REFINEMENTS OF KY FAN'S INEQUALITY

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Abstract. We prove the inequalities

\[ \frac{A'_n}{G'_n} \leq \frac{(1 - G'_n)}{(1 - A'_n)} \leq \frac{A_n}{G_n} \]

and

\[ \frac{A'_n}{G'_n} \leq \frac{(1 - G_n)}{(1 - A_n)} \leq \frac{A_n}{G_n}, \]

where \( A_n \) and \( G_n \) (respectively, \( A'_n \) and \( G'_n \)) denote the unweighted arithmetic and geometric means of \( x_1, \ldots, x_n \) (respectively, \( 1 - x_1, \ldots, 1 - x_n \)) with \( x_i \in (0, \frac{1}{2}] \) \( (i = 1, \ldots, n; \ n \geq 2) \). Further we show that the ratios \( (1 - G'_n)/(1 - A'_n) \) and \( (1 - G_n)/(1 - A_n) \) can be compared if and only if \( n = 2 \).

1. Introduction

In 1961 the following remarkable inequality, due to Ky Fan, was published for the first time in the well-known book Inequalities by Beckenbach and Bellman [3, p. 5]:

If \( A_n \) and \( G_n \) (respectively, \( A'_n \) and \( G'_n \)) denote the unweighted arithmetic and geometric means of the real numbers \( x_1, \ldots, x_n \) (respectively, \( 1 - x_1, \ldots, 1 - x_n \)), i.e.,

\[ A_n = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \quad G_n = \prod_{i=1}^{n} x_i^{1/n} \]

(respectively, \( A'_n = \frac{1}{n} \sum_{i=1}^{n} (1 - x_i) \) and \( G'_n = \prod_{i=1}^{n} (1 - x_i)^{1/n} \)), then we have for all \( x_i \in (0, \frac{1}{2}] \) \( (i = 1, \ldots, n; \ n \geq 2) \),

\[ G_n/G'_n \leq A_n/A'_n. \]

Equality holds in (1.1) if and only if \( x_1 = \cdots = x_n \).

Inequality (1.1) has evoked the interest of several mathematicians and many papers have been published providing new proofs, noteworthy extensions, and sharpenings as well as intriguing counterparts and variants; see [2] and the references therein.

Among the different refinements of Fan's inequality we could not find one presenting a sharpening of the equivalent inequality

\[ A'_n/G'_n \leq A_n/G_n. \]

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The aim of this paper is to prove two refinements of inequality (1.2). In §2 we establish that the ratios \((1 - G'_n)/(1 - A'_n)\) and \((1 - G'')/\(1 - A''\) both separate the left-hand side and the right-hand side of (1.2). It is natural to ask whether all four quotients can be included in a chain of three inequalities. This is indeed possible if \(n = 2\); but if \(n > 2\), then the expressions \((1 - G'_n)/(1 - A'_n)\) and \((1 - G'')/(1 - A'')\) cannot be compared. These results will be proved in §3.

2. TWO REFINEMENTS

In the proof of Theorem 1 the following additive analogue of inequality (1.1) plays a central role.

If \(x_i \in (0, 1/2] \ (i = 1, \ldots, n; \ n \geq 2)\) then

\[
G_n - G'_n \leq A_n - A'_n,
\]

with equality holding if and only if \(x_1 = \cdots = x_n\).

A proof for this proposition can be found in [1].

**Theorem 1.** If \(x_i \in (0, 1/2] \ (i = 1, \ldots, n; \ n \geq 2)\) then

\[
A'_n/G'_n \leq (1 - G'_n)/(1 - A'_n) \leq A_n/G_n.
\]

Equality is valid if and only if \(x_1 = \cdots = x_n\).

**Proof.** The function \(f(x) = x(1-x)\) is strictly decreasing on \([1/2, \infty)\). Because of \(1/2 \leq G'_n \leq A'_n < 1\), we obtain \(f(A'_n) \leq f(G'_n)\) with equality holding if and only if all the \(x_i\)'s are equal. This establishes the left-hand side of (2.2).

