TOPOLOGICAL COMPLETIONS OF METRIZABLE SPACES

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(Communicated by Dennis Burke)

Abstract. For a pair of metrizable spaces $X$ and $Y$, we investigate conditions under which there is a dense embedding $h: X \to Z$, where $Z$ is completely metrizable and $Z \setminus h(X)$ is homeomorphic to $Y$. In such a case, $Z$ is called a topological completion of $X$ and $Y$ is called a completion remainder of $X$. In case $X$ and $Y$ are completely metrizable, we give necessary and sufficient conditions that $Y$ be a completion remainder of $X$. We characterize the completion remainders of $\mathbb{R}$ and those of the rationals, $\mathbb{Q}$. We also characterize the remainders of $\mathbb{Q}(\kappa)$, a nonseparable analogue of $\mathbb{Q}$.

1. Introduction

Wilansky [4] asked whether there is a 3-point completion of the reals, i.e., is there a dense embedding $h$ of $\mathbb{R}$ into a Polish space $Z$ such that $|Z \setminus h(\mathbb{R})| = 3$? More generally, one might ask under what conditions on metrizable spaces $X$ and $Y$ does there exist a homeomorphism $h$ of $X$ into a completely metrizable space $Z$ such that $h(X)$ is dense in $Z$ and $Z \setminus h(X)$ is homeomorphic to $Y$. In such a case, $Z$ is called a topological completion of $X$ and $Y$ is called a completion remainder of $X$. In case $h(X)$ is open in $Z$, $Y$ is called a closed completion remainder of $X$. Throughout, if $X$ is a space, $d(X)$, $w(X)$, and $e(X)$ denote, respectively, the density, weight, and extent of $X$ and $\text{l dp}(x)$ denotes the local density at $p$ of $X$:

$$d(X) = \omega + \inf \{|M| : M \text{ is dense in } X\};$$
$$w(X) = \omega + \inf \{|B| : B \text{ is a basis for } X\};$$
$$e(X) = \omega + \sup \{|D| : D \text{ is a closed, discrete set in } X\};$$
$$\text{l dp}(x) = \omega + \inf \{|D| : D \text{ is dense in some open } U \text{ in } X, p \in U\}.$$ 

For metrizable spaces, $d(X) = w(X) = e(X)$. This and other relevant properties are to be found in Engelking [1].

In §2 we characterize those pairs $(X, Y)$ of completely metrizable spaces such that $Y$ is a closed completion remainder of $X$ and those pairs such that $Y$ is a completion remainder of $X$. Section 3 provides a characterization of the completion remainders of $\mathbb{Q}$, the rationals, and gives both necessary and sufficient conditions (neither being necessary and sufficient) for a space to...
be a completion remainder of $P$, the irrationals. Section 4 studies completion remainders of nonseparable analogs $Q(\kappa)$ and $P(\kappa)$ of $Q$ and $P$. We conclude in §5 with two open questions.

2. The completely metrizable case

It is easy to see that if $X$ and $Y$ are metrizable and $Y$ is a completion remainder of $X$, then $d(Y) \leq d(X)$ and that if $Y$ is nonempty then $X$ is not compact. Whether or not $e(X)$ is achieved, that is, whether there exists a closed discrete set in $X$ whose cardinality is $e(X)$, plays an important role. Note that, for metrizable spaces $X$, if $e(X)$ is not achieved then $e(X)$ has countable cofinality. We begin with an example.

Example 1. A completely metrizable space $Z = X \cup Y$, where $X \cap Y = \emptyset$, $X$ is dense in $Z$, and there is no closed discrete set in $X$ of cardinality $d(Y)$. Consider a hedgehog $H$, centered at a point $\emptyset$, where $H = \bigcup_{i \in \Lambda_i} H_i$ and each $H_i$ has $\aleph_i$ spines of length 1. That is, $H_i = \{\emptyset\} \cup \bigcup_{\lambda \in \Lambda_i} (\{0\} \times \lambda)$, where $\Lambda_i$ is an indexing set of cardinality $\aleph_i$. Take $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$. Let $H'_i = \{\emptyset\} \cup \bigcup_{\lambda \in \Lambda_i} (\{0\} \times \{\lambda\})$; let $Z = \bigcup_{i < \omega} H'_i$; let $Y = \{(2^{-i}, \lambda) : i < \omega, \lambda \in \Lambda_i\}$; and let $X = Z \setminus Y$. Note that $Y$ is a discrete space of cardinality $\aleph_\omega$ but every discrete subset of $X$ of cardinality $\aleph_\omega$ has $\emptyset$ as a limit point.

This example motivates the following useful lemma.

Lemma 1. Suppose $X$ is a metrizable space and $e(X)$ is not achieved. Then there exists a point $x$ of $X$ such that $ld_x(X) = e(X)$. Moreover, the set of all such points is compact.

