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MORE SMOOTHLY REAL COMPACT SPACES

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ABSTRACT. A topological space X is called $\mathscr{A}$-real compact if every algebra homomorphism from $\mathscr{A}$ to the reals is an evaluation at some point of X, where $\mathscr{A}$ is an algebra of continuous functions. Our main interest lies on algebras of smooth functions. Arias-de-Reyna has shown that any separable Banach space is smoothly real compact. Here we generalize this result to a huge class of locally convex spaces including arbitrary products of separable Fréchet spaces.

In [KMS] the notion of real compactness was generalized by defining a topological space X to be $\mathscr{A}$-real-compact if every algebra homomorphism $\alpha: \mathscr{A} \to \mathbb{R}$ is just the evaluation at some point $a \in X$, where $\mathscr{A}$ is some subalgebra of $C(X, \mathbb{R})$. In case $\mathscr{A}$ equals the algebra $C(X, \mathbb{R})$ of all continuous functions this condition reduces to the usual real-compactness. Our main interest lies on algebras $\mathscr{A}$ of smooth functions. In particular we showed in [KMS] that every space admitting $\mathscr{A}$-partitions of unity is $\mathscr{A}$-real-compact. Furthermore any product of the real line $\mathbb{R}$ is C^∞-real-compact. A question we could not solve was whether ℓ^1 is C^∞-real-compact, despite the fact that there are no smooth bump functions. [AdR] had already shown that this is true not only for ℓ^1, but for any separable Banach space.

The aim of this paper is to generalize this result of [AdR] to a huge class of locally convex spaces, including arbitrary products of separable Fréchet spaces.

Convention. All subalgebras $\mathscr{A} \subseteq C(X, \mathbb{R})$ are assumed to be real algebras with unit and with the additional property that for any $f \in \mathscr{A}$ with $f(x) \neq 0$ for all $x \in X$ the function $1/f$ lies also in $\mathscr{A}$.

1. **Lemma.** *Let $\mathscr{A} \subset C(X, \mathbb{R})$ be a finitely generated subalgebra of continuous functions on a topological space X. Then X is $\mathscr{A}$-real-compact.*

Proof. Let $\alpha: \mathscr{A} \to \mathbb{R}$ be an algebra homomorphism. We first show that for any finite set $\mathscr{F} \subset \mathscr{A}$ there exists a point $x \in X$ with $f(x) = \alpha(f)$ for all $f \in \mathscr{F}$.

For $f \in \mathscr{A}$ let $Z(f) := \{x \in X : f(x) = \alpha(f)\}$. Then $Z(f) = Z(f - \alpha(f)1)$, since $\alpha(f - \alpha(f)1) = 0$. Hence we may assume that all $f \in \mathscr{F}$ are even contained in $\ker\alpha = \{f : \alpha(f) = 0\}$. Then $\bigcap_{f\in\mathscr{F}} Z(f) = Z(\sum_{f\in\mathscr{F}} f^2)$. The

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sets $Z(f)$ are not empty, since otherwise $f \in \ker\alpha$ and $f(x) \neq 0$ for all x, so $1/f \in \mathscr{A}$ and hence $1 = f/f \in \ker\alpha$, a contradiction to $\alpha(1) = 1$.

Now the lemma is valid, whether the condition "finitely generated" is meant in the sense of an ordinary algebra or even as an algebra with the additional assumption on non-vanishing functions, since then any $f \in \mathscr{A}$ can be written as a rational function in the elements of $\mathscr{F}$. Thus α applied to such a rational function is just the rational function in the corresponding elements of $\alpha(\mathscr{F}) = \mathscr{F}(x)$ and is thus the value of the rational function at x. □

2. **Corollary.** *Any algebra-homomorphism $\alpha: \mathscr{A} \to \mathbb{R}$ is monotone.*

Proof. Let $f_1 \leq f_2$. By Lemma 1 there exists an $x \in X$ such that $\alpha(f_i) = f_i(x)$ for $i = 1, 2$. Thus $\alpha(f_1) = f_1(x) \leq f_2(x) = \alpha(f_2)$. □

3. **Corollary.** *Any algebra-homomorphism $\alpha: \mathscr{A} \to \mathbb{R}$ is bounded, for every convenient algebra structure on $\mathscr{A}$.*

By a convenient algebra structure we mean a convenient vector space structure for which the multiplication $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is a bilinear bornological mapping. A convenient vector space is a separated locally convex vector space that is Mackey complete, see [FK].

