A NOTE ON ALMOST SUBNORMAL SUBGROUPS
OF LINEAR GROUPS

B. A. F. WEHRFRITZ

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Abstract. Following Hartley we say that a subgroup $H$ of a group $G$ is almost
subnormal in $G$ if there is a series of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_r = G$
of $G$ of finite length such that for each $i < r$ either $H_i$ is normal in $H_{i+1}$
or $H_i$ has finite index in $H_{i+1}$. We extend a result of Hartley's on arithmetic
groups (see Theorem 2 of Hartley's Free groups in normal subgroups of unit
groups and arithmetic groups, Contemp. Math., vol. 93, Amer. Math. Soc.,
Providence, RI, 1989, pp. 173–177) to arbitrary linear groups. Specifically, we
prove: let $G$ be any linear group with connected component of the identity
$G^0$ and unipotent radical $U$. If $H$ is any soluble-by-finite, almost subnormal
subgroup of $G$ then $[H \cap G^0, G^0] \leq U$.

Following Hartley [1] we say that a subgroup $H$ of a group $G$ is almost
subnormal in $G$, and write $H \trianglelefteq G$ for short, if there is a series of subgroups
$H = H_0 \leq H_1 \leq \cdots \leq H_r = G$
of $G$ of finite length such that for each $i < r$ either $H_i$ is normal in $H_{i+1}$
or $H_i$ has finite index in $H_{i+1}$. Theorem 2 of [1] reads as follows. Let $H$
be a connected reductive algebraic $\mathbb{Q}$-group and $\Gamma$ an arithmetic subgroup of
$H$. Then $\Gamma$ contains a normal subgroup $\Gamma_0$ of finite index such that if $\Delta$
is an almost subnormal subgroup of $\Gamma$, then either $\Delta$ contains a nonabelian free
group or $[\Delta \cap \Gamma_0, \Gamma] = \{1\}$.

In this note we show that this result can be viewed as a special case of a
basically elementary result about arbitrary linear groups. Note first that by
Tits's theorem [2, 10.17] either the $\Delta$ above contains a nonabelian free subgroup
or $\Delta$ is soluble-by-finite. Thus we are concerned with soluble-by-finite almost
subnormal subgroups of linear groups.

Throughout this note $F$ denotes a field, $n$ a positive integer, and $G$
a subgroup of $\text{GL}(n, F)$. Then $G$ carries its Zariski topology and topological
terms below refer to this topology. $G$ has a unique minimal closed subgroup of
finite index; we denote this subgroup by $G^0$. Also $u(G)$ denotes the unipotent
radical of $G$. Our main result is the following.

Proposition 1. Let $G$ be a subgroup of $\text{GL}(n, F)$ and $H$ a soluble-by-finite,
almost subnormal subgroup of $G$. Then $[H \cap G^0, G^0] \leq u(G^0)$.
If $\Gamma$ is as in [1, Theorem 2] quoted above, then the centre of $\Gamma^0$ lies in the centre of $\Gamma$ (indeed of $H$) and $u(\Gamma^0) = u(\Gamma) = (1)$. Consequently Hartley’s theorem follows from Tits’s theorem and Proposition 1 (with $\Gamma^0$ for $\Gamma_0$). (These facts about $\Gamma$ are easy to deduce from the first part of Hartley’s proof of [1, Theorem 2].) If we recast Proposition 1 in the format of [1] we obtain

**Corollary.** Let $H \lhd G \leq \text{GL}(n, F)$. Suppose either that $\text{char} F = 0$ or that the entries of the elements of $G$ lie in some finitely generated subring of $F$. Then either $H$ contains a noncyclic free subgroup or $[H \cap G^0, G^0] \leq u(G^0)$.

In characteristic zero we can strengthen the conclusion of Proposition 1.