Proof. Assume that $e(X)$ is not achieved. Let $\alpha_0 = 0$. Let $\{\alpha_n : n < \omega\}$ be an increasing sequence of cardinals whose sum is $e(X)$. Suppose $ld_x(X) < e(X)$ for all $p \in X$. There is a minimal, locally finite open cover $\mathcal{U}$ of $X$ such that for all $U \in \mathcal{U}$, $d(U) < e(X)$. Then $|\mathcal{U}| < e(X)$. If there is a cardinal $\alpha < e(X)$ such that $d(U) < \alpha$ for all $U \in \mathcal{U}$, then $d(X) \leq \alpha \cdot |\mathcal{U}| < e(X)$, which is impossible. For each $n < \omega$ there is a $U_n \in \mathcal{U}$ such that $d(U_n) \geq \alpha_n$. Since $d(U_n) = d(\overline{U_n}) = e(\overline{U_n})$, there is a closed discrete set $D_n$ in $\overline{U_n}$ such that $|D_n| \geq \alpha_{n-1}$. Then $D = \bigcup_{n < \omega} D_n$ is closed and discrete and has cardinality $e(X)$, a contradiction.

Next, assume that there is an infinite closed and discrete set $A = \{a_n : n < \omega\}$ such that $ld_a(X) = e(X)$ for every $a \in A$. There is a discrete collection $\{U_n : n < \omega\}$ of open sets screening $A$. For each $n < \omega$, there is a closed discrete set $D_n$ in $\overline{U_n}$ of cardinality $\geq \alpha_{n-1}$. Then $D = \bigcup_{n < \omega} D_n$ is closed and discrete in $X$ and has cardinality $e(X)$, which is impossible.

Theorem 1. Let $X$ and $Y$ be completely metrizable spaces. Then $Y$ is a closed completion remainder of $X$ if and only if there is a closed discrete subset of $X$ of cardinality $d(Y)$.

Proof. Suppose first that $Y$ is a closed completion remainder of $X$, $h : X \to Z$ is a dense embedding, $Z$ is completely metrizable, $Z \setminus h(X)$ is closed in $Z$ and is homeomorphic to $Y$. Since $d(Y) \leq d(X) = e(X)$, it follows that if $e(X)$ is achieved or if $d(Y) < e(X)$ then there is a closed discrete set in $X$ of cardinality $d(Y)$. Assume then that $d(Y) = e(X)$ and that $e(X)$ is not achieved. The set $K$ of all points of $X$ at which $X$ has local density $e(X)$ is
compact. Thus \( h(K) \) and \( Z \setminus h(X) \) are disjoint closed sets in \( Z \) and can be enclosed in open sets \( U_X \) and \( U_Y \) with disjoint closures. Let

\[
Z' = Z \cap \overline{U_Y}, \quad X' = h(X) \cap \overline{U_Y}, \quad Y' = (Z \setminus h(X)) \cap \overline{U_Y}.
\]

Then \( Z' \) is completely metrizable, \( Y' \simeq Y \), \( Y' \) is closed in \( Z' \), and \( X' \) is dense in \( Z' \). Now, since \( X' \cap U_X = \emptyset \), then for all \( p \in X' \) we have \( \ell d_p(X') < e(X) = e(Y) + e(X') \); therefore \( e(X') \) is achieved. Let \( D \) be a closed discrete set in \( X' \) of cardinality \( e(X') \). Then \( h^{-1}(D) \) is closed and discrete in \( X \).

Next, assume that \( X \) has a closed discrete subset of cardinality \( d(Y) = \alpha \). For some (perhaps finite) cardinal \( \beta \), \( Y \) has a dense subset \( K \) of cardinality \( \beta \). We further assume that \( X \cap Y = \emptyset \) (otherwise, take disjoint copies \( X' \) and \( Y' \)). Since \( \alpha \) is infinite and \( \beta \leq \alpha \), there is a discrete collection \( \mathcal{H} \) of open sets in \( X \) of cardinality \( \beta \cdot \omega \). Let \( H \) be an Axiom of Choice set for \( \mathcal{H} \), and let \( T \) be the induced mapping from \( H \) onto \( \mathcal{H} \). Now, \( H = \bigcup_{n<\omega} H_n \), where \( H_m \cap H_n = \emptyset \) for \( m \neq n \), and \( |H_n| = \beta \), \( n < \omega \). For each \( n \), let \( T_n \) denote a bijection from \( K \) onto \( H_n \).

There exists a sequence \( \{G'_n : n < \omega \} \) for \( X \) as in Moore's metrization theorem [3], i.e., for each \( n \), \( G'_n \) is an open covering of \( X \), \( G'_{n+1} \subseteq G'_n \), and for every \( p \in X \), \( \{G'(p, n) : n < \omega \} \) forms a local base for the topology at \( p \). For each \( p \in H \), denote by \( R_0(p) \) an element of \( G'_0 \) containing \( p \) whose closure is a subset of \( T(p) \) and, having defined \( R_{n-1}(p) \), denote by \( R_n(p) \) an element of \( G'_n \) containing \( p \) whose closure is a subset of \( R_{n-1}(p) \). For each \( n < \omega \), let \( G_n \) be the collection of all elements \( g \) of \( G'_n \) such that if \( p \in H \) and \( i \leq n \) then \( g \) does not intersect both \( R_i(p) \) and \( X \setminus R_{i-1}(p) \).

