_i | 1 \leq i \leq m\}$ of $\zeta := \zeta_1$ over $\mathscr{L}$ is $\{\gamma\zeta | \gamma \in G\}$. Moreover, we have $\zeta_i - \zeta_j = \gamma \cdot (\zeta_k - \zeta)$ for some $\gamma \in G$ and k. Thus the set of all possible characteristic monomials of ζ is

$$\{M_k := M_{k1} | 2 \leq k \leq m\},$$

where M_{k1} is defined as in (1) and (2). We may have $M_k = M_{k'}$ for some $k \neq k'$. The identity

$$M_i \varepsilon_{i1} - M_j \varepsilon_{i1} = (\zeta_i - \zeta_1) - (\zeta_j - \zeta_1) = M_{ij} \varepsilon_{ij}$$

implies

Lemma 2.2 [L2, Lemma (5.6)]. *The set $\{M_k | 2 \leq k \leq m\}$ of characteristic monomials of quasi-ordinary branch ζ is totally ordered by divisibility (i.e., $M_i \leq M_j$ if M_i divides M_j in $\mathbb{C}\{t_1^{1/n}, \dots, t_d^{1/n}\}$).*

This lemma implies that we can reindex the set of distinct characteristic monomials in such a way that

$$M_1 < M_2 < \cdots < M_g \tag{3}$$

where g is the number of the distinct characteristic monomials. In this case, M_1 is called the *minimal characteristic monomial* of ζ. Moreover, the following holds:

Lemma 2.3 [L1, Lemma (5.7) and Remark (5.8)]. *Let $\{M_k | 1 \leq k \leq g\}$ be the set of distinct characteristic monomials of a quasi-ordinary branch ζ indexed as in (3). Then we have the following chain of the subfields of $\mathscr{L}(\zeta)$:*

$$\mathscr{L} \subsetneqq \mathscr{L}(M_1) \subsetneqq \mathscr{L}(M_1, M_2) \subsetneqq \cdots \subsetneqq \mathscr{L}(M_1, M_2, \dots, M_g) = \mathscr{L}(\zeta).$$

In particular, each characteristic monomial M_k must appear with nonzero coefficient in the fractional power series ζ.

The previous two lemmas gives

Proposition 2.4. *For the characteristic monomials M_i of ζ indexed as in (3), let*

$$R_i := \mathbb{C}[\![t_1, t_1^{-1}, \dots, t_d, t_d^{-1}, M_1, M_1^{-1}, \dots, M_i, M_i^{-1}]\!] \cap \mathbb{C}\{t_1^{1/n}, \dots, t_d^{1/n}\}$$

be the ring of the intersection of a formal power series ring and a convergent power series ring. Then for every quasi-ordinary branch ζ, *there exist unique nonunit elements* f_i *in* R_i *such that*

$$\zeta = f_0 + c_1 M_1 + M_1 f_1 + c_2 M_2 + M_2 f_2 + \cdots + c_g M_g + M_g f_g . \tag{4}$$

Notation 2.5. Let ζ be a quasi-ordinary branch as in (4). For $0 \le l \le g$, $\zeta^{[l]}$ denote a quasi-ordinary branch such that

$$\zeta^{[l]} = \zeta - M_l f_l , \tag{5}$$

and $U(\zeta, k)$ denotes the set of conjugates ζ_{k_i} of ζ such that

$$\zeta - \zeta_{k_i} = M_k \varepsilon_{k_i}(t_1^{1/n}, \dots, t_d^{1/n}), \qquad \varepsilon_{k_i}(0, \dots, 0) \neq 0 . \tag{6}$$

Remark 2.6. Note that ζ and $\zeta^{[l]}$ have the same set of characteristic monomials and that $\mathscr{L}(\zeta) = \mathscr{L}(\zeta^{[l]})$. Moreover we have $(\gamma \cdot \zeta)^{[l]} = \gamma \cdot \zeta^{[l]}$ for $\gamma \in \mathrm{Aut}(\mathscr{L}(\zeta)/\mathscr{L})$.

The following distance estimates between the conjugates of ζ and $\zeta^{[l]}$ will be used in the proof of the main theorem.