**Proposition 2.** There exists an integer-valued function $f(n)$ of $n$ only such that whenever $G \leq \text{GL}(n, F)$ with $\text{char} F = 0$ there is a closed normal subgroup $C$ of $G$ with $(G : C) \leq f(n)$ such that if $H \lhd G$ then either $H$ contains a noncyclic free subgroup or $[H \cap C, C] \leq u(C)$.

Although we have been discussing almost subnormal subgroups, the core of our proofs is that one needs only consider characteristic subgroups. This is the content of the following proposition.

**Proposition 3.** Let $G$ be a subgroup of $\text{GL}(n, F)$. Then $G$ has a unique maximal soluble-by-finite, almost subnormal subgroup $T$ and $T$ contains every soluble-by-finite, almost subnormal subgroup of $G$.

**The proofs**

1. Let $G \leq \text{GL}(n, F)$. Then $G$ has a unique maximal soluble-by-finite normal subgroup $T$. The subgroup $T$ contains every soluble-by-finite normal subgroup of $G$.

*Proof.* The product $S$ of the soluble normal subgroups of $G$ is soluble and closed in $G$, see [2, 3.8 and 5.11]. Let $T$ be the product of the soluble-by-finite normal subgroups of $G$. Then $T/S$ is an $FC$-group and hence is centre-by-finite, see [2, 5.5 and 5.14]. Consequently, $T$ is soluble-by-finite, $T/S$ is finite, and the lemma is proved.

2. Let $H \lhd G \leq \text{GL}(n, F)$. Then $u(H^0) \leq u(G^0) \leq u(G)$ and $u(H)^0 \leq u(G)^0$. If $\text{char} F = 0$ then $u(H) \leq u(G)$.

*Proof.* Let $H = H_0 \leq H_1 \leq \cdots \leq H_r = G$ be as in the definition above of the statement ‘$H \lhd G$.’ If $H$ is normal in $H_1$ then $H^0$ is normal in $H_1^0$; hence $u(H^0)$ is normal in $H_1^0$ and so $u(H^0) \leq u(H_1^0)$. Suppose $(H_1 : H)$ is finite. Use a star to denote closures of subsets in $H_1$. Then $H_0^*$ is connected and $(H_1 : H_0^*) \leq (H_1 : H^0)$ is also finite. Consequently, $H_0^* = H_1^0$. Also $u(H^0)^*$ is unipotent and normal in $H_0^*$, see [2, 5.9 and 1.21]. Therefore,

$$u(H^0) \leq u(H^0)^* \leq u(H_0^*) = u(H_1^0).$$

We have now proved that in all cases $u(H^0) \leq u(H_0^0)$. The same argument yields that $u(H^0) \leq u(H_{i+1}^0)$ for each $i$ and hence $u(H^0) \leq u(G^0)$.

Now $u(H)^0 \leq H^0$, so $u(H)^0 \leq u(H^0) \leq u(G^0) \leq u(G)$. It is also connected, so $u(H)^0 \leq u(G)^0$. If $\text{char} F = 0$ then unipotent subgroups of $\text{GL}(n, F)$ are connected and $u(H) = u(H^0) \leq u(G)^0 = u(G)$.
3. Suppose $H \triangleleft G \leq \text{GL}(n,F)$ with $H$ soluble-by-finite. Then $H^G = \langle g^{-1}hg : h \in H, g \in G \rangle$ is also soluble-by-finite.

**Proof.** We may assume that $F$ is algebraically closed. Let $H = H_0 \leq H_1 \leq \cdots \leq H_r = G$ be as in the definition of $H \triangleleft G$. We induct on $r$. If $r = 0$ there is nothing to prove, so assume otherwise and set $K = H_{r-1}$. By induction $H^K$ is soluble-by-finite. Replacing $H^K$ by $H$ we may assume that $H$ is normal in $K$. If $K$ is normal in $G$ then $HG$ is soluble-by-finite by [5, 5.1]. Hence consider the case $(G : K) < \infty$. Replace $H$ and $K$ by their closures in $G$, so now $H$ and $K$ are closed subgroups of $G$.