Since \(A_n + A'_n = 1\), we obtain from (2.1) that

\[
G_n(1 - G'_n) \leq G_n(2A_n - G_n) \leq A_n^2,
\]

which yields the second inequality of (2.2). If \(G_n(1 - G'_n) = A_n^2\) then we conclude from the right-hand inequality of (2.3): \(A_n = G_n\); hence \(x_1 = \cdots = x_n\). \(\Box\)

**Remark.** From double-inequality (2.3) we get the following sharpening of the right-hand side of (2.2):

\[
(1 - G'_n)/(1 - A'_n) \leq 2 - G_n/A_n \leq A_n/G_n.
\]

Equality is valid if and only if all the \(x_i\)'s are equal. This is obvious for the second inequality of (2.4), and since equality holds in (2.1) only if \(x_1 = \cdots = x_n\), the same is true for the first inequality of (2.4).

**Theorem 2.** If \(x_i \in (0, 1/2] \ (i = 1, \ldots, n; \ n \geq 2)\) then

\[
A'_n/G'_n \leq (1 - G'')/(1 - A'') \leq A_n/G_n,
\]

with equality holding if and only if \(x_1 = \cdots = x_n\).

**Proof.** The validity of the second inequality follows immediately from \(0 < G_n \leq A_n \leq 1/2\) and the fact that \(f(x) = x(1-x)\) is strictly increasing on \((0, 1/2]\).

To establish the left-hand inequality of (2.5) we define

\[
g: [0, 1/2]^n \to \mathbb{R},
\]

\[
g(x_1, \ldots, x_n) = \left(1 - \prod_{i=1}^n x_i^{1/n}\right) \prod_{i=1}^n (1 - x_i)^{1/n} - \left(1 - \frac{1}{n} \sum_{i=1}^n x_i\right)^2.
\]
Let \( a = (a_1, \ldots, a_n) \in [0, \frac{1}{2}]^n \) be the absolute minimum of \( g \). We prove \( a_1 = \cdots = a_n \), which implies

\[
g(x_1, \ldots, x_n) \geq g(a_1, \ldots, a_1) = 0 \quad \text{for all } (x_1, \ldots, x_n) \in [0, \frac{1}{2}]^n
\]

with equality holding if and only if \( x_1 = \cdots = x_n \).

If \( a \) is an interior point of \([0, \frac{1}{2}]^n\), then we obtain

\[
\nabla g(a_1, \ldots, a_n) = 0
\]

such that \( a_1, \ldots, a_n \) solve the equation

\[
P(x) = -G_nG'_n(1 - x) - (1 - G_n)G'x + 2(1 - A_n)x(1 - x) = 0.
\]

Since \( P \) is a polynomial of degree 2, we conclude from

\[
P(0) < 0 \quad \text{and} \quad 2P\left(\frac{1}{2}\right) = 1 - G'_n - A_n \geq 1 - A'_n - A_n = 0
\]

that \( P \) has at most one zero on \((0, \frac{1}{2})\); hence \( a_1 = \cdots = a_n \).

Next we assume that \( a \) is a boundary point of \([0, \frac{1}{2}]^n\). We consider two cases.

Case 1. No component of \( a \) is equal to 0. Then \( l \geq 1 \) components of \( a \)
are equal to \( \frac{1}{2} \). Without loss of generality, we may suppose

\[
a_k+1 = \cdots = a_n = \frac{1}{2}, \quad 1 \leq n - k = l \leq n - 1.
\]

We define

\[
h: [0, \frac{1}{2}]^k \to \mathbb{R}, \quad h(x_1, \ldots, x_k) = g(x_1, \ldots, x_k, \frac{1}{2}, \ldots, \frac{1}{2})
\]

\[
= \frac{1}{2}[1 - \frac{1}{2}(2G_k)^{k/n}](2G'_k)^{k/n} - \frac{1}{2} + k \left( \frac{1}{2} - A_k \right) / n]^2.
\]