Proposition 2.7. *Let* $\zeta(t_1^{1/n}, \dots, t_d^{1/n}) \in \mathbb{C}\{t_1^{1/n}, \dots, t_d^{1/n}\}$ *be a quasi-ordinary branch,* $\zeta_{k_i} \in U(\zeta, k)$ *and* $\zeta_{k_i'} \in U(\zeta, k')$. *Then for any large* $c > 0$, *there exist positive numbers* $\delta_1, \dots, \delta_d$ *such that for* $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{C}^d$ *with* $|s_i| \le \delta_i$, *we have the following inequalities*:

$$\zeta(\mathbf{s}) - \zeta^{[l]}(\mathbf{s}) = \zeta_{k_i}(\mathbf{s}) - \zeta_{k_i}^{[l]}(\mathbf{s}) \quad \text{if } k > l , \tag{7}$$

$$c|\zeta(\mathbf{s}) - \zeta^{[l]}(\mathbf{s})| \le |\zeta(\mathbf{s}) - \zeta_{k_i}(\mathbf{s})| \quad \text{if } k \le l , \tag{8}$$

$$c|\zeta_{k_i}(\mathbf{s}) - \zeta_{k_i}^{[l]}(\mathbf{s})| \le |\zeta(\mathbf{s}) - \zeta_{k_i}(\mathbf{s})| \quad \text{if } k \le l . \tag{9}$$

Proof. Let γ be an element of $\mathrm{Aut}(\mathscr{L}(\zeta)/\mathscr{L})$ such that $\gamma \cdot \zeta = \zeta_{k_i}$. By Remark 2.6, we also have $\gamma \cdot \zeta^{[l]} = \zeta_{k_i}^{[l]}$. We will show each inequality separately.

Proof of (7). Since $\zeta - \zeta_{k_i} = M_k \cdot \varepsilon_{k_i}$, we have $\gamma \cdot f = f$ for $f \in R_i$, $i < k$. Thus

$$\begin{aligned} \zeta_{k_i} - \zeta_{k_i}^{[l]} &= \gamma \cdot (\zeta - \zeta^{[l]}) \quad \text{(by Remark 2.6)} \\ &= \gamma \cdot M_l f_l = M_l f_l \quad (\text{since } M_l f_l \in R_l \text{ and } l < k) \\ &= \zeta - \zeta^{[l]} . \end{aligned}$$

Proof of (8). Since $\zeta - \zeta^{[l]} = M_l f_l$ and $\zeta - \zeta_{k_i} = M_k \varepsilon_{k_i}$, it is sufficient to show that for $s_1, \dots, s_d$ with sufficiently small $|s_i|$,

$$c|M_l f_l(\mathbf{s})| \le |M_k \varepsilon_{k_i}(\mathbf{s})| .$$

This inequality holds because the analytic function

$$(c M_l f_l / M_k \varepsilon_{k_i})(\mathbf{s})$$

vanishes at the origin. Note that $M_k \varepsilon_{k_i}$ divides M_l and f_l is a nonunit.

Proof of (9). By Remark (2.6), we have

$$\zeta_{k_i} - \zeta_{k_i}^{[l]} = \gamma \cdot (\zeta - \zeta^{[l]}) = \gamma \cdot M_l f_l .$$

Since the action of γ on fractional monomials is given by multiplication of complex numbers, there is a nonunit element g_l such that $\gamma \cdot M_l f_l = M_l g_l$. Hence the same argument as in the above proof works. □

For a quasi-ordinary branch ζ, let

$$\{\zeta_1 = \zeta, \dots, \zeta_m\} \quad (\text{resp. } \{\zeta_1^{[l]} = \zeta^{[l]}, \dots, \zeta_m^{[l]}\})$$

be the set of conjugates of ζ (resp. $\zeta^{[l]}$ defined as in (5)). For $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{C}^d$ and i, we define a line segment $\overline{\zeta_i \zeta_i^{[l]}}(\mathbf{s})$ by

$$\overline{\zeta_i \zeta_i^{[l]}}(\mathbf{s}) = \{t\zeta_i(\mathbf{s}) + (1-t)\zeta_i^{[l]}(\mathbf{s}) | 0 \le t \le 1\}. \tag{10}$$

Corollary 2.8. *For* $\mathbf{s}$ *in* $\mathbb{C}^d$ *outside the coordinate hyperplanes with sufficiently small* $\sum |s_i|^2$, *the line segments* $\overline{\zeta_i \zeta_i^{[l]}}(\mathbf{s})$, $i = 1, \dots, m$, *do not intersect each other unless* $\zeta_i = \zeta_i^{[l]}$.