Certainly $u(G)$ is soluble and $G/u(G)$ embeds into $\text{GL}(n,F)$ in the standard way. Thus we may pass to $G/u(G)$ and assume that $u(G) = \langle 1 \rangle$. In particular by 2 we now have $u(H^0) = \langle 1 \rangle$. It follows from the Lie-Kolchin theorem [2, 5.8] that $H^0$ is abelian. Now $K/u(K)$ has a faithful completely reducible representation in $\text{GL}(n,F)$, see [2, Chapter 1], and hence [2, 1.12 and 5.4] the centralizer $C_K(H^0)$ is closed in $K$ of finite index. So too is $C_K(H/H^0)$, see [2, 5.10]. Hence $G^0 = K^0$ stabilizes the series $H \supseteq H^0 \supseteq \langle 1 \rangle$ and consequently $L = (G^0)'$ centralizes $H$. Thus $H$ lies in the normal subgroup $C_G(L)$ of $G$ and hence $H^G \leq C_G(L)$. Finally $G^0 \cap C_G(L)$ is soluble and of finite index in $C_G(L)$. The proof is complete.

4. *The Proof of Proposition 3.* Let $T$ be as in 1. The result then follows from 1 and 3.

5. Suppose $H \triangleleft G \leq \text{GL}(n,F)$ with $H$ soluble-by-finite and $u(G) = \langle 1 \rangle$. Then $[H \cap G^0, G^0] = \langle 1 \rangle$.

**Proof.** Let $T$ be as in Proposition 3. Then it suffices to prove that $[T \cap G^0, G^0] = \langle 1 \rangle$. We may assume that $F$ is algebraically closed and, since $u(G) = \langle 1 \rangle$, that $G$ is completely reducible. Then $T^0$ is abelian and $(G : C_G(T^0))$ divides $n!$ by [2, 5.11, 5.8, and 1.12]. Certainly $T$ and $T^0$ are closed in $G$. Hence $L = C_G(T^0) \cap C_G(T/T^0)$ is a closed normal subgroup of $G$ of finite index. By stability theory, $L/CL(T \cap L)$ embeds into

$\text{Der}((T \cap L)/(T^0 \cap L), T^0 \cap L) = \text{Hom}((T \cap L)/(T^0 \cap L), T^0 \cap L),$

since $T^0 \cap L$ is central in $T \cap L$. Also $(T \cap L)/(T^0 \cap L)$ is finite and $T^0$ is abelian with its torsion subgroup of finite rank (at most $n$, see [2, 2.2]). Hence this Hom group is finite and therefore $C = CL(T \cap L)$ is a closed normal subgroup of $G$ of finite index. Consequently we have $G^0 \leq C$ and $[T \cap C, C] \leq [T \cap L, C] = \langle 1 \rangle$. The proof is complete.

6. *The Proof of Proposition 1.* There is a continuous homomorphism $\phi$ of $G$ into $\text{GL}(n,F)$ with kernel $u(G)$ and image completely reducible. Then $(G\phi)^0 \supseteq G^0\phi$. Also $[H\phi \cap (G\phi)^0, (G\phi)^0] = \langle 1 \rangle$ by 5. Therefore $[H \cap G^0, G^0] \leq \ker \phi \cap G^0 = u(G) \cap G^0 = u(G^0)$.

7. *The Proof of Proposition 2.* We modify the proof of 5. Again we may assume that $F$ is algebraically closed and that $G$ is completely reducible. Then $T$ has a closed diagonalizable subgroup $A$ normal in $G$ such that $(T : A)$ is bounded by an integer-valued function of $n$ only [2, 10.11; 3, Proposition 1]. The proof of 5 yields that if $L = C_G(A) \cap C_G(T/A)$ and $C = CL(T \cap L)$, then $C$ is
a closed normal subgroup of $G$ with $(G : C)$ bounded by an integer-valued function of $n$ only and $[T \cap C, C] = \langle 1 \rangle$.

