PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 2, February 1993

INTEGRABLE MEAN PERIODIC FUNCTIONS ON LOCALLY COMPACT ABELIAN GROUPS

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(Communicated by J. Marshall Ash)

ABSTRACT. Let G be a locally compact abelian group with a Haar measure λ_G. A function f on G is said to be mean-periodic if there exists a nonzero finite regular measure μ of compact support on G such that $f * \mu = 0$. It is known that there exist no nontrivial integrable mean periodic functions on $\mathbf{R}^n$. We show that there exist nontrivial integrable mean periodic functions on G provided G has nontrivial proper compact subgroups. Let $f \in L_1(G)$ be mean periodic with respect to a nonzero finite measure μ of compact support. If $\mu(G) \neq 0$ and $\lambda_G(\operatorname{supp}(\mu)) > 0$, then there exists a compact subgroup K of G such that $f * \lambda_K = 0$, i.e., f is mean periodic with respect to λ_K, where λ_K denotes the normalized Haar measure of K. When G is compact, abelian and meterizable, we show that there exists continuous (hence integrable and almost periodic) functions on G that are not mean periodic.

1. INTRODUCTION

The theory of mean periodic functions on the real line $\mathbf{R}$ was initiated by Delsarte [2, 3] and were analyzed in detail by Schwartz [11], Kahane [7], and Malgrange [8] on $\mathbf{R}^n$ (see also Berenstein and Taylor [1] for mean periodic functions on $\mathbf{R}^n$ and other references). Ehrenpreis and Mautner [5] started the study of mean periodic functions on the group $\mathrm{SL}(2, \mathbf{R})$. It is well known that there does not exist nontrivial integrable mean periodic functions on $\mathbf{R}^n$. However, Sitaram [12] has given an example of a nontrivial integrable mean periodic function on $\mathrm{SL}(2, \mathbf{R})$. The aim of this note is to analyze mean periodic integrable functions on locally compact abelian groups.

Let G be a locally compact abelian group with a Haar measure λ_G. We shall denote by $M(G)$ the set of all finite regular measures on G, and $M_c(G)$ will denote the set of all $\mu \in M(G)$, μ with compact support. $L_1(G)$ denotes the space of all integrable functions on G.

1.1. **Definition.** We call $f \in L_1(G)$ *mean periodic* if there exists $\mu \in M_c(G)$, $\mu \neq 0$, such that $f * \mu = 0$; here $*$ denotes the convolution operation. We shall also say f is mean periodic with respect to μ.

Received by the editors September 24, 1990 and, in revised form, June 6, 1991.
1991 *Mathematics Subject Classification*. Primary 43A70; Secondary 42A75.
Key words and phrases. Locally compact abelian groups, mean periodic functions on groups, almost periodic functions, character group, annihilator, Fourier transform.

1.2. **Example.** Let K be any compact abelian group, and let $\widehat{K}$ denote the dual group of K. Let $\gamma \in \widehat{K}$ be any nontrivial character. Then for $x \in K$,

$$(\gamma * \lambda_K)(x) = \int \langle x - y, \gamma \rangle \, d\lambda_K(y) = \langle x, \gamma \rangle \int \langle \overline{y, \gamma} \rangle \, d\lambda_K(y).$$

Also $\int_K \langle x - y, \gamma \rangle \, d\lambda_K(x) = \int_K \langle \overline{y, \gamma} \rangle \, d\lambda_K(y)$. Since γ is nontrivial, we have $\int_K \langle \overline{y, \gamma} \rangle \, d\lambda_K(y) = 0$, i.e., $\gamma * \lambda_K = 0$.

Thus every nontrivial $\gamma \in \widehat{K}$ is mean periodic. Also if f is any constant function on K and K has at least two distinct points $k_1, k_2 \in K$, then $f * (\delta_{k_1} - \delta_{k_2}) = 0$, where δ_k denotes the unit mass at k.