Proof. Suppose that f_n is a bounded sequence, but $|\alpha(f_n)|$ is unbounded. Replacing f_n by f_n^2 we may assume that $f_n \geq 0$ and hence also $\alpha(f_n) \geq 0$. Choosing a subsequence we may even assume that $\alpha(f_n) \geq 2^n$. Now consider $\sum_n f_n/2^n$. This series converges in the sense of Mackey, and since the bornology on $\mathscr{A}$ is complete, the limit is an element $f \in \mathscr{A}$. Applying α yields

$$\begin{aligned}\alpha(f) &= \alpha\left(\sum_{n=0}^{N} \frac{1}{2^n} f_n + \sum_{n>N} \frac{1}{2^n} f_n\right) = \sum_{n=0}^{N} \frac{1}{2^n}\alpha(f_n) + \alpha\left(\sum_{n>N} \frac{1}{2^n} f_n\right) \\ &\geq \sum_{n=0}^{N} \frac{1}{2^n}\alpha(f_n) + 0 = \sum_{n=0}^{N} \frac{1}{2^n}\alpha(f_n),\end{aligned}$$

where we applied to the function $\sum_{n>N} \frac{1}{2^n} f_n \geq 0$ that α is monotone. Thus the series $\sum_{n=0}^{N} \alpha(f_n)/2^n$ is bounded and increasing, hence converges, but its summands are bounded by 1 from below. This is a contradiction. □

4. **Definition.** We recall that a mapping $f: E \to F$ between convenient vector spaces is called smooth (C^∞ for short) if the composite $f \circ c: \mathbb{R} \to F$ is smooth for every smooth curve $c: \mathbb{R} \to E$. It can be shown that under these assumptions derivatives $f^{(p)}: E \to L^p(E, F)$ exist. See [FK].

A mapping is called C_c^∞ if in addition all derivatives considered as mappings $d^p f: E \times E^p \to F$ are continuous.

Now we generalize Lemma 5 and Proposition 7 of [AdR] to arbitrary convenient vector spaces.

5. **Definition.** Let $\mathscr{A} \subseteq C(X, \mathbb{R})$ be a set of continuous functions on X. We say that a space X admits large carriers of class $\mathscr{A}$ if for every neighborhood U of a point $p \in X$ there exists a function $f \in \mathscr{A}$ with $f(p) = 0$ and $f(x) \neq 0$ for all $x \notin U$.

Every $\mathscr{A}$-regular space X admits large $\mathscr{A}$-carriers, where X is called $\mathscr{A}$-regular if for every neighborhood U of a point $p \in X$ there exists a function $f \in \mathscr{A}$ with $f(p) > 0$ and $f(x) = 0$ for $x \notin U$. The existence of large $\mathscr{A}$-carriers follows by using the modified function $\bar{f} := f(a) - f$.

In [AdR, Proof of Theorem 8] it is proved that every separable Banach space admits large C_c^∞-carriers. The carrying functions can even be chosen as polynomials as shown in Lemma 7 below.

6. **Lemma.** *Let E be a convenient vector space, $\{x'_n : n \in \mathbb{N}\} \subset E'$ be bounded, $(\lambda_n) \in \ell^1(\mathbb{N})$. Then the series $(x, y) \mapsto \sum_{n=1}^\infty \lambda_n x'_n(x) x'_n(y)$ converges to a continuous symmetric bilinear function on $E \times E$.*

Proof. Clearly the function converges pointwise. Since the sequence $\{x'_n\}$ is bounded, it is equicontinuous, hence bounded on some neighborhood U of 0, so there exists a constant $M \in \mathbb{R}$ such that $|x'_n(U)| \leq M$ for all $n \in \mathbb{N}$. For $x, y \in U$ we have $|\sum_{n=1}^\infty \lambda_n x'_n(x) x'_n(y)| \leq \sum_{n=1}^\infty |\lambda_n| M^2$, which suffices for continuity of a bilinear function. □

7. **Lemma.** *Let E be a Banach space that is separable or whose dual is separable for the topology of pointwise convergence. Then E admits large carriers for continuous polynomials of degree* 2.

Proof. If E is separable there exists a dense sequence (x_n) in E. By the Hahn-Banach theorem [J, 7.2.4] there exist $x'_n \in E'$ with $x'_n(x_n) = |x_n|$ and $|x'_n| \leq 1$.

Claim. $\sup_n |x'_n(x)| = |x|$.

