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THE TIME SPENT BY THE WIENER PROCESS IN A NARROW TUBE BEFORE LEAVING A WIDE TUBE

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ABSTRACT. The almost sure behavior of the time spent by the Wiener process in a "small interval" before first leaving a "bigger interval" is investigated.

1. INTRODUCTION

In this paper we investigate the almost sure behavior of the time spent by the Wiener process in a "small" interval, before first leaving a "bigger interval". Let $W(t)$ $(t \geq 0)$ be an ordinary Wiener process, and define

$$m_b = \inf\{s \geq 0, \ W(s) \notin [-b, b]\}.$$

Moreover, let

$$\mu_x(\alpha, b) = \mu_x\{0 \leq s; \ |W(s)| \leq \alpha b, \ s \leq m_b\}$$

be the Lebesgue measure of the time spent by $W(s)$ in $[-\alpha b, \alpha b]$ until m_b, where the lowercase subscript x indicates that $W(0) = x$. We will prove the following theorems.

Theorem 1. *For any* $0 < \alpha \leq 1$ *and for any fixed* x,

$$\overline{\lim_{b\to\infty}} \frac{\mu_x(\alpha, b) c_0^2(\alpha)}{2\alpha^2 b^2 \log\log b} = 1 \tag{1.1}$$

where $c_0(\alpha)$ *is the unique root of the equation*

$$u \tan u = \alpha/(1-\alpha)$$

in the interval $(0, \pi/2]$.

Remark 1. Observe that the case $\alpha = 1$ states that

$$\overline{\lim_{b\to\infty}} \frac{m_b \cdot \pi^2}{8b^2 \log\log b} = 1,$$

which is a natural reformulation of Chung's law of iterated logarithm for the maximum of the Wiener process.

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Theorem 2. *For any* $0 < \alpha < 1$ *and for any fixed* x,

$$\varliminf_{b\to\infty} \frac{\mu_x(\alpha, b)}{\alpha^2 b^2} 2 \log\log b = 1 . \tag{1.2}$$

Remark 2. Theorem 2 can be proved (just like Theorem 1) based on the asymptotic behaviour of the Laplace transform of $\mu_x(\alpha, b)$ and then applying the Erdös-Rényi extension of the Borel-Cantelli lemma, however, a much simpler proof reveals that it can be deduced from the law of iterated logarithm.

2. Proof of the results

In our first lemma, we give the Laplace-Stieltjes transform of

$$F_x(t) = P(\mu_x(\alpha, b) < t)$$

where the lowercase subscript x means again that $W(0) = x$.

Lemma 1. *Let, for* $0 < \alpha \le 1$,

$$H(x) = H(x, \lambda) = E(\exp\{-\lambda\mu_x(\alpha, b)\})$$

and denote $q = \alpha b$. *Then*

$$H(x) = \begin{cases} \dfrac{\cosh(q\sqrt{2\lambda}) - (x+q)\sqrt{2\lambda}\sinh(q\sqrt{2\lambda})}{\cosh(q\sqrt{2\lambda}) + (b-q)\sqrt{2\lambda}\sinh(q\sqrt{2\lambda})} & \textit{if } -b \le x \le -q, \\ \dfrac{\cosh(x\sqrt{2\lambda})}{\cosh(q\sqrt{2\lambda}) + (b-q)\sqrt{2\lambda}\sinh(q\sqrt{2\lambda})} & \textit{if } -q \le x \le q, \\ \dfrac{\cosh(q\sqrt{2\lambda}) + (x-q)\sqrt{2\lambda}\sinh(q\sqrt{2\lambda})}{\cosh(q\sqrt{2\lambda}) + (b-q)\sqrt{2\lambda}\sinh(q\sqrt{2\lambda})} & \textit{if } q \le x \le b. \end{cases} \tag{2.1}$$

Remark 3. Observe the symmetry of $H(x)$, $H(x) = H(-x)$, a fact that will be used frequently without mentioning it.

Remark 4. The special cases $x = 0$ and $x = q$ are treated in [7, Theorem 4.2.16].

Remark 5. The method needed to compute the Laplace-Stieltjes transform is very general and goes back to Kac [6] (see also [7, Theorem 4.2.11, p. 77; 1; 5]).

Theorem. *Let* $f(x)$, $x \in [a, b]$, *be a piecewise continuous function and let* $A(t) = \int_0^t f(W(s))\,ds$. *Then the function*

$$H(x) = E_x(\exp\{-A(m_b)\})$$

(*where* $E_x(\ )$ *means expectation under the condition that* $W(0) = x$) *is the unique continuous solution of the problem*

$$\begin{gathered} \tfrac{1}{2}H''(x) - f(x)H(x) = 0, \\ H(-b+0) = H(b-0) = 1, \qquad x \in (-b, b). \end{gathered} \tag{2.2}$$

Proof of Lemma 1. We apply the above theorem for

$$f(x) = \begin{cases} \lambda & \text{if } |x| \le q, \\ 0 & \text{otherwise}. \end{cases}$$

Then we get the general solution of (2.2) in the following form:

$$H(x) = H(x\,,\lambda) = \begin{cases} vx + z & \text{if } -b < x \le -q\,, \\ \gamma \exp(\sqrt{2\lambda}x) + \beta \exp(-\sqrt{2\lambda}x) & \text{if } |x| \le q\,, \\ lx + r & \text{if } q \le x \le b\,. \end{cases}$$

Using the continuity of $H(x)$ and $H'(x)$ and the boundary conditions in (2.2), we get a system of equations for v, z, γ, β, l, and r. The unique solution of this system produces (2.1). □

Consequently the ordinary Laplace transform of $F_x(t)$ is $H(x\,,\lambda)/\lambda$.

