MAPPING SPACES OF COMPACT LIE GROUPS
AND \( p \)-ADIC COMPLETION

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Abstract. If \( BG, BH \) are the classifying spaces of compact Lie groups, with \( H \) connected, then the mapping space functor \( \text{map}(BG, -) \) commutes with \( p \)-completion on \( BH \): i.e., for each \( f: BG \to BH \) the component \( (\text{map}(BG, BH)_f)_p \) is \( p \)-complete, and is homotopy equivalent to \( \text{map}(BG, BH)_p \).

1. Introduction

In studying \( \text{map}(BG, BH) \), the space of maps between the classifying spaces of two compact Lie groups, it is often useful to know whether the \( p \)-adic completion commutes with the functor \( \text{map}(BG, -) \); special cases where this occurs were used, for example, in [DZ, JMO, N2, NS]. Here we present a more general result in this direction:

1.1. Theorem. Let \( G \) and \( H \) be compact Lie groups, with \( H \) connected, let \( p \) be a prime, and \( i: BH \to BH^\wedge \) the natural inclusion. Then for any map \( f: BG \to BH \), the corresponding component of the mapping space, \( \text{map}(BG, BH^\wedge)_i \), is \( p \)-complete, and

\[
(\text{map}(BG, BH)_f)^\wedge_p \cong \text{map}(BG, BH^\wedge)_i \]

is a homotopy equivalence.

The \( p \)-adic completion of a space \( X \) that we refer to is the \( (F_p)_\infty X \) of [BK, I, §4.2], which we denote by \( X^\wedge \). However, unless \( X \) is nilpotent (e.g., simply-connected), \( X^\wedge_p \) need not be \( p \)-complete in the sense of [BK, I, §5 & VII, §2], and so it enjoys few of the properties associated with completion. In particular, unless \( X^\wedge_p \) is \( p \)-complete, the natural map \( i: X \to X^\wedge_p \) will not induce an isomorphism in \( F_p \)-homology, so \( X^\wedge_p \) will not be the \( H_*(-; F_p) \)-localization of \( X \) (cf. [BK, §2.1]) and \( (X^\wedge_p)^\wedge_p \neq X^\wedge_p \).

In §2 we list some facts about \( \mathbb{Z}^\wedge \)-modules needed to prove the theorem. In §3...
In §4 the Bousfield-Kan spectral sequence is used to prove \( p \)-completeness. The required homotopy equivalence is shown in §5.

2. Finitely generated \( \mathbb{Z}_p^\wedge \)-modules

Let \( \mathcal{F} \) denote the class of finitely generated \( \mathbb{Z}_p^\wedge \)-modules, where \( \mathbb{Z}_p^\wedge \) is the ring of \( p \)-adic integers, and let \( \mathcal{F}' = \mathcal{F} \cup \{ G : G \text{ is a finite } p \text{-group} \} \).

2.1. **Lemma.** If \( X \) is a connected space with \( \pi_k X \in \mathcal{F}' \) for each \( k \geq 1 \), then

1. \( H_*(X; \mathbb{F}_p) \) is of finite type, that is, \( H_k(X; \mathbb{F}_p) \) is finite for each \( k \geq 0 \);
2. \( X \) is \( p \)-complete and \( \mathbb{F}_q \)-acyclic for any prime \( q \neq p \), that is, \( \tilde{H}_*(X; \mathbb{F}_q) = 0 \).

**Proof.** Any \( M \in \mathcal{F} \) is isomorphic to \( N \otimes \mathbb{Z}_p^\wedge \), where \( N \) is a finitely generated abelian group. Thus \( K(M, n) \simeq K(N, n)_p^\wedge \), which is \( p \)-complete (see [BK, VI, 5.2]), and so \( H_*(K(M, n); \mathbb{F}_p) \) is of finite type for all \( n \geq 1 \). Therefore, if \( Y \) is a simply-connected space with each \( \pi_* Y \in \mathcal{F} \), by induction on its Postnikov system, we see \( H_*(Y; \mathbb{F}_p) \) is of finite type.