Let \( \rho \) denote a metric on \( Y \). For \( y \in Y \) and \( \delta > 0 \) let

\[
B(y, \delta) = \{z \in Y : \rho(z, y) < \delta\}.
\]

For \( i < \omega \) define \( M_i = \{E_i(q) : q \in Y \text{ and } j \geq i\} \). Let \( L_i = M_i \cup G_i \), and let \( Z = X \cup Y \). Then \( L_0 \) is a basis for a \( T_1 \)-topology \( \Omega \) on \( Z \) and \( \{L_n : n < \omega \} \) satisfies the conditions of Moore's theorem, so that \( (Z, \Omega) \) is metrizable. Clearly, the inclusion maps \( \Phi_X : X \to Z \) and \( \Phi_Y : Y \to Z \) are homeomorphisms, \( X \) is a dense open set in \( Z \), and \( Z \setminus X = Y \).

Now, let \( Z' \) be a completely metrizable space containing \( Z \). Since \( X \) and \( Z \setminus X \) are completely metrizable, they are \( G_\delta \)-sets in \( Z' \). The union of two \( G_\delta \)-sets is a \( G_\delta \)-set, so \( Z \) is a \( G_\delta \)-set in the complete space \( Z' \) and is itself complete.

**Corollary 1.** If \( X \) and \( Y \) are completely metrizable, \( X \) is not compact, and \( Y \) is separable, then \( Y \) is a completion remainder of \( X \).

**Remark.** It is clear that if \( X \) and \( Y \) are metrizable and there is a dense embedding \( h \) of \( X \) into a completely metrizable space \( Z \) such that \( Z \setminus h(X) \simeq Y \) and such that \( h(X) \) is open in \( Z \), then \( X \) and \( Y \) must be completely metrizable. It is also clear that the metrizable space \( X \) is an absolute \( F_\sigma \) if and only if every completion remainder of \( X \) is complete. Thus we have

**Corollary 2.** The completion remainders of \( \mathbb{R} \), or, indeed, of any separable, locally compact, noncompact, metrizable space, are the nonempty Polish spaces.
Corollary 3. Suppose $X$ and $Y$ are completely metrizable and $X$ is not compact. Let $e(X)$ be achieved, i.e., let $X$ have a closed, discrete set of cardinality $e(X)$. Then the following are equivalent.

(A) $d(Y) \leq e(X)$.

(B) $Y$ is a completion remainder of $X$.

(C) $Y$ is a closed completion remainder of $X$.

Theorem 2. Suppose $X$ and $Y$ are completely metrizable and $X$ is not compact. Suppose $e(X)$ is not achieved. Then $Y$ is a completion remainder of $X$ if and only if $d(Y) \leq e(X)$ and $ld_y(Y) < e(X)$ for every $y \in Y$.

Proof. Let $\{\alpha_n : n < \omega\}$ be an increasing sequence of cardinals whose supremum is $e(X)$. As before, we can assume that $X \cap Y = \emptyset$. Suppose first that $Z$ is completely metrizable, $Z = X \cup Y$, and $X$ is dense in $Z$. Suppose there is a point $y \in Y$ such that $ld_y(Y) = e(X)$. For each open set $U$ in $Z$ containing $y$, $d(U \cap X) = e(X)$. Take a sequence $\{U_n : n < \omega\}$ of open sets in $Z$, $U_{n+1} \subseteq U_n$, $y \in U_n$ for every $n < \omega$, and diam$(U_n) < 2^{-n}$. For each $n < \omega$, there is a closed discrete set $D_n$ in $Z \cap U_n$, $|D_n| = \alpha_n$. Then $D = \bigcup_{n < \omega} D_n$ is closed and discrete in $X$ and $|D| = e(X)$, a contradiction. This completes the necessity proof.

Next, suppose $d(Y) \leq d(X)$ and $ld_y(Y) < e(X)$ for all $y \in Y$. As before, we assume that $X \cap Y = \emptyset$. We first consider the special case in which $Y = \bigcup_{n < \omega} Y_n$ is a countable discrete union of closed subsets, each of density less than $e(X)$. We show in this case that $Y$ is a completion remainder of $X$ in such a way that in the completion $Z = X \cup Y$, the sequence $\{Y_n : n < \omega\}$ converges to a point $p$ of $X$.