Since $|x'_n| \leq 1$ we have $(\leq)$. For the converse direction let $\delta > 0$ be given. By denseness there exists an $n \in \mathbb{N}$ such that $|x_n - x| < \frac{\delta}{2}$. So we have

$$\begin{aligned} |x| &\leq |x_n| + |x - x_n| < |x'_n(x_n)| + \tfrac{\delta}{2} \\ &\leq |x'_n(x)| + \underbrace{|x'_n(x - x_n)|}_{<|x-x_n|<\frac{\delta}{2}} + \tfrac{\delta}{2} < |x'_n(x)| + \delta. \end{aligned}$$

If the dual E' is separable for the topology of pointwise convergence, then let x'_n be a sequence that is weakly dense in the unit ball of E'. Then $|x| = \sup_n |x'_n(x)|$.

In both cases the continuous polynomials of Lemma 6

$$x \mapsto \sum_{n=1}^{\infty} x'_n(x - a)^2 / n^2$$

vanish exactly at a. □

8. **Lemma.** *Let $\alpha: \mathscr{A} \to \mathbb{R}$ be an algebra homomorphism and assume that some subset $\mathscr{A}_0 \subset \mathscr{A}$ exists and a point $a \in X$ such that $\alpha(f_0) = f_0(a)$ for all $f_0 \in \mathscr{A}_0$ and such that X admits large carriers of class $\mathscr{A}_0$.*

Then $\alpha(f) = f(a)$ for all $f \in \mathscr{A}$.

Proof. Let $f \in \mathscr{A}$ be arbitrary. Since X admits large $\mathscr{A}_0$-carriers, there exists for every neighborhood U of a a function $f_U \in \mathscr{A}_0$ with $f_U(a) = 0$ and $f_U(x) \neq 0$ for all $x \in U$. By Lemma 1 there exists a point a_U such that $\alpha(f) = f(a_U)$ and $\alpha(f_U) = f_U(a_U)$. Since $f_U \in \mathscr{A}_0$, we have $f_U(a_U) =$

$\alpha(f_U) = f_U(a) = 0$, hence $a_U \in U$. Thus the net a_U converges to a and consequently $f(a) = f(\lim_U a_U) = \lim_U f(a_U) = \lim_U \alpha(f) = \alpha(f)$ since f is continuous. □

Now we generalize Proposition 2 and Lemma 3 of [BBL]. For every convenient vector space E, let a subalgebra $\mathscr{A}(E)$ of $C(E, \mathbb{R})$ be given, such that for every $f \in L(E, F)$ the image of f^* on $\mathscr{A}(F)$ lies in $\mathscr{A}(E)$. Examples are C_c^∞, $C^\infty \cap C$, $C_c^\omega := C_c^\infty \cap C^\omega$, $C^\omega \cap C$, where C^ω denotes the algebra of real analytic functions in the sense of [KM] and suitable algebras of functions of finite differentiability like Lip^m (see [FK]) or C_c^m.

9. **Theorem.** *Let E_i be $\mathscr{A}$-real-compact spaces that admit large carriers of class $\mathscr{A}$. Then any closed subspace of the product of the spaces E_i and, in particular, every projective limit of these spaces, has the same properties.*

Proof. First we show that this is true for the product E. We use Lemma 8 with $\mathscr{A}(E)$ for $\mathscr{A}$ and the vector space generated by $\bigcup_i \{f \circ \mathrm{pr}_i : f \in \mathscr{A}(E_i)\}$ for $\mathscr{A}_0$, where $\mathrm{pr}_j : E = \prod_i E_i \to E_j$ denotes the canonical projection. Let the finite sum $f = \sum_i f_i \circ \mathrm{pr}_i$ be an element of $\mathscr{A}_0$. Since $\alpha \circ \mathrm{pr}_i{}^* : \mathscr{A}(E_i) \to \mathscr{A}(E) \to \mathbb{R}$ is an algebra homomorphism, there exists a point $a_i \in E_i$ such that $\alpha(f_i \circ \mathrm{pr}_i) = (\alpha \circ \mathrm{pr}_i{}^*)(f_i) = f_i(a_i)$. Let a be the point in E with coordinates a_i. Then

$$\alpha(f) = \alpha\left(\sum_i f_i \circ \mathrm{pr}_i\right) = \sum_i \alpha(f_i \circ \mathrm{pr}_i) = \sum_i f_i(a_i) = \sum_i (f_i \circ \mathrm{pr}_i)(a) = f(a).$$

Now let U be a neighborhood of a in E. Since we consider the product topology on E, we may assume that $a \in \prod U_i \subset U$, where U_i are neighborhoods of a_i in E_i and are equal to E_i except for i in some finite subset F of the index set. Now choose $f_i \in \mathscr{A}(E_i)$ with $f_i(a_i) = 0$ and $f_i(x_i) \neq 0$ for all $x_i \notin U_i$. Consider $f = \sum_{i \in F} (f_i \circ \mathrm{pr}_i)^2 \in \mathscr{A}_0$. Then $f(a) = \sum_{i \in F} f_i(a_i)^2 = 0$. Furthermore $x \notin U$ implies that $x_i \notin U_i$ for some i, which turns out to be in F, and hence $f(x) \geq f_i(x_i)^2 > 0$. So we may apply Lemma 8 to conclude that $\alpha(f) = f(a)$ for all $f \in \mathscr{A}(E)$.

