THE SUBALGEBRA OF $L^1(AN)$ GENERATED BY THE LAPLACIAN

WALDEMAR HEBISCH

(Communicated by J. Marshall Ash)

Abstract. We prove that for the Iwasawa $AN$ groups corresponding to complex semisimple Lie groups, the subalgebra of $L^1(AN)$ associated to a distinguished laplacian is isomorphic with the algebra of integrable radial functions on $\mathbb{R}^n$.

In [1] Cowling et al. derived a formula for the heat semigroup generated by a distinguished laplacian on a large class of Iwasawa $AN$ groups and proved that the maximal function constructed from the semigroup is of weak type $(1, 1)$.

In this paper we show that in the case of the $AN$ groups corresponding to complex semisimple Lie groups the results of [1] can be strengthened once we notice that the subalgebra of $L^1(AN)$ associated to a distinguished laplacian is isomorphic with the algebra of integrable radial functions on $\mathbb{R}^n$. This implies that the maximal operators associated to the Riesz means are of weak type $(1, 1)$. Also the functional calculus for the laplacian on $\mathbb{R}^n$ can be transferred to the distinguished laplacian on $AN$. This seems to be the first construction of a nonanalytic functional calculus on groups of exponential growth.

Let $G$ denote a connected, complex semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Denote by $\theta$ a Cartan involution of $\mathfrak{g}$, and write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

for the associated Cartan decomposition. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$; this determines a root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha,$$

$\Lambda$ denoting the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Corresponding to a choice of the ordering of the roots, we have an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}.$$

Let $G = ANK$ be the corresponding Iwasawa decomposition of $G$. A distinguished laplacian on $AN$ can be constructed as follows. Let $\pi: \mathfrak{g} \to \mathfrak{p}$ be the projection (defined by the Cartan decomposition). We define a positive definite form $\tilde{B}$ on $\mathfrak{a} \oplus \mathfrak{n}$ setting $\tilde{B}(X, Y) = B(\pi X, \pi Y)$ where $B$ is the Killing form.
on \( g \). Put \( n = \dim(AN) \). Choose an orthonormal (with respect to \( \tilde{B} \)) basis in \( a \oplus n \), say \( \{X_1, \ldots, X_n\} \).

Define the laplacian \( \Delta \) by setting

\[
\Delta f(x) = \sum_{j=1}^{n} \left( \frac{d}{dt} \right)^2 f(x \exp(tX_j))|_{t=0}.
\]

Denote by \( |x| \) the riemannian distance between \( e \) and \( x \) corresponding to the left-invariant riemannian structure induced by \( \tilde{B} \). Let \( \phi_0 \) be the distinguished spherical function (restricted to \( AN \))—\( \phi_0 \) may be defined as the unique function that is radial, i.e., we have \( \phi_0(x) = \psi(|x|) \) for some \( \psi \), and that satisfies the equation \( \Delta(\delta^{-1/2}\phi_0) = 0 \) where \( \delta \) is the modular function of \( AN \).

The heat kernel \( p_t \) corresponding to \( \Delta \) is given by the formula (see [1])

\[
p_t(x) = C_0 t^{-n/2} \phi_0(x) \delta^{-1/2}(x) e^{-|x|^2/(4t)}.
\]

Let \( A \) be the subalgebra of \( L^1(AN) \) (with respect to right-invariant Haar measure) generated by \( p_t \), \( t > 0 \).

1. **Theorem.** The operator \( T \) given by the formula

\[
(Tf)(x) = C_0(4\pi)^{n/2} \phi_0(x) \delta^{-1/2}(x) f(|x|)
\]

is an isometric algebra isomorphism between \( L^1_{\text{rad}}(\mathbb{R}^n) \) and \( A \).

**Proof.** Let \( V \) denote the vector space of all linear combinations of \( q_t \), \( t > 0 \), where \( q_t \) is the heat kernel on \( \mathbb{R}^n \). By the formula above, \( T \) restricted to \( V \) is an isomorphism of algebras. Moreover, for all \( f \in V \) we have

\[
\int_{AN} Tf = \int_{\mathbb{R}^n} f.
\]

On the other hand, if we denote by \( E \) the space \( L^1_{\text{rad}}(\mathbb{R}^n) \) for sufficiently large \( C \), then \( T \) is continuous from \( E \) into \( L^1(AN) \). Since \( V \) is dense in \( E \), it follows that \( T(E) \subset A \) and for all \( f \in E \) we have

\[
\int_{AN} Tf = \int_{\mathbb{R}^n} f.
\]

Decomposing a function \( f \) in \( E \) into its positive and negative part (which of course belong to \( E \)) we have

\[
\|f\|_{L^1_{\text{rad}}} = \int_{\mathbb{R}^n} f_+ + \int_{\mathbb{R}^n} f_- = \int_{AN} Tf_+ + \int_{AN} Tf_- = \|Tf\|_{L^1(AN)}.
\]

Now the closure of \( T|_V \) is an isometry of \( L^1_{\text{rad}}(\mathbb{R}^n) \) with \( A \). Obviously this closure is equal to \( T \), which ends the proof.

2. **Corollary.** The spectrum of \( \Delta \) on \( L^p(AN) \), \( \infty > p \geq 1 \), does not depend on \( p \) and equals \( R_+ \).

Let \( R_\alpha(x) = (1-x)_+^\alpha \). We put

\[
S_\alpha f = \sup_{t>0} |R_\alpha(-t\Delta)f|
\]

where \( R_\alpha(-t\Delta) \) is well defined by the spectral theorem.
3. **Corollary.** For \( \alpha > (n - 1)/2 \) the maximal operator \( S_\alpha \) for Riesz means of order \( \alpha \) is of weak type \((1, 1)\).

**Proof.** Let \( Mf = \sup_{t > 0} t^{-1} \int_0^t |f| * p_s \, ds \). On \( \mathbb{R}^n \), \( S_\alpha \) is majorized by \( M \). \( T \) preserves pointwise majorization of kernels, so this majorization holds also on \( AN \). But \( M \) is of weak type \((1, 1)\) by the Dunford-Schwartz maximal ergodic theorem.

4. **Corollary.** Let \( k > (n - 1)/2, \ t > 0, \ \varepsilon > 0, \ F \in C^k(\mathbb{R}), \) and
\[
\sup_{\lambda > 0} (1 + \lambda)^{l+\varepsilon} |F^{(l)}(\lambda)| < \infty, \quad l = 0, \ldots, k.
\]

Then the operator \( F(-t\Delta) \) (defined by the spectral theorem on \( L^1(\cap L^2) \)) is bounded on \( L^1(AN) \).

**References**