Now assume \( \pi_1 X = G \in \mathcal{F}' \) and consider the universal covering fibration for \( X \);

\[
(\ast) \quad \tilde{X} \to X \to K(G, 1) .
\]

The action of \( G \) on the universal covering space \( \tilde{X} \) makes \( H_*(\tilde{X}; \mathbb{F}_p) \) into a \( G \)-module, and one has a Leray-Cartan spectral sequence (cf. [CE, XVI, §9]), with

\[
E_2^{s, t} \cong H_s(G; H_t(X; \mathbb{F}_p)) \Rightarrow H_{t+s}(X; \mathbb{F}_p) .
\]

Now for fixed \( t \), let \( V = H_t(X; \mathbb{F}_p) \) and let \( \phi: G \to \text{Aut}(V) \) describe the \( \pi_1 \)-action. \( \text{Aut}(V) \) is finite, and if \( G \in \mathcal{F} \) then \( G \) is \( q \)-divisible for any \( q \) prime to \( p \), so in any case \( \text{Im}(\phi) \subseteq \text{Aut}(V) \) is a finite \( p \)-group. Thus \( G \) acts nilpotently on \( V \) (cf. [BK, II, 5.2]): that is, there is a filtration \( 0 = V_0 \subset V_1 \subset \cdots \subset V_i \cdots \subset V_n = V \) of \( G \)-modules such that \( G \) acts trivially on each \( V_i/V_{i-1} \).

Using the short exact sequences \( 0 \to V_{i-1} \to V_i \to V_i/V_{i-1} \to 0 \), we see by induction on \( i \) that each \( H_*(G; V_i) \)—and so in particular \( H_*(G; V) \cong E_{s,t}^2 \)—is finite. Thus \( H_*(X; \mathbb{F}_p) \) is of finite type.

Furthermore, because \( G \) acts nilpotently on \( V \), by the mod-\( \mathbb{F}_p \) fiber lemma of [BK,II, 5.1] the universal covering \((\ast)\) remains a fibration after \( p \)-completion:

\[
(\ast)_p^\wedge \quad \tilde{X}^\wedge \to X^\wedge \to K(G, 1)^\wedge_p .
\]

Since \( G = \pi_1 X \in \mathcal{F}' \), \( K(G, 1) \) is \( p \)-complete; similarly \( \tilde{X} \simeq \tilde{X}^\wedge_p \) (being nilpotent, with \( \pi_k \tilde{X} \in \mathcal{F} \)). The Five Lemma, applied to the natural map from the long exact sequence of \((\ast)\) to that of \((\ast)_p^\wedge \), shows \( \pi_* X \to \pi_*(X^\wedge_p) \) is an isomorphism, so \( X \) is \( p \)-complete. Since \( \tilde{H}_* \tilde{X} = 0 = \tilde{H}_* K(G, 1) \) for \( q \neq p \) by [BK, VI, 5.6], the same holds for \( X \). \( \square \)

2.2. **Corollary.** Let \( X \) be a pointed connected space such that \( \pi_k X \in \mathcal{F} \) for \( k \geq 2 \), and suppose that \( \pi_1 X \) has a finite normal series

\[
1 = G_0 < G_1 < \cdots < G_{n-1} < G_n = \pi_1 X ,
\]

where each \( G_i/G_{i-1} \in \mathcal{F}' \). Then \( X \) is \( p \)-complete and \( \mathbb{F}_q \)-acyclic for \( q \neq p \).
Proof. For each $i = 1, \ldots, n$, let $X_{i-1} \to X_i \to K(G_i/G_{i-1}, 1)$ be the covering fibration corresponding to the short exact sequence $1 \to G_{i-1} \to G_i \to G_i/G_{i-1} \to 1$ (where $X_0 = \tilde{X}$ and $X_n = X$). As above, $K(G_i/G_{i-1}, 1)$, and by induction also $X_{i-1}$, are $p$-complete and $\mathbb{F}_p$-acyclic, with $\mathbb{F}_p$-homology of finite type. The same then holds for $X_i$, too, by the covering-space argument in the proof of Lemma 2.1, and thus for $X$. □

2.3. Lemma. For $A, C \in \mathcal{F}$:

(1) If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then $B \in \mathcal{F}$.