Let $p$ be a point of $X$ such that $ld_p(X) = d(X)$. Let $U_0$ be an open set in $X$ containing $p$ with diam $U < 2^{-0}$. There is a closed discrete set $D_0$ in $X$, $D_0 \subseteq U_0$, $p \notin D_0$, $|D_0| = \alpha_0$. There exists a discrete collection $\mathcal{G}_0$ of open sets screening $D_0 \cup \{p\}$ such that $\bigcup\{\overline{G} : G \in \mathcal{G}_0\} \subseteq U_0$. Let $G_{0,p}$ be the element of $\mathcal{G}_0$ containing $p$. Having chosen $U_{n-1}, D_{n-1}, \mathcal{G}_{n-1}$, and $G_{n-1,p}$, take $U_n$ to be an open set in $X$, $p \in U_n$, diam $U_n < 2^{-n}$, $\overline{U_n} \subseteq G_{n-1,p}$; take $D_n$ to be a closed discrete set in $X$, $D_n \subseteq U_n$, $p \notin D_n$, $|D_n| = \alpha_n$; take $\mathcal{G}_n$ to be a discrete collection of open sets screening $D_n \cup \{p\}$ such that $\bigcup\{\overline{G} : G \in \mathcal{G}_n\} \subseteq U_n$; and let $G_{n,p}$ denote the element of $\mathcal{G}_n$ containing $p$.

As in the sufficiency proof of Theorem 1, there is a topology $T_n$ on $(U_n \setminus \overline{U_{n+1}}) \cup Y_n$ such that $Y_n$ is closed and nowhere dense in $(U_n \setminus \overline{U_{n+1}}) \cup Y_n$, and where $\mathcal{G}_n$ plays the role of the discrete collection $\mathcal{H}$. This latter condition implies that there is a base for $T_n$ that is $\sigma$-discrete in $X$. For each $n < \omega$, let $U_n^* = U_n \cup \{Y_n : m \geq n\}$. Then $T = (\bigcup_{n < \omega} T_n) \cup \{U_n^* : n < \omega\} \cup \{U : U$ open in $X$, $p \notin U\}$ is a basis for a completely metrizable topology on $X \cup Y$, $X \setminus \{p\}$ is dense and open in $X \cup Y$, and $\{Y_n : n < \omega\}$ converges in $X \cup Y$ to $\{p\}$.

We now proceed to the general case. There is a minimal, locally finite open cover $\mathcal{U}$ of $Y$ such that if $U \in \mathcal{U}$ then $d(U) < e(X)$. Now, $|\mathcal{U}| \leq d(Y) \leq e(X)$, so $\mathcal{U}$ is a countable union, $\mathcal{U} = \bigcup_{n < \omega} \mathcal{U}_n$, where $|\mathcal{U}_n| \leq \alpha_n$. Let $\mathcal{U}_{n,m} = \{U \in \mathcal{U}_n : d(U) \leq \alpha_m\}$, and let $U_{n,m} = \bigcup \mathcal{U}_{n,m}$. Note that $d(U_{n,m}) \leq \alpha_n \cdot \alpha_m < e(X)$. By Engelking [1, Lemma 5.2.4], the countable open cover
this cover can be shrunk, \( \{ \text{Cl}(V_n) : n < \omega \} \) can be taken to be a star-finite cover of \( Y \). For \( n < \omega \), let \( Y_n = \overline{V_n} \), let \( \{ Y'_n : n < \omega \} \) be a sequence of disjoint spaces, \( Y'_n \cong Y_n \), and let \( Y^* \) be the free union of the \( Y'_n \)'s. Then there is a completely metrizable topology on \( X \cup Y^* \) as in the special case, with \( \{ Y'_n : n < \omega \} \) converging to a point \( p \in X \). Let \( f : X \cup Y^* \to X \cup Y \) be the obvious quotient map. Note that \( f^{-1}(q) \) is finite for all \( q \in X \cup Y \).

We claim that \( f \) is a closed and therefore perfect mapping. For, let \( H \subseteq X \cup Y^* \) be closed. If \( p \in H \) then \( H \cap Y'_n = \emptyset \) for all sufficiently large \( n \). Since \( f^{-1}(f(H)) = H \cup \bigcup_{n,m<\omega} f^{-1}(f(H \cap Y'_n \cap Y_m)) \), it follows that \( f(H) \) is closed in this case. But it is also clearly closed if \( p \in H \). Thus \( X \cup Y \) is completely metrizable, since it is a perfect image of a completely metrizable space. Clearly, \( X \) is dense in \( X \cup Y \).

3. Completion remainders of \( Q \) and of \( P \)

**Theorem 3.** The completion remainders of \( Q \) are the nowhere locally compact Polish spaces.

**Proof.** Assume \( Y \) is a nowhere locally compact Polish space, regarded as a subset of the Hilbert cube. Let \( K = \overline{Y} \). Then \( Y \) is a dense \( G_\delta \)-set in the compact metric space \( K \), and, since \( Y \) is nowhere locally compact, \( K \setminus Y \) is dense in \( K \). Let \( G = \{ G_n : n < \omega \} \) be a countable basis for \( K \). For \( n < \omega \), let \( U_n \) be open in \( K \), with \( Y = \bigcap_{n<\omega} U_n \), \( U_n \supseteq U_{n+1} \). Choose \( a_n \in (U_n \setminus Y) \cap G_n \). Let \( A = \{ a_n : n < \omega \} \). Then \( A \), being a countable metric space with no isolated points, is homeomorphic to \( Q \). Let \( Z = A \cup Y \). Then \( A \) is dense in \( Z \). It remains to be shown that \( Z \) is completely metrizable. We show that \( Z \) is a \( G_\delta \)-set in \( K \). Let \( V_n = U_n \cup \{ a_i : i < n \} \). Each \( V_n \), as the union of two \( G_\delta \)-sets, is a \( G_\delta \)-set, so \( V_n \) is one and \( Z = \bigcap_{n<\omega} V_n \) is thus a \( G_\delta \)-set.