1.3. **Example.** Let $\mathbf{T}$ denote the circle group and $f(x) = x$, $x \in \mathbf{T}$. We have already seen in Example 1.2 that f is mean periodic with respect to $\lambda_{\mathbf{T}}$, the normalized Haar measure on $\mathbf{T}$. Let ν denote the measure on $\mathbf{T}$ given by $d\nu(x) = 1/x d\lambda_{\mathbf{T}}(x)$. Then it is easy to see that $f * \nu = 0$, i.e., f is mean periodic with respect to ν.

1.4. **Example.** Let G be a locally compact abelian group such that G has a nontrivial proper compact subgroup K. Let $\Lambda = \{\gamma \in \widehat{G} | \langle x, \gamma \rangle = 0$ for all $x \in K\}$ denote the annihilator of K. Then Λ is a proper closed subgroup of $\widehat{G}$. Let $\gamma_0 \in \widehat{G} \backslash \Lambda$. Choose a neighbourhood U of γ_0 and V of Λ such that $U \cap V = \varnothing$. Then $\gamma_0 \in U \subseteq V^c$. Now choose $f \in L_1(G)$ such that $\hat{f}(\gamma_0) = 1$ and $\hat{f}(\gamma) = 0$ for every $\gamma \in U^c$ (see [11, Theorem 2.6.2]). Thus $\hat{f}(\gamma) = 0$ for every $\gamma \in \Lambda$, and hence $f * \lambda_K = 0$ (see [10, Theorem 2.7.4]). Thus, we have shown that there exists nontrivial $f \in L_1(G)$ that are mean periodic provided G has nontrivial proper compact subgroups.

We next analyze integrable mean periodic functions on compact abelian groups.

2. Integrable mean periodic functions on compact abelian groups

Let K be a nontrivial compact abelian group. In view of Example 1.2, one may ask: is every $f \in L_1(K)$ mean periodic? We shall see that this is not so in general. However, every $f \in L_1(K)$ can be written as a sum of two integrable mean periodic functions on K.

2.1. **Proposition.** *Let K be a compact abelian group and $f \in L_1(K)$. Then $f = g + h$, where g, h are mean periodic functions in $L_1(K)$.*

Proof. Let λ_K denote the normalized Haar measure of K. Let $h = f * \lambda_K$ and $g = f - h$. Then clearly $g * \lambda_K = 0$ and $h(x) = \int_K f \, d\lambda_K$ for every $x \in K$. Thus g, h are mean periodic functions in $L_1(K)$ and $f = g + h$.

2.2. *Remark.* The above proposition says that $L_1(K) = S + C$, where $S, C \subseteq L_1(K)$ and every $f \in S \cup C$ is a mean periodic function. In fact, C is the space of all constant functions on K. One can say something more about S. We claim that $S = \{g \in L_1(K) | g * \lambda_K = 0\}$ is a maximal ideal of $L_1(K)$. To see this we first note that S is a closed subalgebra of $L_1(K)$. Next, if $\gamma_0 \in \widehat{K}$ and $\gamma_0 \not\equiv 1$, we choose a neighbourhood W of γ_0 such that $1 \notin W$. Now we choose $g \in L_1(K)$ such that $\hat{g}(\gamma_0) = 1$ and $\hat{g}(\gamma) = 0$ for every $\gamma \in W^c$ (see [10, Theorem 2.6.2]). Thus, $\hat{g}(\gamma_0) = 1$ and $\hat{g}(1) = 0$. But then

$(g * \lambda_K)(x) = \int_K g(x-y)\,d\lambda_K(y) = \int_K g(y)\,d\lambda_K(y) = \hat{g}(1) = 0$. Thus, for $g \in S$, $\hat{g}$ separates points of $\widehat{K}$. Hence by a corollary [10, p. 232] either $S = L_1(K)$ or S is a maximal ideal of $L_1(K)$. Since $S \neq L_1(K)$, S is a maximal ideal of $L_1(K)$.

We show next that not every $f \in L_1(K)$ is mean periodic.