(2) Any group homomorphism $f : C \to A$ is $\mathbb{Z}_p$-linear.

Proof. It is enough to show that the forgetful functor induces isomorphisms

(1) $\text{Ext}^1_{\mathbb{Z}_p}(C, A) \cong \text{Ext}^1_{\mathbb{Z}}(C, A)$ and $\text{Hom}_{\mathbb{Z}_p}(C, A) \cong \text{Hom}_{\mathbb{Z}}(C, A)$.

As above, write $A \cong A' \otimes \mathbb{Z}_p^\wedge$, $C \cong C' \otimes \mathbb{Z}_p^\wedge$, for finitely generated abelian groups $A'$, $C'$. Since Ext and Hom commute with finite direct sums, it is enough to consider cyclic $C$ and $A$, that is, each either $\mathbb{Z}_p^\wedge$ or $\mathbb{Z}/p^r$ for some $r$.

By the Change of Rings Theorem (see [HS, IV, Theorem 12.2]) we know

$$\text{Ext}^n_{\mathbb{Z}_p}(C, A) \cong \text{Ext}^n_{\mathbb{Z}}(C' \otimes \mathbb{Z}_p^\wedge, A) \cong \text{Ext}^n_{\mathbb{Z}}(C', A) \quad (n \geq 0),$$

so (1) is satisfied when $C$ is torsion and thus $C = C'$.

Now let $C = \mathbb{Z}_p^\wedge$.

(1) If $A = \mathbb{Z}_p^\wedge$ then $\text{Ext}^1_{\mathbb{Z}}(C, A) = 0$ by [Ha, Proposition 2.1].

(2) If $A = \mathbb{Z}/p^r$, tensor $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ with $\mathbb{Z}_p^\wedge$ to get the exact sequence $0 \to \text{Tor}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}_p^\wedge) \to \mathbb{Z}_p^\wedge \to \mathbb{Q} \otimes \mathbb{Z}_p^\wedge \to (\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}_p^\wedge \to 0$. Applying $\text{Ext}^1_{\mathbb{Z}}(-, A)$ to this, we see that $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, A) = 0$ since $\mathbb{Z}_p^\wedge \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector space and $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/p^r) = 0$.

We clearly also have $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \mathbb{Z}/p^r) = 0$ for any $A$.

Finally, $\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^\wedge, A) \cong A$ for any $A \in \mathcal{F}$ while $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \mathbb{Z}/p^r) \cong \mathbb{Z}/p^r$, so

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \mathbb{Z}_p^\wedge) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \lim \mathbb{Z}/p^r) \cong \lim \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, \mathbb{Z}/p^r) \cong \lim \mathbb{Z}/p^r \cong \mathbb{Z}_p^\wedge.$$

Thus $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_p^\wedge, A) \cong A$ for any $A \in \mathcal{F}$, too. The required isomorphism is readily verified. □

3. The mod-p approximation of $BG$

In order to prove Theorem 1.1 for more general $G$, we start with the known case when $G$ is $p$-toral, i.e., $\pi_0 G$ is a finite $p$-group and the identity component of $G$ is a torus. Then we have
3.1. Lemma. If \( P \) is a \( p \)-toral group and \( H \) is a connected compact Lie group, then for any \( f : BP \to BH \), \( (\text{map}(BP, BH)_f)^{\wedge}_p \to \text{map}(BP, BH^{\wedge}_p)_i \) is a homotopy equivalence.