Next, assume that \( Y \) is a remainder of \( Q \). Let \( Z = A \cup Y \), where \( Z \) is completely metrizable, \( A \cong Q \), \( A \cap Y = \emptyset \), \( A \) is dense in \( Z \). It is immediate that \( Y \) is a Polish space. Suppose \( Y \) is locally compact at some point \( p \in Y \). Let \( U_Y \) be an open set in \( Y \) containing \( p \), \( J = \text{Cl}_Y(U_Y) \) is compact. Then \( J \) is closed in \( Z \). There is an open set \( U \) in \( Z \) such that \( U \cap Y = U_Y \). Then \( U \setminus J \) is open in \( Z \) and therefore topologically complete. But \( U \setminus J \) is a subset of \( A \) with no isolated point, so \( U \setminus J \cong Q \), a contradiction.

For the sake of completeness, we include the following. The proofs are immediate.

**Theorem 4.** The topological completions of \( Q \) and those of \( P \) are the Polish spaces with no isolated points.

Since \( Q \) is a completion remainder of \( P \), the irrationals, one might wonder whether every \( \sigma \)-compact metric space is a completion remainder of \( P \). That this is not the case is shown in

**Theorem 5.** If the metrizable space \( S \) contains a nondegenerate continuum then \( Q \times S \) is not a completion remainder of \( P \).

**Proof.** We may assume that \( S \) is separable. Let \( I \) be a nondegenerate continuum in \( S \). Suppose \( \Theta : Q \times S \to Z \) is an embedding, where \( Z \) is a Polish space. We will prove the theorem by showing that \( Z \setminus \Theta(Q \times S) \) contains a nondegenerate connected set and hence is not homeomorphic to \( P \). \( Z \) is a
dense $G_δ$-set in some compact metric space $K$. Let $K \setminus Z = \bigcup_{n<\omega} K_n$, where each $K_n$ is compact. For each $t \in \mathbb{R}$, let

$$L_t = \bigcap_{n<\omega} \text{Cl}_K(\Theta((t-2^{-n}, t+2^{-n}) \cap Q) \times I).$$

Note that if $t \in \mathbb{Q}$ then $L_t = \Theta\{t\} \times I$, and if $t \notin \mathbb{Q}$ then $L_t \cap \Theta(Q \times S) = \emptyset$.

Let $W_n = \{t \in \mathbb{R} : L_t \cap K_n \neq \emptyset\}$. Suppose some $W_k$ is dense in some open interval $(a, b)$ in $\mathbb{R}$. Let $q \in (a, b) \cap \mathbb{Q}$, and let $t_n \in W_k$ with $t_n \to q$. Let $x_n \in L_{t_n} \cap K_k$. There is a limit point $x$ of $\{x_n : n \in \omega\}$ in $K_k$. Clearly $x \in L_q = \Theta\{q\} \times I$ is in $Z$, a contradiction. Thus each $W_k$ is nowhere dense.

Now pick distance points $s$ and $p$ of $I$, pick $q \in Q$, and let

$$\delta = \rho_K(\Theta(q, p), \Theta(q, s)),$$

where $\rho_K$ is a metric on $K$. Note that $M = \{r \in \mathbb{Q} : \rho_K(\Theta(r, p), \Theta(r, s)) > \delta/2\}$ is a nonempty open subset of $\mathbb{Q}$. Let $t \in M \setminus (\mathbb{Q} \cup \bigcup_{n<\omega} W_n)$. Then $L_t \cap K_n = \emptyset$ for all $n < \omega$, i.e., $L_t \subset Z$. Since $t \notin \mathbb{Q}$, $L_t \subset Z \setminus \Theta(Q \times S)$. Choose $q_n \in M$, $q_n \to t$. By passing to a subsequence if necessary, we may assume that $\{\Theta\{q_n\} \times I : n \in \omega\}$ converges in the Vietoris topology on $2^K$ to some set $J$, which is nondegenerate, connected, and contained in $L_t$. Thus $Z \setminus \Theta(Q \times S)$ contains a nondegenerate connected set and so is not homeomorphic to $P$.