Because of

\[
(2.6) \quad h(x_1, \ldots, x_k) \geq h(a_1, \ldots, a_k) \quad \text{for all } (x_1, \ldots, x_k) \in [0, \frac{1}{2}]^k,
\]

we conclude that \( h \) attains its absolute minimum at \( \bar{a} = (a_1, \ldots, a_k) \). Since
\( 0 < a_i < \frac{1}{2} \) \( (i = 1, \ldots, k) \), we obtain \( \nabla h(a_1, \ldots, a_k) = 0 \), which implies that
\( a_1, \ldots, a_k \) solve the equation

\[
Q(x) = \frac{1}{4}(4G_kG'_k)^{k/n}(2x - 1) - \frac{1}{2}(2G'_k)^{k/n}x
+ (-2kA_k/n + 1 + k/n) x(1 - x) = 0.
\]

We have \( Q(0) < 0 \) and

\[
(2.7) \quad 4Q\left(\frac{1}{2}\right) = -(2G'_k)^{\alpha} - 2A_k\alpha + 1 + \alpha
\]

with \( \alpha = k/n \in (0, 1) \). If we designate the right-hand side of (2.7) by \( \tilde{Q}(\alpha) \),
then \( \tilde{Q} \) is strictly concave on \([0, 1]\) and, since \( \tilde{Q}(0) = 0 \) and

\[
\tilde{Q}(1) = 2(1 - A_k - G'_k) \geq 2(1 - A_k - A'_k) = 0,
\]

we conclude

\[
Q\left(\frac{1}{2}\right) = \frac{1}{4}\tilde{Q}(k/n) > 0.
\]

Thus, \( Q \) has precisely one root on \((0, \frac{1}{2})\), which leads to \( a_1 = \cdots = a_k \). Now
we prove that the function

\[
\tilde{h}(x) = h(x, \ldots, x)
\]
is strictly decreasing on \([0, \frac{1}{2}]\). This implies
\[ h(a_1, \ldots, a_k) = \tilde{h}(a_1) > \tilde{h}(\frac{1}{2}) = h(\frac{1}{2}, \ldots, \frac{1}{2}), \]
which contradicts inequality (2.6). Differentiation of \(\tilde{h}\) yields, for \(x \in (0, \frac{1}{2})\),
\[
(2.8) \quad \frac{1}{\alpha} \tilde{h}'(x) = \frac{1}{4} \left( \frac{1}{1-x} - \frac{1}{x} \right) [4x(1-x)]^\alpha - \frac{1}{2(1-x)} [2(1-x)]^\alpha + 1 + \alpha - 2\alpha x
\]
with \(\alpha = k/n \in (0, 1)\). We denote the right-hand side of (2.8) by \(p(\alpha)\). Differentiation of \(p\) leads to
\[
p''(\alpha) = (2x - 1)[4x(1-x)]^\alpha \log(4x(1-x)) - \log(2(1-x)) - 2x
\]
Hence we obtain, for \(\alpha \in (0, 1)\):
\[
(2.9) \quad p'(\alpha) = (2x - 1)[4x(1-x)]^\alpha \log(4x(1-x)) - \log(2(1-x)) + 1 - 2x.
\]
We designate the right-hand side of (2.9) by \(q(x)\). Because of \(q''(x) > 0\) for \(x \in (0, \frac{1}{2})\) and \(q(\frac{1}{2}) = q'(\frac{1}{2}) = 0\), we conclude \(p'(1) > 0\). Therefore \(p(\alpha) < p(1) = 0\) for \(\alpha \in (0, 1)\), which proves that \(\tilde{h}\) is strictly decreasing on \([0, \frac{1}{2}]\).