This is contained in [JMO, Theorem 3.2]; we give an outline of the proof:

By [N1, Theorem 1.1], \( f \simeq B\rho \) for some homomorphism \( \rho : P \to H \); let \( C(\rho) \) denote its centralizer. The homomorphism \( C(\rho) \times P \to H \) passes to classifying spaces and has an adjoint \( BC(\rho) \to \text{map}(BP, BH)_{BP} \), or if we first complete,

\[
BC(\rho)^{\wedge}_p \to \text{map}(BP, BH^{\wedge}_p)_{i\circ BP}.
\]

The first map induces an \( H_*(-; \mathbb{F}_p) \)-isomorphism by [N1], and so a homotopy equivalence after completion (see [BK, I, 5.5]), while the second is shown in [JMO, loc. cit.] to be a homotopy equivalence. \( \Box \)

3.2. Remark. Since \( C(\rho) \) is compact and \( \pi_0 C(\rho) \) is a finite \( p \)-group (cf. [JMO, Proposition A.4]), the homotopy groups \( \pi_k(\text{map}(BP, BH^{\wedge}_p)_{i\circ BP}) \) are finitely generated \( \mathbb{Z}_p \)-modules for \( k \geq 2 \) and a finite \( p \)-group for \( k = 1 \).

We now recall some results of Jackowski, McClure, and Oliver on the mod-\( p \) approximation of \( BG \):

For any compact Lie group \( G \), let \( \mathcal{C}_p(G) \) denote the full subcategory of the orbit category \( \mathcal{O}(G) \) whose objects are homogenous spaces \( G/P \) where \( P \) is a \( p \)-toral group and whose morphisms are \( G \)-maps. In [JMO, 1.3], Jackowski, McClure, and Oliver define a full subcategory \( \mathcal{R}_p(G) \subset \mathcal{C}_p(G) \) (containing \( G/P \) only for certain "\( p \)-stubborn" \( P \)'s), which has the property that

\[
\text{holim}_{\mathcal{R}_p(G)} EG \times_G (G/P) \to BG
\]

is a \( H_*(-; \mathbb{F}_p) \)-isomorphism. Here \( \text{holim} \) denotes the homotopy direct limit of [BK, XII, §2], and \( EG \times_G (G/P) \cong EG/P \simeq BP \).

Recall from [BK, I, §4] that for any space \( X \), the \( p \)-completion is obtained as the total space (i.e., homotopy inverse limit) of a certain cosimplicial space:

\[
X^\wedge_p \overset{\text{def}}{=} \text{Tot}(\mathbb{F}_p X)^*, \text{where each space } (\mathbb{F}_p X)^k \text{ is homotopy equivalent to an } \mathbb{F}_p\text{-GEM, i.e., a product of } K(\mathbb{F}_p, n_i)'s. \text{ Therefore, for any space } Z, \text{ we have}
\]

\[
\text{map}(Z, X^\wedge_p) = \text{map}(Z, \text{Tot}(\mathbb{F}_p X)^*) \cong \text{Tot}(\text{map}(Z, (\mathbb{F}_p X)^*))
\]

(see [BK, XI, 4.4, 7.6]), so the space of maps into a \( p \)-completion is the total space of a cosimplicial \( \mathbb{F}_p\)-GEM, too.

Now if \( f : Y \to Z \) is an \( H_*(-; \mathbb{F}_p) \)-isomorphism, it induces a homotopy equivalence \( \text{map}(Z, K(\mathbb{F}_p, n)) \overset{\sim}{\to} \text{map}(Y, K(\mathbb{F}_p, n)) \), and so \( \text{map}(Z, (\mathbb{F}_p X)^k) \overset{\sim}{\to} \text{map}(Y, (\mathbb{F}_p X)^k) \) is a homotopy equivalence for each \( k \geq 0 \). Therefore, by [BK, XI, 5.6] the same is true for the Tot's, and thus \( \text{map}(Z, X^\wedge_p) \overset{\sim}{\to} \text{map}(Y, X^\wedge_p) \) is a homotopy equivalence. Since

\[
\text{map}(\lim Y_i, X) = \lim \text{map}(Y_i, X)
\]
for any diagram \( \{ Y_i \} \) (cf. [BK, XII, 4.1]), we have a natural homotopy equivalence
\[
\text{map}(BG, BH^\wedge_p) \to \lim_{\mathcal{R}_p(G)} \text{map}(EG/P, BH^\wedge_p).
\]

Thus, if we restrict a map \( f : BG \to BH \) to \( BP \hookrightarrow BG \) (for some \( G/P \) in \( \mathcal{R}_p(G) \)), we see that
\[
(2) \quad \text{map}(BG, BH^\wedge_p)_{i \circ f} \to \lim_{\mathcal{R}_p(G)} \text{map}(EG/P, BH^\wedge_p)_{i \circ f|BP}
\]
is the inclusion of a component (the homotopy inverse limit need not be connected!).