Now we prove the result for a closed subspace $F \subset E$. Again we want to apply Lemma 8, this time with $\mathscr{A}(F)$ for $\mathscr{A}$ and $\{f|_F : f \in \mathscr{A}(E)\}$ for $\mathscr{A}_0$. Since $\alpha \circ \mathrm{incl}^* : \mathscr{A}(E) \to \mathscr{A}(F) \to \mathbb{R}$ is an algebra homomorphism there exists an $a \in E$ with $\alpha(f|_F) = f(a)$ for all $f \in \mathscr{A}(E)$. Now let U be a neighborhood of a in E; then there exists an $f_U \in \mathscr{A}(E)$ with $f_U(a) = 0$ and $f_U(x) \neq 0$ for all $x \notin U$. By Lemma 1 there exists a point $a_U \in F$ such that $f_U(a_U) = \alpha(f_U|_F) = f_U(a) = 0$. Hence a_U is in U, and thus is a net in F that converges to a. In particular, $a \in F$ since F is closed in E. If V is a neighborhood of a in F then there exists a neighborhood U of a in E with $U \cap F \subset V$ and hence an $f \in \mathscr{A}_0$ with $f(a) = 0$ and $f(x) \neq 0$ for all $x \notin U$. So again Lemma 8 applies. □

10. *Remark.* Theorem 9 shows that a closed subspace of a product of certain $\mathscr{A}$-real-compact spaces is again $\mathscr{A}$-real-compact. Of course the natural question arises of whether the result remains true for arbitrary $\mathscr{A}$-real-compact spaces.

The question is even open—whether the product of two $\mathscr{A}$-real-compact spaces is $\mathscr{A}$-real-compact, or whether a closed subspace of an $\mathscr{A}$-real-compact space is $\mathscr{A}$-real-compact, or whether a projective limit of a projective system of $\mathscr{A}$-real-compact spaces is $\mathscr{A}$-real-compact.

11. **Corollary.** *Let E be a separable Fréchet space (e.g., a Fréchet-Montel space); then every algebra homomorphism on $C^\infty(E, \mathbb{R})$ or on $C_c^\infty(E, \mathbb{R})$ is a point evaluation. The same is true for any product of separable Fréchet spaces.*

Proof. Any Fréchet space has a countable Basis $\mathscr{U}$ of absolutely convex 0-neighborhoods, and since it is complete, it is a closed subspace of the product $\prod_{u\in\mathscr{U}} \widetilde{E_{(U)}}$. The $E_{(U)}$ are the normed spaces formed by E modulo the kernel of the Minkowski functional generated by U. As quotients of E the spaces $E_{(U)}$ are separable if E is such. So the completion $\widetilde{E_{(U)}}$ is a separable Banach space, and hence by [AdR, Theorem 8] $\widetilde{E_{(U)}}$ is C_c^∞-real-compact and admits large C_c^∞-carriers. By Theorem 9 the same is true for the given Fréchet space. So the result is true for $C_c^\infty(E, \mathbb{R})$. Since E is metrizable this algebra coincides with $C^\infty(E, \mathbb{R})$, see [K, 82].

Now for a product E of metrizable spaces the two algebras $C^\infty(E, \mathbb{R})$ and $C_c^\infty(E, \mathbb{R})$ again coincide. This can be seen as follows. For every countable subset A of the index set, the corresponding product is separable and metrizable, hence C^∞-real-compact. Thus there exists a point x_A in this countable product such that $\alpha(f) = f(x_A)$ for all f that factor over the projection to that countable subproduct. Since for $A_1 \subset A_2$ the projection of x_{A_2} to the product over A_1 is just x_{A_1} (use the coordinate projections composed with functions on the factors for f), there is a point x in the product whose projection to the subproduct with index set A is just x_A. Every Mackey continuous function and, in particular, every C^∞-function, depends only on countably many coordinates, thus factors over the projection to some subproduct with countable index set A, hence $\alpha(f) = f(x_A) = f(x)$. This can be shown by the same proof as for a product of factors $\mathbb{R}$ in [FK, Theorem 6.2.9] since the result of [M, 1952] is valid for a product of separable metrizable spaces. □

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