**Case 2.** \(l \geq 1\) components of \(a\) are equal to 0. We assume
\[ a_{k+1} = \cdots = a_n = 0, \quad 1 \leq n - k = l \leq n - 1, \]
and define
\[ \varphi: [0, \frac{1}{2}]^k \to \mathbb{R}, \]
\[ \varphi(x_1, \ldots, x_k) = g(x_1, \ldots, x_k, 0, \ldots, 0) = \prod_{i=1}^{k} (1-x_i)^{1/n} - \left(1 - \frac{1}{n} \sum_{i=1}^{k} x_i \right)^2. \]
We have, for \(j = 1, \ldots, k\),
\[ \frac{n}{2} \varphi_{x_j}(x_1, \ldots, x_k) = -\frac{1}{2(1-x_j)} (G'_k)^\alpha + 1 - \alpha A_k \geq -(G'_k)^\alpha + 1 - \alpha A_k \]
with \(\alpha = k/n \in (0, 1)\). Since the function
\[ \varphi(\alpha) = -(G'_k)^\alpha + 1 - \alpha A_k \]
is strictly concave on \([0, 1]\) and because of
\[ \varphi(0) = 0 \quad \text{and} \quad \varphi(1) = -G'_k + 1 - A_k = -G'_k + A'_k \geq 0, \]
we obtain
\[ \varphi(\alpha) > 0 \quad \text{for} \ \alpha \in (0, 1). \]
Hence we have
\[ \varphi(x_1, \ldots, x_k) \geq \varphi(0, \ldots, 0) = 0 \quad \text{for all} \ (x_1, \ldots, x_k) \in [0, \frac{1}{2}]^k. \]
Since \(\varphi\) attains its absolute minimum at \(\hat{a} = (a_1, \ldots, a_k)\), we conclude \(a_1 = \cdots = a_k = 0\). This completes the proof of Theorem 2. \(\square\)

### 3. The case \(n = 2\)
In this section we prove that the ratios \((1-G'_n)/(1-A'_n)\) and \((1-G_n)/(1-A_n)\) can be compared if and only if \(n = 2\).
Theorem 3. If \( x_1, x_2 \in (0, \frac{1}{2}] \) then

\[
(3.1) \quad \frac{A_2}{G_2} \leq \frac{(1 - G_2)}{(1 - A_2)} \leq \frac{(1 - G_2)}{(1 - A_2)} \leq A_2 / G_2,
\]

with equality holding if and only if \( x_1 = x_2 \).

Proof. It remains to establish the second inequality of (3.1). We define

\[
f: [0, \frac{1}{2}]^2 \to \mathbb{R},
\]

\[
f(x, y) = \left(1 - \sqrt{xy}\right) \frac{x + y}{2} - \left(1 - \frac{x + y}{2}\right) \left(1 - \sqrt{(1 - x)(1 - y)}\right),
\]

and denote the absolute minimum of \( f \) by \( a = (a_1, a_2) \). We prove \( a_1 = a_2 \).

If \( a \) is an interior point of \( [0, \frac{1}{2}]^2 \) then we have

\[
\nabla f(a_1, a_2) = 0,
\]

which leads to

\[
-\sqrt{a_2/a_1} A_2 + 2 - G_2 - G'_2 - \sqrt{(1 - a_2)/(1 - a_1)} A'_2 = 0
\]

and

\[
-\sqrt{a_1/a_2} A_2 + 2 - G_2 - G'_2 - \sqrt{(1 - a_1)/(1 - a_2)} A'_2 = 0.
\]

From these equations we obtain

\[
(A_2/G_2 - A'_2/G'_2)(a_1 - a_2) = 0.
\]

Suppose \( a_1 \neq a_2 \). Then we get \( A_2/G_2 = A'_2/G'_2 \), and from Fan's theorem we conclude \( a_1 = a_2 \).

Next we assume that \( a \) is a boundary point of \( [0, \frac{1}{2}]^2 \). We distinguish two cases.

Case 1. One component of \( a \) is equal to 0. If \( a_1 = 0 \) and \( a_2 = z \in (0, \frac{1}{2}] \), then we have

\[
F(z) = f(0, z) = z - 1 + \sqrt{1 - z} \left(1 - \frac{z}{2}\right)
\]

and

\[
(3.2) \quad 2\sqrt{1 - z} F'(z) = \frac{3z}{2} - 2 + 2\sqrt{1 - z}.
\]

Since the right-hand side of (3.2) is increasing on \( [0, \frac{1}{2}] \) we get, for \( z \in (0, \frac{1}{2}] \), \( F'(z) > 0 \) and \( F(z) > F(0) = 0 \).