4. Cosimplicial spaces

Let \( \text{sk} \mathcal{R}_p(G) \) be a skeleton of \( \mathcal{R}_p(G) \), that is, a full subcategory of \( \mathcal{R}_p(G) \), containing a single representative of each isomorphism type of its objects. This is a finite category, since \( \mathcal{R}_p(G) \) has finitely many isomorphism types of objects, and finitely many morphisms between them (cf. [JMO, Proposition 1.6]).

Given a map \( f : BG \to BH \) as above, consider the finite diagram of spaces
\[
X = \{ X_p \}_{G/P \in \text{sk} \mathcal{R}_p(G)}, \quad \text{where } X_p = \text{map}(BP, BH^\wedge_p)_{i \circ f|BP}.
\]
By cosimplicial replacement (see [BK, XI, §5]) we obtain a cosimplicial space \( Y^* \), with
\[
Y^n = \prod_{G/P_0 \to \cdots \to G/P_n} X_{P_0} \quad \text{for } n \leq \infty,
\]
(where the product, over all possible sequences of \( n \) composable morphisms in \( \text{sk} \mathcal{R}_p(G) \), is finite), such that \( \lim_{\text{sk} \mathcal{R}_p(G)} \{ X_p \} \cong \text{Tot} Y^* \).

Now if \( Z^* \) is the cosimplicial replacement of the analogous infinite diagram of \( X_p \)'s for the full category \( \mathcal{R}_p(G) \), then the equivalence of categories \( \text{sk} \mathcal{R}_p(G) \to \mathcal{R}_p(G) \) (with noncanonical inverse \( \mathcal{R}_p(G) \to \text{sk} \mathcal{R}_p(G) \)) induces a homotopy equivalence \( \text{Tot} Y^* \cong \text{Tot} Z^* \), so that up to homotopy the natural map of (2) above is the inclusion of one component in \( \text{Tot} Y^* \):
\[
\text{map}(BG, BH^\wedge_p)_{i \circ f} \to \lim_{\mathcal{R}_p(G)} \{ X_p \} \cong \text{Tot} Y^*.
\]

We choose a basepoint \( y_0 \in \text{Tot} Y^* \) corresponding to the map \( i \circ f \).

4.1. Lemma. For any \( f : BG \to BH \), the space \( \text{map}(BG, BH^\wedge_p)_{i \circ f} \) is \( p \)-complete and \( \mathbb{F}_q \)-acyclic for \( q \neq p \).

**Proof.** Consider the Bousfield-Kan spectral sequence for \( Y^* \) as above (more precisely, for the component of \( y_0 \) in \( \text{Tot} Y^* \) (cf. [B2, §2])) with \( E_2^{s,t} \cong \pi^s \pi_t Y^* \).

For \( t \geq 2 \), the construction of \( Y^* \) and Remark 3.2 imply that \( \pi_t Y^s \in \mathscr{F} \) and all the cosimplicial morphisms of \( \pi_t Y^* \) are \( \mathbb{Z}_p^\wedge \)-linear by Lemma 2.3(b); hence \( E_2^{s,t} \in \mathscr{F} \). For \( t = 1 \), \( E_2^{0,1} \) is a subgroup of \( \pi_1 Y^0 \cong \prod \pi_1 X_p \), and so is itself a finite \( p \)-group by Remark 3.2.

Moreover, if \( t \geq 2 \), the differentials \( d_r : E_r^{s,t} \to E_r^{s+r,t+r-1} \) are homomorphisms, and thus \( \mathbb{Z}_p^\wedge \)-linear, for \( t > s \geq 0 \). Therefore, \( E_r^{s,t} \in \mathscr{F} \) for \( r \leq \infty \).
if \( t > s \geq 0 \) or \( t = s \geq r \). For \( t = 1 \) we have \( E^{0,1}_r \subseteq E^{0,1}_{r-1} \subseteq E^{0,1}_2 \) (cf. [B2, §2.4]), so \( E^{0,1}_r \) is a finite \( p \)-group.