2.3. **Proposition.** *Let K be a compact metrizable abelian group and let $\widehat{K}$ be its dual group. Let $f = \sum_{\gamma \in \widehat{K}} a_\gamma \gamma$, where $a_\gamma \in \mathbf{C}$ and the series converges uniformly on K. Then there exists a nonzero finite measure μ on K such that $f * \mu = 0$ iff $a_\gamma = 0$ for at least one $\gamma \in \widehat{K}$.*

Proof. Clearly $a_\gamma = \hat{f}(\gamma)$ for every $\gamma \in \widehat{K}$. Further $f * \mu = \sum_{\gamma \in \widehat{K}} a_\gamma \hat{\mu}(\gamma)\gamma$. Now if $f * \mu = 0$, then for every $\gamma \in \widehat{K}$, $0 = (f \hat{*} \mu)(\gamma) = \hat{f}(\gamma)\hat{\mu}(\gamma) = a_\gamma \hat{\mu}(\gamma)$. Since $\mu \neq 0$, there exists at least one $\gamma_0 \in \widehat{K}$ such that $\hat{\mu}(\gamma_0) \neq 0$. Thus, $a_{\gamma_0} = 0$. Conversely, if $a_{\gamma_0} = 0$ for some $\gamma_0 \in \widehat{K}$, then $f * \mu = 0$ for $d\mu(x) = \gamma_0(x) d\lambda_K(x)$.

2.4. **Corollary.** *Let K be a compact metrizable abelian group and let $\widehat{K}$ be its dual group. Let $f = \sum_{\gamma \in \widehat{K}} a_\gamma \gamma$ where $0 \neq a_\gamma \in \mathbf{C}$ for every $\gamma \in \widehat{K}$ and the series converges uniformly. Then f is a continuous hence integrable (and almost periodic) function on K and f is not mean periodic.*

2.5. *Remark.* Corollary 2.4 extends a remark of Kahane [7] that on $\mathbf{R}^n$ there exist almost periodic functions that are not mean periodic.

3. Integrable mean periodic functions on locally compact abelian groups

Let G be a locally compact abelian group. As seen in Example 1.4, if G has a nontrivial compact subgroup K, then there exists $0 \neq f \in L_1(G)$ that is mean periodic. To be more precise, $f * \lambda_K = 0$, where λ_K denotes the normalized Haar measure of K. We ask the question: if $f \in L_1(G)$ and is mean periodic, under what conditions can one say $f * \lambda_K = 0$ for some compact subgroup K of G? The answer is given by the following

3.1. **Theorem.** *Let G be a locally compact abelian group and $f \in L_1(G)$ be mean periodic with respect to $0 \neq \mu \in M_c(G)$. If $\mu(G) \neq 0$ and $\lambda_G(\operatorname{supp}(\mu)) > 0$, then there exists a compact subgroup K of G such that $f * \lambda_K = 0$.*

To prove the theorem, we need some preliminary results. Let G be a locally compact abelian group and K a compact subgroup of g. Let λ_G, λ_K, and $\lambda_{G/K}$ denote the Haar measures of G, K, and G/K, respectively, such that $\lambda_K(K) = 1$ and such that for every $f \in L_1(G)$

$$\int_G f\,d\lambda_G = \int_{G/K} \left(\int_K f(x+y)\,d\lambda_K(y) \right) d\lambda_{G/K}.$$

Let $M(G)$ and $M(G/K)$ denote the space of all bounded regular measures on G and G/K, respectively. Following [10, §2.7], we define

$$T : M(G) \to M(G/K),$$

$$(T\mu)(f) = \int_G (f \circ \Pi)(x)\,d\mu(x), \qquad f \in C_0(G/K),\ \mu \in M(G),$$

where $\Pi\colon G \to G/K$ is the natural homomorphism and $C_0(G/K)$ denotes the space of all continuous functions on G/K vanishing at ∞. Given μ, $T\mu \in M(G/K)$ is well defined and

$$\int_G (f \circ \Pi)(x)\, d\mu(x) = \int_{G/K} f\, d(t\mu)$$

for every bounded Borel function f on G/K. The following properties of T are easy to check.