8. The corollary to Proposition 1 is an immediate consequence of Proposition 1, Tits's theorem, and the following, no doubt well-known, fact:

If $H$ is a soluble-by-periodic subgroup of $\text{GL}(n, R)$, where $R$ is a finitely generated integral domain, then $H$ is soluble-by-finite.

To see this note first that $H$ is soluble by locally-finite by [2, 5.9, 5.11, 6.4, and 4.9]. Suppose first that $H$ is absolutely irreducible. Then by [2, 1.12] the group $H$ has a centre by locally-finite normal subgroup $K$ of finite index. Then $K'$ is locally finite. Since $u(K') \leq u(H) = \langle 1 \rangle$, it follows from [2, 4.8] that $K'$ is finite. Consequently $H$ is soluble-by-finite. In general, by adjoining the (finitely many) entries and the inverse determinant of a suitable change-of-basis matrix to $R$, we may assume that $H/u(H)$ is isomorphic to an absolutely completely reducible subgroup of $\text{GL}(n, R)$. Then $H/u(H)$ is soluble-by-finite by the first case and consequently so is $H$.

9. Remarks. (i) Consider the situation of 5. Clearly we cannot prove in general that $[H, G^0] = \langle 1 \rangle$, for we can choose $H = G$ and $G$ not centre-by-finite. However, we cannot even show that $G^0$ normalizes $H$, i.e., that $[H, G^0] \leq H$. For the infinite dihedral group,

$$G = \langle x, y \mid x^y = x^{-1}, y^2 = 1 \rangle$$

has an embedding in $\text{GL}(2, \mathbb{Q})$ (and also in $\text{GL}(4, \mathbb{Z})$) with $G^0 = \langle x \rangle$ and $u(G) = \langle 1 \rangle$. Set $H = \langle x^3, y \rangle$. Then $(G : H) = 3$ and so $H$ hasn $G$. Also

$$H[H, G^0] \geq \langle x^3, y, [y, x] = x^2 \rangle = G.$$

(ii) There is no analogue of Proposition 2 if $\text{char} F = p > 0$. For if $G$ is a finite simple linear group of degree $n$ and characteristic $p$, then the order of $G$ is not boundable and, since $G$ is an allowable choice for $H$, the only possibility for $C$ is $\langle 1 \rangle$.

(iii) If $\text{char} F = 0$ in 3 then $H^G$ has a soluble normal subgroup with finite index bounded by a function of $n$ only [2, 10.11]. If $\text{char} F > 0$ this conclusion is false. One can at least prove the following (cf. [5, 5.1]).

Let $H$ hasn $G \leq \text{GL}(n, F)$. Suppose $H$ has a soluble normal subgroup of finite index $m$. Then $H^G$ has a soluble normal subgroup with index bounded by a function of $m, n$ and the indices $(H_{i+1} : H_i)$ for those $i$ for which $H_i$ is not normal in $H_{i+1}$. (The $H_i$ here are as in the definition of $H$ hasn $G$.)

(iv) Using the techniques and main theorem of [4, §6], including a converse of [4, 6.4], one can extend the results of this note as follows.

Let $R$ be a commutative ring, $M$ a Noetherian $R$-module, and $G$ a group of $R$-automorphisms of $M$.

(a) There exists a normal subgroup $C$ of $G$ of finite index such that whenever $H$ is a soluble-by-finite, almost subnormal subgroup of $G$, we have $[H \cap C, C] \leq u(C)$.

(b) If $R$ is finitely generated as a ring and $C$ is as in (a), then for every almost subnormal subgroup $H$ of $G$ either $H$ contains a noncyclic free subgroup or $[H \cap C, C] \leq u(C)$.

(c) $G$ has a unique maximal soluble-by-finite, almost subnormal subgroup $T$ and $T$ contains every soluble-by-finite, almost subnormal subgroup of $G$. 

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School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London E1 4NS, Great Britain