Since \( E^{s,t}_2 \cong \lim_{\mathcal{R}_p(G)} \pi_t X \) by [BK, XI, 7.1], Lemma 4.2 below, applied to the functors

\[
\pi_t(EG \times_G -) : \mathcal{R}_p(G) \to \mathbb{Z}_p^\wedge, \]

shows that there is an \( N \) such that \( E^{s,t}_\infty = 0 \) for \( s > N \) and \( t \geq 2 \).

This in turn implies the complete convergence of the spectral sequence (see [B2, §4.5]): thus, for each \( t \geq 1 \) there is a finite tower of epimorphisms

\[
\pi_t(\Tot Y^*, y_0) = Q^n \pi_t \to Q^{n-1} \pi_t \to \cdots Q_0 \pi_t \to Q_{-1} \pi_t = 1,
\]

where \( Q_s \pi_t = \text{im}\{\pi_t(\Tot Y^*, y_0) \to \pi_t(\Tot Y^*, y_0)\} \) (cf. [BK, IX, §5.3]), and for each \( s \geq 0 \) there is a short exact sequence

\[
1 \to E^{s,s+t}_\infty \to Q_s \pi_t \to Q_{s-1} \pi_t \to 1.
\]

Now for \( t \geq 2 \) we have \( E^{s,t}_\infty \in \mathcal{F} \). Therefore, Lemma 2.3(a) implies (by induction on \( s \)) that \( Q_s \pi_t \in \mathcal{F} \) for all \( s \), and so \( \pi_t(\Tot Y^*, y_0) \) is in \( \mathcal{F} \), too.

For \( t = 1 \) we obtain a finite normal series

\[
0 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_N \triangleleft G_{N+1} = \pi_1(\Tot Y^*, y_0),
\]

where \( G_i/G_{i-1} = E^{N-i+1,N-i+2}_\infty \) is in \( \mathcal{F} \) for \( 1 \leq i \leq N \) and \( G_{N+1}/G_N = E^{0,1}_\infty \) is a finite \( p \)-group. Thus Corollary 2.2 applies, and the component of \( y_0 \) in \( \Tot Y^* \) is \( p \)-complete, and \( \mathbb{F}_q \)-acyclic for \( q \neq p \). \( \square \)

The following lemma appeared in an earlier version of [JMO].

4.2. Lemma. If \( G \) is any compact Lie group and \( p \) a prime, there is an \( N \) such that for any contravariant functor

\[
F : \mathcal{R}_p(G) \to \mathbb{Z}_p^\wedge
\]

we have \( \lim_{\mathcal{R}_p(G)}^s F = 0 \) for \( s > N \).

Proof. The homotopy direct limit \( E_{\mathcal{R}_p(G)} = \holim_{\mathcal{R}_p(G)} G/P \) is a \( G \)-space, and

\[
\holim_{\mathcal{R}_p(G)}^s F \cong H^s_G(E_{\mathcal{R}_p(G)}; F) \quad \text{for all} \quad s \geq 0 \quad \text{by [JMO, Theorem 1.7].}
\]

Here \( H^s_G(-; F) \) denotes equivariant cohomology with the functor \( F \) as coefficient system (see [I, 2.2]).

By [JMO, Proposition 1.2, Theorem 2.14], there exists a finite dimensional \( G \)-complex \( X \) with finitely many orbit types and a \( G \)-\( \mathbb{F}_p \)-isomorphism \( f : X \to E_{\mathcal{R}_p(G)} \); that is, a \( G \)-equivariant map \( f \) such that \( f^H : X^H \to (E_{\mathcal{R}_p(G)})^H \) is an \( H \)-\( \mathbb{F}_p \)-isomorphism on \( H \)-fixed point sets for any \( H \subseteq G \).