3.2. **Proposition.** (i) *If $\mu \in M(G)$ is such that either $\mu(G) \neq 0$ or $\mu \neq 0$ and μ is K-invariant, then $T(\mu) \neq 0$.*

(ii) *If $\mu \in M(G)$ has compact support, then so does $T\mu$.*

(iii) *For $f \in L_1(G)$, $Tf \in L_1(G/K)$ and*

$$(Tf)(\Pi(x)) = \int_K f(x+k)\, d\lambda_K(k), \qquad x \in G.$$

(iv) *For $f \in L_1(G)$ and $\mu \in M_c(G)$, $T(f * \mu) = (Tf) * (\mu)$.*

(v) *If $f \in L_1(G)$ is K-invariant and $(Tf)(\Pi(x)) = 0$ for every $x \in G$, then $f(x) = 0$ for every $x \in G$.*

Our next lemma is motivated by the lemma of [9, p. 311]

3.3. **Lemma.** *Let $f \in L_1(G)$ be a continuous function and $\mu \in M_c(G)$ be such that $\mu(G) \neq 0$, $\lambda_G(\operatorname{supp}(\mu)) > 0$, and $f * \mu = 0$. Then there exists a compact subgroup K of G such that $f * \lambda_K = 0$.*

Proof. Let $E = \operatorname{supp}(\mu)$. Then E is a compact subset of G and $\lambda_G(E) > 0$. Let G_0 be the subgroup of G generated by E. Then G_0 is a locally compact compactly generated abelian group and, by the structure theorem (Hewitt and Ross [6]), G_0 is topologically isomorphic to $\mathbf{R}^n \times \mathbf{Z}^k \times K$, where $\mathbf{Z}$ denotes the integer group, n, k, are nonnegative integers, and K is a compact abelian group. We assume without loss of generality that $G_0 = \mathbf{R}^n \times \mathbf{Z}^k \times K$. To prove the lemma, we first assume that f is K-invariant, i.e., $f(x+k) = f(x)$ for all $x \in G$, $k \in K$. We shall show $f * \lambda_K = f \equiv 0$. Choose $x_0 \in G$ arbitrarily and fix it. Let $f_{x_0}(x) = f(x + x_0)$, $x \in G$. Then f_{x_0} is a K-invariant continuous integrable function on G and $f_{x_0} * \mu = 0$. Since μ is concentrated on G_0, this gives $f_{x_0} * \mu = 0$ on G_0. Using Proposition 3.2,

$$(*) \qquad T(f_{x_0} * \mu) = T(f_{x_0}) * T(\mu) = 0 \quad \text{on } \mathbf{R}^n \times \mathbf{Z}^k.$$

Taking the Fourier transform, we have

$$(**) \qquad (\widehat{Tf_{x_0}})(\gamma) \cdot (\widehat{T(\mu)})(\gamma) = 0 \quad \text{for every } \gamma \in \mathbf{R}^n \times \mathbf{T}^k.$$

Since $\mu \in M_c(G)$ and $\mu(G_0) \neq 0$, again by Proposition 3.2, $0 \neq T\mu \in M_c(G_0/K) = M_c(\mathbf{R}^n \times \mathbf{Z}^k)$. Thus $(\widehat{T\mu})(\gamma) \neq 0$ for almost all $\gamma \in \mathbf{R}^n \times \mathbf{T}^k$. Hence $(**)$ gives $(\widehat{Tf_{x_0}})(\gamma) = 0$ for almost all $\gamma \in \mathbf{R}^n \times \mathbf{T}^k$. Thus, $(Tf_{x_0})(\overline{x}) = 0$ for almost all $\overline{x} \in \mathbf{R}^n \times \mathbf{Z}^k$. Since f_{x_0} is continuous, this implies $Tf_{x_0}(\overline{x}) = 0$ for all $\overline{x} \in G_0/K = \mathbf{R}^n \times \mathbf{Z}^k$. Again, since f_{x_0} is K-invariant, by Proposition 3.2, this implies that $f_{x_0}(x) \equiv 0$. To prove the proposition in the general case, we consider the function $f * \lambda_K$ and apply the above case to conclude $f * \lambda_K \equiv 0$.