**Corollary 4.** $Q \times \mathbb{R}$ is not a completion remainder of $P$.

**Remark.** The following generalization has essentially the same proof as that of Theorem 5.

**Theorem 6.** If $X$ contains a closed subset $Y$ such that there exists an open and closed mapping $f : Y \to Q$ such that each $f^{-1}(q)$ is a nondegenerate continuum, then $X$ is not a completion remainder of $P$.

**Theorem 7.** Every $\sigma$-compact, 0-dimensional metrizable space is a completion remainder of $P$.

**Proof.** Suppose $Y$ is as in the hypothesis. There is an embedding $\phi : Y \to P$. $P \setminus \phi(Y)$ is a $G_δ$-set in $P$; it is separable, 0-dimensional, metrizable, and nowhere locally compact, so it is homeomorphic to $P$.

4. **Completion remainders of $Q(\kappa)$ and $P(\kappa)$**

Throughout this section $\kappa$ denotes an infinite cardinal. Let $Q(\kappa)$ denote a $\sigma$-discrete metric space in which every open set has cardinality $\kappa$. Medvedev [2] has shown that all such spaces are homeomorphic. Let $P(\kappa)$ denote a complete metric space with covering dimension 0 that has density $\kappa$ and local density $\kappa$ at each point and that is nowhere locally $\kappa$-compact. A straightforward argument shows that all such spaces are homeomorphic; in particular, $P(\kappa)$ is homeomorphic to the Baire space $B(\kappa)$, the countable Cartesian product of discrete spaces of cardinality $\kappa$. It follows that $P(\kappa)$ is a completion remainder of $Q(\kappa)$. The 0-dimensionality, however, is not necessary. We have

**Theorem 8.** The completion remainders of $Q(\kappa)$ are the completely metrizable spaces that have density $\kappa$ and local density $\kappa$ at every point but that are nowhere locally $\kappa$-compact.
Before proving Theorem 8 we present a lemma that extends the old result of Niemytzki and Tychonoff [4] that a metrizable space is compact if and only if every compatible metric on the space is complete.

**Lemma 2.** Let $X$ be a metrizable space. Then the following are equivalent.

(A) $X$ is nowhere locally compact.

(B) $X$ can be embedded in a metrizable space $Z$ in such a way that both $X$ and $Z \setminus X$ are dense in $Z$.

(C) $X$ admits a compatible metric that is nowhere locally complete.

**Proof.**

(B) $\Rightarrow$ (C) Let $X$ and $Z$ be as in (B); $Z$ can be densely embedded in a complete metric space $(W, \rho)$. Then $X$ is dense in $W$, and if $\rho_X$ denotes the restriction of $\rho$ to $X \times X$ then $X$ is nowhere locally complete according to $\rho_X$.

(C) $\Rightarrow$ (A) The proof follows immediately from the Niemytzki-Tychonoff Theorem.

(A) $\Rightarrow$ (B) Suppose $X$ is nowhere locally compact. Let $\rho$ be a metric on $X$. We make repeated use of the following observation.

(*) For every nonempty open set $U$ in $X$, there is a sequence $\{U_n : n < \omega\}$ of nonempty open sets in $X$ such that $\bigcup_0 \subset U$, $\bigcup_{n+1} \subset U_n$ for all $n < \omega$, and $\bigcap_{n<\omega} U_n = \varnothing$.

There exists a locally finite open cover $G_0$ of $X$ such that if $g \in G_0$ then $\rho$-diam $g < 2^{-n}$ and $g$ contains a point not in $h$ for any $h \in G_0 \setminus \{g\}$.

For each $g \in G_0$ let $U_g$ be a nonempty open set such that $\overline{U_g} \subset g$ and $\overline{U_g} \cap \overline{h} = \emptyset$ for every $h \in G_0 \setminus \{g\}$. Let $\{U_n(g) : n < \omega\}$ be a sequence as in $(*)$, with $\overline{U_0(g)} \subset U_g$.

Take $G'_0 = G_0 \cup \{U_n(g) : g \in G_0, n < \omega\}$. If $x \in X$ there is an open set $v_0(x)$ containing $x$ that intersects only finitely many elements of $G'_0$. Let $V_0 = \{v_0(x) : x \in X\}$.

There exists, for each $n, 0 < n < \omega$, collections $G_n, G'_n, V_n, \{U_g : g \in G_n\}, \{U_m(g) : g \in G, m < \omega\}$ such that

(1) $G_n$ is a locally finite open cover of $X$ and $\rho$-diam $g < 2^{-n}$ for all $g \in G_n$.

(2) $G_n$ refines both $G'_{n-1}$ and $V_{n-1}$.

(3) If $g \in G_n$ then $U_g$ is a nonempty open set such that

(i) $\overline{U_g} \subset g$ and $\overline{U_g} \cap \overline{h} = \emptyset$ for every $h \in G_n \setminus \{g\}$;

(ii) if $h \in G_0 \cup \cdots \cup G_{n-1}$ and $U_g \cap U_m(h) \neq \emptyset$, $m < \omega$, then $\overline{U_g} \subset U_m(h)$; and

(iii) if $h \in G_0 \cup \cdots \cup G_{n-1}$ then there is an $m < \omega$ such that $\overline{U_g} \cap \overline{U_m(h)} = \emptyset$.