Proof. It is enough to show that for a fixed $\zeta = \zeta_i$, the line $\overline{\zeta \zeta^{[l]}}(\mathbf{s})$ does not intersect with $\overline{\zeta_{k_i} \zeta_{k_i}^{[l]}}(\mathbf{s})$. For $k > l$, this is true because of (7) and the fact that $\zeta(\mathbf{s}) - \zeta^{[l]}(\mathbf{s}) \neq a(\zeta(\mathbf{s}) - \zeta_{k_i}(\mathbf{s}))$ for any real number a. For $k \le l$, the statement holds by inequalities (8) and (9). □

3. Topological type and characteristic monomials

Definition 3.1. Let $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ and $(X', x') \subset (\mathbb{C}^{d+1}, 0)$ be two germs of analytic sets. Then (X, x) and (X', x') are said to have the *same topological type* if there exist open sets $U_1, U_2 \subset \mathbb{C}^{d+1}$ containing the origin 0 and a homeomorphism $h: U_1 \to U_2$ such that $h(0) = 0$ and $h(X \cap U_1) = X' \cap U_2$.

To state our main result, we set up some notation. Let $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ (resp. $(X', x) \subset (\mathbb{C}^{d+1}, 0)$) be the quasi-ordinary singularity given by the parametrization ζ (resp. $\zeta^{[l]}$), and let $\pi|_X: (X, x) \to (\mathbb{C}^d, 0)$ (resp. $\pi|_{X'}: (X', x) \to (\mathbb{C}^d, 0)$) be a quasi-ordinary projection for (X, x) (resp. (X', x)) induced by the map $\pi: \mathbb{C}^{d+1} \to \mathbb{C}^d$ projecting $(t_1, \dots, t_d, z)$ to $(t_1, \dots, t_d)$. Here we assume $\zeta \neq \zeta^{[l]}$ by avoiding the trivial case. The following theorem implies our main result, particularly, that the topological type of quasi-ordinary singularities are determined by their characteristic monomials.

Theorem 3.2. *The analytic germs* $(X, x) \subset (\mathbb{C}^{d+1}, 0)$ *and* $(X', x) \subset (\mathbb{C}^{d+1}, 0)$ *have the same topological type.*

Proof. We set

$$D_i = \{t \in \mathbb{C} | \, |t| \le \delta_i\}, \qquad \delta_i > 0,$$
$$\mathbf{D}(\delta_1, \dots, \delta_{d+1}) = D_1 \times \cdots \times D_{d+1},$$
$$N_i = D_1 \times \cdots \times D_{i-1} \times \partial D_i \times D_{i+1} \times \cdots \times D_{d+1},$$
$$\partial \mathbf{D}^{d+1} = \bigcup_{i=1}^{d+1} N_i, \qquad N_i^c = \partial \mathbf{D}^{d+1} - N_i,$$
$$L = X \cap \partial \mathbf{D}^{d+1} \quad \text{and} \quad L' = X' \cap \partial \mathbf{D}^{d+1}.$$

By the conic structure lemma of Burghelea and Verona [BV], it is enough to prove that the pair $(\partial \mathbf{D}^{d+1}, L)$ is homeomorphic to $(\partial \mathbf{D}^{d+1}, L')$ for sufficiently small $\delta_1, \dots, \delta_{d+1}$. We will prove this by constructing a smooth vector field on $\partial \mathbf{D}^{d+1}$ such that the flow lines send L to L' in a unit time. (We will regard both L and L' as the subset of the *same space* $\partial \mathbf{D}^{d+1}$.)