Proof of Theorem 3.1. Let $f \in L_1(G)$ and $\mu \in M_c(G)$ be such that $\mu(G) \neq 0$ and $f * \mu = 0$. Let $\phi \in C_c(G)$, the space of continuous functions on G with

compact support. Then $(f*\mu)*\phi = (f*\phi)*\mu = 0$ on G. Since $f*\phi \in L_1(G)$ is a continuous function, by Lemma 3.3, $(f*\phi)*\lambda_K = 0$ for all $\phi \in C_c(G)$. Thus, $(f*\lambda)*\phi = 0$ for all $\phi \in C(G)$ and hence $f*\lambda = 0$. This proves the theorem.

3.4. *Remark.* The condition that $\mu(G) \neq 0$ in Theorem 3.1 can be replaced by a weaker condition: $\mu*\lambda_K \neq 0$, where K is the maximal compact subgroup of G_0, the subgroup of G generated by $\operatorname{supp}(\mu)$. To see this we note that in the proof of Lemma 3.3, the equation $(*)$ can be replaced by

$$T(f_{x_0}*\mu*\lambda_K) = T(f_{x_0})*T\tilde{\mu} = 0,$$

where $\tilde{\mu} = \mu*\lambda_K$. Since $\tilde{\mu} \neq 0$ and is K-invariant, by Proposition 3.2, $T(\tilde{\mu}) \neq 0$ has compact support. Now proceeding as in Lemma 3.3, we will get Lemma 3.1 under the condition $\mu*\lambda_K \neq 0$. Hence Theorem 3.1 will hold under the conditions: $\mu*\lambda_K \neq 0$ and $\lambda_G(\operatorname{supp}(\mu)) > 0$.

In the next theorem, we give some equivalent versions of the conclusions of Theorem 3.1.

3.5. **Theorem.** *Let G be a locally compact abelian group. Let $f \in L_1(G)$ be mean periodic with respect to a measure $\mu \in M_c(G)$, where $\mu(G) \neq 0$ and $\lambda_G(\operatorname{supp}(\mu)) > 0$. Then the following hold:*

(i) *f is mean periodic with respect to the normalized Harr measure λ_K of some compact subgroup K of G.*

(ii) *There exists a compact subgroup K of G such that $\hat{f}(\gamma) = 0$ for all $\gamma \in \operatorname{Ann}(K)$.*

(iii) *There exists a compact subgroup K of G such that*

$$f \in \operatorname{cl}\{g \in L_1(G) \mid \operatorname{Ann}(K) \subset \operatorname{Int} Z(g)\},$$

where cl *denotes the closure in L_1-norm, $Z(g) = \{\gamma \in \widehat{G} \mid \hat{g}(\gamma) = 0\}$, and* $\operatorname{Int} Z(g)$ *denotes the interior of $Z(g)$.*

Proof. Under the given conditions, clearly (i) holds by Theorem 3.1. We shall show (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). Implication (i) $\Rightarrow$ (ii) is Theorem 2.7.4 of [10]. Suppose now (ii) holds. Since $\Lambda = \operatorname{Ann}(K)$ is a closed subgroup of G, by Theorem 7.5.2(d) of [10], it follows that for every $\varepsilon > 0$, there exists $g \in L_1(G)$ such that $\hat{g}$ has compact support disjoint from Λ and $\|f - f*g\| < \varepsilon$. Further $f*g \in I_0(\Lambda) = \operatorname{cl}\{\phi \in L_1(G) \mid \Lambda \subset \operatorname{Int} Z(\phi)\}$ by Theorem 7.5.2(a) of [10]. But then clearly $f \in I_0(\Lambda)$. This proves (ii) $\Rightarrow$ (iii). Also if $f \in I_0(\Lambda)$, clearly $\Lambda \subset Z(f)$ and thus (iii) $\Rightarrow$ (ii).

Acknowledgment

The authors would like to thank the referee for the arguments presented in Proposition 2.3.

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