(4) If $g \in G_n$ then $\{U_m(g) : m < \omega\}$ is as in $(*)$ with $\overline{U_0(g)} \subset U_g$.

(5) $G'_n = G'_{n-1} \cup G_n \cup \{U_m(g) : g \in G_n, m < \omega\}$.

(6) $V_n$ is an open cover of $X$ no element of which intersects infinitely many elements of $G'_n$.

Let $A = \bigcup_{n<\omega} A_n$, where $A_n \cap A_m = \emptyset$ for $n \neq m$, $A \cap X = \emptyset$, and, for each $n, |A_n| = |G_n|$. Let $\phi_n : A_n \to G_n$ be a bijection. Let $Z = A \cup X$.

For $V$ open in $X$ let $A_V = \{a \in A :$ for some $m, n < \omega, a \in A_n, \text{ and } \overline{U_m(\phi_n(a))} \subset V\}$, and let $E(V) = V \cup A_V$. We observe that $\{E(V) : V \text{ open in } X\}$ is a cover of $Z$ and that if $V$ and $W$ are open in $X$ then
\[ E(V \cap W) = E(V) \cap E(W). \] Therefore, \( \{E(V) : V \text{ open in } X\} \) is a basis for a topology \( \Omega \) on \( Z \). We also observe that \( E(V) \subset E(W) \) whenever \( V \subset W \) and that \( \text{Cl}_Z(E(V)) = \text{Cl}_Z(V) \) for all open \( V \) in \( X \). It is easily seen that \( \Omega \) is a Hausdorff topology on \( Z \). We list three more observations that are useful in showing \( \Omega \) is regular.

1. \( \mathcal{Z} = \{U_m(a) : m < \omega, a \in A\} \) is non-Archimedean in the sense that if two members of \( \mathcal{Z} \) intersect then one is a subset of the other.

2. If \( p \in A_n \) and \( q \in A \cap \text{Cl}_Z(U_m(\phi_n(p))) \) then \( q \in E(U_m(\phi_n(p))) \).

3. If \( p \in A_n \) then \( \text{Cl}_Z(E(U_{m+1}(\phi_n(p)))) \subset E(U_m(\phi(p))) \).

Next, assume \( p \in A \) and \( E(U) \) is a basic open set containing \( p \). There is an \( n \) such that \( p \in A_n \). There is an \( m \) such that \( \text{Cl}_X(U_m(\phi_n(p))) \subset U \). Let \( V = E(U_{m+1}(\phi_n(p))) \). Then by (3) above, \( \text{Cl}_Z(V) \subset E(U_m(\phi_n(p))) \subset E(U) \).

Therefore, \( \Omega \) is regular at points of \( A \).

Next, assume \( p \in X \) and \( E(U) \) is a basic open set containing \( p \). There is an \( n \) such that if \( p \in g \in G_n \), \( h \in G_n \), and \( g \cap h \neq \emptyset \), then \( h \subset U \). Choose an element \( g \) of \( G_n \) containing \( p \). For each \( i \leq n \) there is an \( m_i < \omega \) such that \( p \not\in \text{Cl}_X(U_m(\phi_i(a))) \) for any \( a \in A_i \). There is an open set \( V \) in \( X \), with \( p \in V \subset g \), and \( \text{Cl}_X(V) \cap \text{Cl}_X(U_m(\phi_i(a))) = \emptyset \) for all \( a \in A_0 \cup \cdots \cup A_n \). So, if \( a \in A_0 \cup \cdots \cup A_n \), then \( a \not\in \text{Cl}_Z(V) \). Suppose \( a \in A_k \), \( k > n \), and \( a \in \text{Cl}_Z(E(V)) = \text{Cl}_Z(V) \). Then \( V \cap E(U_0(a)) \neq \emptyset \) and \( U_0(\phi_k(a)) \) is a subset of some \( h \) in \( G_n \), and \( h \cap g \neq \emptyset \); so \( h \subset U \), which implies \( a \in E(U) \). Therefore, \( \Omega \) is regular at points of \( X \).

Next, we exhibit a \( \sigma \)-locally finite basis for \( \Omega \). Let \( \Sigma_0 = \{E(g) : g \in G_0\} \). Then \( \Sigma_0 \) is locally finite. For \( 1 \leq n < \omega \) and \( k < \omega \), let \( \Sigma(n, k) = \{E(g) : g \in G_n, U_k(h) \cap g = \emptyset \text{ for all } h \in G_0 \cup \cdots \cup G_{n-1}\} \). Then \( \Sigma(n, k) \) is locally finite. Moreover, if \( \Sigma_n = \{E(g) : g \in G_n\} \), then \( \Sigma_n = \bigcup_{k<\omega} \Sigma(n, k) \).