Case 2. Both components of \( a \) are different from 0. Let \( a_1 = \frac{1}{2} \) and \( a_2 = z \in (0, \frac{1}{2}] \). Then we have

\[
G(z) = 4f \left(\frac{1}{2}, z\right) = \left(1 - \sqrt{\frac{z}{2}}\right) (1 + 2z) - (3 - 2z) \left(1 - \sqrt{\frac{1 - z}{2}}\right).
\]
A simple calculation reveals
\[ 2\sqrt{2}(1 - z)^{3/2}G''(z) = \left( \frac{1 - z}{z} \right)^{3/2} \left( \frac{1}{2} - 3z \right) + \frac{5}{2} - 3z > 0 \quad \text{for} \quad z \in \left( 0, \frac{1}{2} \right) \]
and
\[ G \left( \frac{1}{2} \right) = G' \left( \frac{1}{2} \right) = 0, \]
which implies
\[ G(z) \geq 0 \quad \text{for} \quad z \in \left( 0, \frac{1}{2} \right), \]
with equality holding if and only if \( z = \frac{1}{2} \).

Now we show that for every \( n \geq 3 \) the ratios \( (1 - G'_n)/(1 - A'_n) \) and \( (1 - G_n)/(1 - A_n) \) cannot be compared. If we set \( x_1 = \cdots = x_{n-1} = 0 \) and \( x_n = \frac{1}{2} \), then
\[ (1 - G'_n)/(1 - A'_n) > (1 - G_n)/(1 - A_n) \]
is equivalent to
\[ \left( \frac{2n-2}{2n-1} \right)^n > \frac{1}{2}. \]
Since \( \alpha_n = \left( \frac{2n-2}{2n-1} \right)^n \) is strictly increasing, we obtain, for \( n \geq 3 \),
\[ \alpha_n \geq \alpha_3 = \frac{64}{125} > \frac{1}{2}. \]
Next we put \( x_1 = 0 \) and \( x_2 = \cdots = x_n = \frac{1}{2} \). Then
\[ (1 - G'_n)/(1 - A'_n) < (1 - G_n)/(1 - A_n) \]
and \( \frac{1}{2} < \left( \frac{n+1}{4} \right)^n \) are equivalent. The sequence \( \beta_n = \left( \frac{n+1}{4} \right)^n \) is strictly increasing, which implies, for \( n \geq 2 \),
\[ \beta_n \geq \beta_2 = \frac{9}{16} > \frac{1}{2}. \]
The fact that inequality (3.3) is valid for \( n = 2 \), but not for \( n > 2 \), is rather unusual since "most classical inequalities follow inductively from the two-dimensional theorem" [4, p. 206]. Another example of this kind—also in connection with Fan's inequality—was given in 1974 by F. Chan, D. Goldberg, and S. Gonek [4]. They proved that the function
\[
f(r; x_1, \ldots, x_n) = \begin{cases} 
\left( \frac{\sum_{i=1}^{n} x_i^r}{\sum_{i=1}^{n} (1 - x_i)^r} \right)^{1/r}, & r \neq 0, \\
\prod_{i=1}^{n} \left[ x_i/(1 - x_i) \right]^{1/n}, & r = 0,
\end{cases}
\]
satisfies the inequality
\[ f(r; x_1, x_2) < f(s; x_1, x_2) \]
for all real \( r \) and \( s \) with \( r < s \) and for all nonnegative \( x_1, x_2 \) with \( x_1 + x_2 < 1 \) and \( x_1 \neq x_2 \). This result, in particular, leads to
\[ G_2/G'_2 = f(0; x_1, x_2) < f(r; x_1, x_2) < f(1; x_1, x_2) = A_2/A'_2 \]
for all \( r \in (0, 1) \).
Furthermore, Chan, Goldberg, and Gonek investigated the question whether inequality (3.4) also holds for more than two variables. Presenting interesting counterexamples, they established that the implication

\[(3.5) \quad r < s \Rightarrow f(r; x_1, \ldots, x_n) \leq f(s; x_1, \ldots, x_n),\]

\[x_i \in (0, \frac{1}{2}] \quad (i = 1, \ldots, n),\]

is in general not true if \( n > 2 \). However, in a recently published paper [2] it was proved that (3.5) is valid for all \( n \geq 2 \) if \( 0 < r < s \leq 1 \), which yields a refinement of Fan’s inequality written in the form (1.1).

REFERENCES


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