By choosing $\delta_1, \dots, \delta_d$ relatively small compared to δ_{d+1}, we will assume that L and L' are contained in N^c_{d+1}. Let Δ be the union of coordinate hyperplanes in $\mathbb{C}^d$ and $B = \pi^{-1}(\Delta)$. Note that $\pi|_{L-B}\colon L-B \to T := \pi(N^c_{d+1}) - \Delta$ (resp. $\pi|_{L'-B}\colon L'-B \to T$) is an unbranched covering. Thus for every point $\mathbf{t} \in T - \Delta$, the fiber $\pi|^{-1}_{L-B}(\mathbf{t})$ (resp. $\pi|^{-1}_{L'-B}(\mathbf{t})$) consists of $m := \mathrm{Aut}(\mathscr{L}(\zeta)/\mathscr{L})$ distinct points. More precisely, we have

$$\pi|^{-1}_{L-B}(\mathbf{t}) = \{(s_1^n, \dots, s_d^n, \zeta_i(s_1, \dots, s_d)) | i = 1, \dots, m\}$$

and

$$\pi|^{-1}_{L'-B}(\mathbf{t}) = \{(s_1^n, \dots, s_d^n, \zeta_i^{[l]}(s_1, \dots, s_d)) | i = 1, \dots, m\}$$

for some $s_1, \dots, s_d \in \mathbb{C}$. Here $\zeta_1, \dots, \zeta_m$ (resp. $\zeta_1^{[l]}, \dots, \zeta_m^{[l]}$) are conjugates of ζ (resp. $\zeta^{[l]}$).

Define a map $\psi\colon L-B \to L'-B$ by sending $p = (s_1^n, \dots, s_d^n, \zeta_j(s_1, \dots, s_d))$ in the fiber $\pi|^{-1}_{L-B}(\mathbf{t})$ to $\psi(p) = (s_1^n, \dots, s_d^n, \zeta_j^{[l]}(s_1, \dots, s_d))$ in the fiber $\pi|^{-1}_{L'-B}(\mathbf{t})$. This map is independent of the choice of $s_1, \dots, s_d$, since we have $\zeta_i(\omega_1 s_1, \dots, \omega_d s_d) = (\gamma \cdot \zeta_i)(s_1, \dots, s_d)$, $\omega_j^n = 1$ for some $\gamma \in \mathrm{Aut}(\mathscr{L}(\zeta)/\mathscr{L})$ and $(\gamma \cdot \zeta)^{[l]} = \zeta^{[l]}$. Moreover, it is a diffeomorphism because the map ψ is continuous and the covering maps $\pi|_{L-B}$ and $\pi|_{L'-B}$ are local diffeomorphisms in the following commutative diagram:

$$\begin{tikzcd} L-B \arrow[rr, "\psi"] \arrow[dr, "\pi|_{L-B}"'] & & L'-B \arrow[dl, "\pi|_{L'-B}"] \\ & T & \end{tikzcd}$$

Let $\overline{p\psi(p)}$ denote the real line segment joining p and $\psi(p)$ in $\mathbb{R}^{2d+2} = \mathbb{C}^{d+1}$. Note that these lines lie in $\partial \mathbf{D}^{d+1}$ by the assumption that $L, L' \subset N^c_{d+1}$. For sufficiently small $\delta_1, \dots, \delta_{d+1}$, these lines never intersect each other by Corollary 2.8. Thus the set

$$J(L-B, L'-B) = \bigcup_{p \in L-B} \overline{p\psi(p)}$$

is a smooth manifold with boundary in $\partial \mathbf{D}^{d+1}$. In fact, $(L-B) \times [0, 1]$ is diffeomorphic to $J(L-B, L'-B')$ via the map $(p, t) \mapsto (p, (1-t)p + t\psi(p))$. Now define a vector field $\mathscr{V}$ on $J(L-B, L'-B)$ by

$$\mathscr{V}(q) = \psi(p) - p \quad \text{when } q \in \overline{p\psi(p)}.$$

Then the vector field $\mathscr{V}$ is well defined and smooth. Using a partition of unity, we can extend $\mathscr{V}$ to a vector field on $\partial \mathbf{D}^{d+1}$ supported on a proper subset of $\partial \mathbf{D}^{d+1}$. Since every vector field with a compact support generates an isotopy [H, Theorem 1.2, p. 179], we have a homeomorphism of $\partial \mathbf{D}^{d+1}$, which maps

$L-B$ to $L'-B$. By the continuity, this homeomorphism maps L to L'. This completes the proof. □

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Department of Mathematics, University of Missouri, St. Louis, Missouri 63121
E-mail address: oh@arch.umsl.edu