Similarly, it follows that for all \( m, n < \omega \), \( \Delta_m,n = \{E(U_m(g)) : g \in G_n\} \) is \( \sigma \)-locally finite.

Then \( (\bigcup_{n<\omega} \Sigma_n) \cup (\bigcup_{m,n<\omega} \Delta_m,n) \) is a \( \sigma \)-locally finite basis for \( \Omega \), so that \((Z, \Omega)\) is metrizable by the Nagata-Smirnov theorem.

Clearly, \( A \) is dense in \( Z \) and so is \( X \). This completes the proof of Lemma 2.

We now return to the proof of Theorem 8. Note that Theorem 3 is Theorem 8 in the special case \( \kappa = \omega \). From now on we assume \( \kappa > \omega \).

Assume \( Y \) is a completely metrizable space with density \( \kappa \) and local density \( \kappa \) at each point. It follows directly from Lemma 1 that \( Y \) is nowhere locally \( \kappa \)-compact and therefore nowhere locally compact. We apply Lemma 2 to get a metrizable space \( Z \) such that both \( Y \) and \( Z \setminus Y \) are dense in \( Z \). We may assume that \( Z \) is completely metrizable, since it can be densely embedded in a completely metrizable space \( Z' \), and that both \( Y \) and \( Z \setminus Y \) are dense in \( Z' \).

Let \( G = \bigcup_{n<\omega} G_n \) be a \( \sigma \)-discrete basis for \( Z \), where \( |G_n| = \kappa \) and \( G_n \) is discrete, \( n < \omega \).

Since \( Y \) is completely metrizable, it is a \( G_\delta \)-set in \( Z \); let \( \{V_n : n < \omega\} \) be a sequence of open sets in \( Z \), with \( V_n \supset V_{n+1}, \bigcap_{n<\omega} V_n = Y \).

For each \( n < \omega \), let \( A_n \) be an Axiom of Choice set for \( \{(g \cap V_n) \setminus Y : g \in G_n\} \). Then \( A_n \) is closed and discrete and \( A = \bigcup_{n<\omega} A_n \) is \( \sigma \)-discrete and has density \( \kappa \) and local density \( \kappa \) at every point. It follows that \( A \simeq \mathbb{Q}(\kappa) \). Moreover, \( A \) is dense in \( Z \). For \( n = 0 \), let \( W_0 = V_0 \), and for \( n > 0 \), let
\[ W_n = A_0 \cup \cdots \cup A_{n-1} \cup V_n. \] Since each \( A_i \) is closed and \( V_n \) is open, \( W_n \) is a \( \mathcal{G}_\delta \)-set in \( Z \), so \( A \cup Y = \bigcap_{n<\omega} W_n \) is a \( \mathcal{G}_\delta \)-set in \( Z \) and therefore completely metrizable.

Next assume that \( Y \) is a completion remainder of \( \mathbb{Q}(\kappa) \). Then there exist \( A \) and \( Z \), \( A \cong \mathbb{Q}(\kappa) \), \( Z \) completely metrizable, \( Z = A \cup Y \), and \( A \cap Y = \varnothing \). Since \( A \) is an absolute \( F_{\sigma} \), \( Y \) is a \( \mathcal{G}_\delta \)-set in \( Z \) and thus completely metrizable. Since \( A \) is dense in \( Z \), we have \( d(Y) \leq d(Z) \leq d(A) = \kappa \). Since \( Y \) is dense in \( Z \), we have \( \kappa = d(A) \leq d(Z) \leq d(Y) \). Therefore, \( d(Y) = \kappa \). Similarly, \( ld_p(Y) = \kappa \) for each \( p \in Y \). It follows from Lemma 1 that \( Y \) is nowhere locally \( \kappa \)-compact.

**Theorem 9.** If \( Y \) is metrizable, \( \dim Y = 0 \), and \( Y \) is the union of countably many sets, each the union of a discrete collection of compact sets, then \( Y \) is a completion remainder of \( \mathbb{P}(\kappa) \).

**Proof.** The proof is very similar to that of Theorem 6. Firstly, we know that there is an embedding \( \Phi: Y \rightarrow \mathbb{P}(\kappa) \). Secondly, \( \mathbb{P}(\kappa) \setminus \Phi(Y) \) is a \( \mathcal{G}_\delta \)-set in \( \mathbb{P}(\kappa) \); it has covering dimension 0 and density \( \kappa \) and local density \( \kappa \) at every point and is nowhere locally \( \kappa \)-compact, so it is homeomorphic to \( \mathbb{P}(\kappa) \).

5. Open questions

**Question 1.** What are the completion remainders of \( \mathbb{P}(\kappa) \)? We do not have a characterization even in case \( \kappa = \omega \).

**Question 2.** In the class of Moore spaces, what are the completion remainders of \( \mathbb{Q} \) or of \( \mathbb{Q}(\kappa) \)?

Acknowledgments

We are indebted to I. Reclaw, S. Baldwin, and the referee for helpful suggestions.

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