LINEAR DIFFERENTIAL EQUATIONS
WITH EXCEPTIONAL FUNDAMENTAL SETS. II

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ABSTRACT. We prove a sharp order estimate for entire functions of completely regular growth, whose zeros are distributed near finitely many rays \( \arg z = \omega_j \) in terms of the angles \( \omega_j \). This result then leads immediately to a proof of a conjecture of Hellerstein and Rossi concerning the distribution of zeros of the solutions of linear differential equations with polynomials coefficients.

1. Introduction

Let \( w_1, w_2 \) be linearly independent solutions of

\[
w'' + p(z)w = 0,
\]

where \( p \) is a nonconstant polynomial. From the well-known results of Hille [6, Chapter 5] on the asymptotic distribution of zeros, Gundersen [4] deduced that the exponent of convergence of the nonreal zeros of \( E = w_1 w_2 \) equals \( 1 + \frac{1}{2} \deg p \). In an earlier paper, Hellerstein, Shen, and Williamson [5] showed that \( E \) has infinitely many nonreal zeros.

The methods in [4, 5] do not apply to the \( n \)th order case. In this paper we prove an analogon of Gundersen’s result for the solutions of \( n \)th order linear differential equations

\[
w^{(n)} + p_{n-2}(z)w^{(n-2)} + \cdots + p_0(z)w = 0
\]

with polynomial coefficients, which is even slightly sharper. In particular, we solve Problem 2.72 in [1], posed by Hellerstein and Rossi.

This is done by proving a sharp order estimate (Theorem 1) for entire functions of completely regular growth, which seems to be of independent interest.

2. Notation

Let \( E \) be an entire function of finite positive order \( \lambda \) and denote the counting function of zeros of \( E \) lying in the planar set \( S \) by \( N(r, 1/E, S) \). We say that the zeros of \( E \) are distributed near the rays

\[
\arg z = \omega_j, \quad 0 \leq \omega_1 < \omega_2 < \cdots < \omega_k < \omega_{k+1} = \omega_1 + 2\pi,
\]

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if for every sufficiently small \( \delta > 0 \),

\[
N(r, 1/E, \omega_j + \delta < \arg z < \omega_{j+1} - \delta) = o(r^\lambda) \quad \text{as } r \to \infty
\]

\((j = 1, 2, \ldots, k)\).

The basic results in [8] for solutions of equation (2) can be summarized as follows:

Let \( w \) be a transcendental solution of (2), where \( p_0, \ldots, p_{n-2} \) are polynomials. Then \( w \) is a completely regular growing entire function of order \( \lambda \) (in the sense of Lewin [7]), \( 0 < \lambda < \infty \), whose zeros are distributed near finitely many rays (3).

3. Results

We first prove a result on completely regular growing entire functions with radially distributed zeros, from which the main result concerning the distribution of zeros of a fundamental set of (3) follows.

**Theorem 1.** Let \( E \) be a completely regular growing entire function of order \( \lambda \), whose zeros are distributed near the rays (3). Then either

\[
\lambda < \frac{\pi}{\min(\omega_{j+1} - \omega_j)}
\]

or else

\[
\delta(0, E) > 0.
\]

**Remark.** There is a similar result [3, Corollary 1.2]: If the zeros of the entire function \( E \) are lying on the rays \( \arg z = \omega_j \) and if the order of \( E \) is finite, but sufficiently large (depending on the geometry of the rays), then \( \delta(0, E) > 0 \). For us it is important to have an explicit (and sharp) bound.

**Theorem 2.** Let \( \{w_1, w_2, \ldots, w_n\} \) be a fundamental set of (2), where \( p_0, \ldots, p_{n-2} \) are polynomials. Suppose that the zeros of \( \{w_1, \ldots, w_n\} \) are distributed near the rays (3). Then either

\[
\lambda(w_\nu) \leq \frac{\pi}{\min(\omega_{j+1} - \omega_j)}, \quad \nu = 1, \ldots, n,
\]

or else the coefficients in (2) are constants.

**Remark.** The product \( E = w_1w_2\cdots w_n \) is a completely regular growing entire function of order \( \lambda \), \( 0 < \lambda < \infty \), or a polynomial, and the radial distribution of the zeros of \( \{w_1, \ldots, w_n\} \) corresponds to the function \( E \) (and its order).

For the sake of completeness we reformulate Theorem 2 in two special cases.

**Corollary 1.** Let \( \{w_1, \ldots, w_n\} \) be a fundamental set of (2), whose zeros are distributed near the rays \( \arg z = 2\pi j/k \) \((1 \leq j \leq k, \quad k \geq 2)\). Then

\[
\lambda(w_\nu) \leq k/2, \quad \nu = 1, \ldots, n.
\]

**Corollary 2.** Let \( \{w_1, \ldots, w_n\} \) be a fundamental set of (2), whose zeros are distributed near the real axis. Then the coefficients in (2) are constants.
4. Proof of Theorem 1

Since the function $E$ has completely regular growth and since its zeros are distributed near the rays $\arg z = \omega_j$, there are complex numbers $c_j$ such that

$$\log |E(z)| = \text{Re}(c_j z^\lambda) + o(|z|^\lambda).$$

This is true as $|z| \to \infty$ outside a set $\mathcal{E} \subseteq (0, \infty)$ of linear density zero:

$$\lim_{r \to \infty} \frac{\text{mes}(\mathcal{E} \cap [0, r])}{r} = 0$$

and uniformly in $\omega_j + \eta \leq \arg z \leq \omega_{j+1} - \eta$, $\eta > 0$ arbitrary.

Thus the contribution of the sector $\omega_j < \arg z < \omega_{j+1}$ to $m(r, E)$ and $m(r, 1/E)$ is asymptotically $M_j r^\lambda$ and $m_j r^\lambda$, respectively, where

$$M_j = \frac{1}{2\pi} \int_{\omega_j}^{\omega_{j+1}} \text{Re}(c_j e^{i\lambda \theta})^+ \, d\theta$$

and

$$m_j = \frac{1}{2\pi} \int_{\omega_j}^{\omega_{j+1}} \text{Re}(c_j e^{i\lambda \theta})^- \, d\theta$$

(as usual $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$).

If we assume $\delta(0, E) = 0$, then we must have $m_j = 0$ ($1 \leq j \leq k$), but $M_j \neq 0$ for at least one $j$ since $E$ has order $\lambda$.

Thus $c_j \neq 0$ and $|c_j| \cos(\arg c_j + \lambda \theta) \geq 0$ in $\omega_j < \theta < \omega_{j+1}$ for at least one $j$, which gives

$$\lambda(\omega_{j+1} - \omega_j) \leq \pi$$

and so (7).

5. Proof of Theorem 2

Set $E = w_1 w_2 \cdots w_n$ and assume first that $E$ is transcendental of order $\lambda$.

The Wronskian

$$W = W(w_1, w_2, \ldots, w_n)$$

is a constant $c$, say. Thus

$$m\left(r, \frac{1}{E}\right) = m\left(r, \frac{W}{E}\right) + O(1) = O(\log r)$$

by the lemma on the logarithmic derivative and so

$$\lambda \leq \pi / \min(\omega_{j+1} - \omega_j)$$

holds by Theorem 1.

However, if $\lambda < \max(\lambda(w_1), \ldots, \lambda(w_n))$, then by [9, Theorem 2] the coefficients of (2) are constants, and this is of course also true if $E$ is a polynomial. Thus, either $\lambda = \max(\lambda(w_1), \ldots, \lambda(w_n))$ and (16) implies (7) or the coefficients of (2) are constants.
References


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