Since each \( H_k((E_{\mathcal{R}_p(G)})^H; \mathbb{Z}) \) is finitely generated (see [JMO, Proposition 1.11]), \( f^H \) is in fact an isomorphism in \( \mathbb{Z}_p^\wedge \)-homology for each \( H \), and therefore \( f \) is a \( G \)-\( \mathbb{Z}_p^\wedge \)-homology isomorphism; by [JMO, A.13] this implies that \( H^s_G(E_{\mathcal{R}_p(G)}; F) \cong H^s_G(X; F) \) for any \( \mathbb{Z}_p^\wedge \)-module valued coefficient system.

Now one can filter \( X \) by \( G \)-skeleta \( X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = X \) so that \( X_i/X_{i-1} \) contains a single orbit type \( G/P_i \). If \( N \) is the dimension of \( X \), by induction on the \( X_i \) one then shows (as in the proof of [JMO, A.13]) that \( H^s_G(X; F) = 0 \) for \( s > N \). \( \square \)
5. The homotopy equivalence

For a connected compact Lie group $H$, consider the arithmetic square

$$
\begin{array}{ccc}
BH & \xrightarrow{i} & BH^p \\
\downarrow j & & \downarrow j' \\
BH_Q & \xrightarrow{i_Q} & (BH^p)_Q \\
\end{array}
$$

(3)

(see [BK, VI, 8.1]), where $X^p = \prod X_p^p$ is the product over all primes $p$ of the $p$-completions and $X_Q$ is the $Q$-localization.

Without loss of generality, $i_Q$ is a fibration and (3) is a pullback diagram, so both horizontal maps have the same fiber $F$. Since $H$ is compact and $BH_Q$, $(BH^p)_Q$ are rational $H$-spaces, they are even-dimensional rational GEMs (that is, products of even-dimensional rational Eilenberg-Mac Lane spaces) and $F$ is an odd-dimensional rational GEM.

For any map $f : BG \to BH$ (where $G$ is a compact Lie group), (3) induces another pullback diagram

$$
\begin{array}{ccc}
map(BG, BH)_f & \longrightarrow & map(BG, BH^p)_{i_Qf} \\
\downarrow & & \downarrow \\
map(BG, BH_Q)_{i_Qf} & \longrightarrow & map(BG, (BH^p)_Q)_{i_Qf} \\
\end{array}
$$

(4)

As for any compact Lie group, $H^{2k-1}(BG; Q) = 0$ for all $k \geq 1$ (cf. [Bo, Theorem 19.1]). Since $F \simeq \prod K(Q, 2r_i - 1)$ is an odd-dimensional rational GEM, $map(BG, F)$ is an odd-dimensional rational GEM, too, by a direct calculation of its homotopy groups. In particular, $map(BG, F)$ is connected, and $F_p$-acyclic for any prime $p$.

Thus $map(BG, F)$ is the fiber of $map(BG, BH_Q)_c \to map(BG, (BH^p)_Q)_c$, where $c$ is the constant map. Because $BH_Q$ is an $H$-space and $i_Q$ is an $H$-map, this is in fact the fiber for all components and thus for the two horizontal maps in (4).

Therefore, applying the $q$-completion functor to the top fibration sequence in the diagram

$$
map(BG, F) \to map(BG, BH)_f \to map(BG, BH^p)_{i_Qf},
$$

we get another fibration (by [BK, II, 5.2]):

$$
map(BG, F)^q \to (map(BG, BH)_f)^q \xrightarrow{g} (map(BG, BH^p)_{i_Qf})^q,
$$

with $g$ a homotopy equivalence (since the fiber is contractible).

Finally, Lemma 4.1 implies that $(map(BG, BH^p)_{i_Qf})^q$ is homotopy equivalent to $(map(BG, BH^p)_{i_Qf})_p$ for $q = p$, and is contractible for $q \neq p$, so we get the desired homotopy equivalence

$$
(map(BG, BH)_f)^q \simeq map(BG, BH^p)_{i_Qf}.
$$

This completes the proof of Theorem 1.1. □
REFERENCES


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