To investigate the asymptotic behavior of $F_x(t)$, when $t \to \infty$, we have to study $H(x\,,\lambda)/\lambda$ when $\lambda \to 0$.

The following lemma is most probabily well known; however, we give a brief sketch of its proof.

Lemma 2. *The function*

$$U(\lambda) = \cosh(q\sqrt{2\lambda}) + (b-q)\sqrt{2\lambda}\sinh(q\sqrt{2\lambda})$$

(*where* $q = \alpha b$, $0 < \alpha < 1$) *has roots only on the negative real axis and among them the root of the smallest absolute value is*

$$\lambda_0 = -c_0^2(\alpha)/2\alpha^2 b^2$$

where $c_0(\alpha)$ *is the unique root of* $u \tan u = \alpha/(1-\alpha)$ *in the interval* $0 < u \le \pi/2$.

Proof. To find the roots on the negative real axis, let $2\lambda = -z^2$ where $z > 0$. Applying the well-known formulas $\sinh(ix) = i \sin x$ and $\cosh ix = \cos x$, our equation $U(\lambda) = 0$ transforms into

$$zq\tan(zq) = \frac{q}{b-q} \qquad \left(= \frac{\alpha}{1-\alpha}\right).$$

Now let $u = zq$ (where $z > 0$, $q > 0$, hence $u > 0$). Then we want the positive solutions of

$$u \tan u = \alpha/(1-\alpha)\,. \tag{2.3}$$

Equation (2.3) has one root in each of the intervals $(k\pi\,, k\pi + \pi/2]$ $(k \ge 0)$. Denote the smallest one by $c_0(\alpha)$, $0 < c_0(\alpha) \le \pi/2$. Then $\lambda_0 = -c_0^2(\alpha)/2q^2$ is the negative real root of $U(\lambda)$ of smallest absolute value.

To see that all roots of $U(\lambda)$ are on the negative real axis, first observe that $U(\lambda) > 0$ when $\lambda > 0$ and real, so we have only to show that no root of the form

$$q\sqrt{2\lambda} = A + Bi\,, \qquad A \ne 0\,,\ B \ne 0 \text{ real}\,, \tag{2.4}$$

exists. Arguing indirectly, if there were a root of the form (2.4), then from $U(\lambda) = 0$ we would get

$$(A + Bi)\tanh(A + Bi) = -q/(b-q)\,.$$

Elementary calculations lead to the point that $(A + Bi)[e^{4A} - 1 + i(2e^A \sin 2B)]$ has to be real, which is the same as (being $A \ne 0$, $B \ne 0$)

$$B(e^{4A} - 1) + 2Ae^{2A}\sin 2B = 0\,,$$

implying that

$$\frac{\sin 2B}{2B} = -\frac{\sinh 2A}{2A}. \tag{2.5}$$

But for all $x \neq 0$, $|\frac{\sin x}{x}| < 1$ and $|\frac{\sinh x}{x}| > 1$, which proves that (2.5) cannot be true for any real $A \neq 0$, $B \neq 0$. □

Based on Lemma 2, using standard methods (see, e.g., [3]) we get the following result:

Let $x = \delta b$, $|\delta| < 1$ and $\widetilde{F}_x(t) = P(\frac{\mu_x(\alpha, b)}{b^2} < t)$.

Lemma 3. *For* $t \to \infty$, $0 < \alpha < 1$,

$$1 - \widetilde{F}_x(t) \sim c_1(\alpha) \cdot c_2(\alpha, x) \exp\left\{-\frac{c_0^2(\alpha)t}{2\alpha^2}\right\} \tag{2.6}$$

where $c_0(\alpha)$ *(as before) is the unique root of* $u \tan u = \alpha/(1-\alpha)$ *in* $(0, \pi/2)$,

$$c_1(\alpha) = \frac{2}{1/(1-\alpha) + c_0^2(\alpha)(1-\alpha)/\alpha}, \tag{2.7}$$

and

$$c_2(\alpha, x) = \begin{cases} (1+\delta)/(1-\alpha) & \text{if } -1 < \delta \leq -\alpha, \\ \cos\left(c_0(\alpha)\frac{\delta}{\alpha}\right) / \cos c_0(\alpha) & \text{if } |\delta| \leq \alpha, \\ (1-\delta)/(1-\alpha) & \text{if } \alpha \leq \delta \leq 1 \end{cases}$$

in the case that $\alpha = 1$,

$$1 - \widetilde{F}_x(t) \sim \frac{4}{\pi} \cos\left(\frac{\delta\pi}{2}\right) \exp\left\{-\frac{\pi^2 t}{8}\right\} \quad \text{as } t \to \infty.$$

(Here $\sim$ *means that the ratio of the two sides* $\to 1$ *as* $t \to \infty$.*)*

Remark 7. Observe that $0 < c_2(\alpha, x) \leq 1/\cos c_0(\alpha)$.

Proof of Theorem 1. Convergent part. Let $b_k = \theta^k$ where $\theta > 1$ will be specified later on. For any arbitrary $\varepsilon > 0$, define

$$A_x(k) = \left\{\mu_x(\alpha, b_k) < \frac{2\alpha^2 b_{k-1}^2 \log\log b_{k-1}}{c_0^2(\alpha)}(1+\varepsilon)\right\}. \tag{2.8}$$

x being fixed, let k be big enough such that $x \in (-b_k, b_k)$. Applying Lemma 3

$$\begin{aligned} P(A_x(k)) &\leq 2c_1(\alpha) \exp\left\{-\frac{b_{k-1}^2}{b_k^2} \log\log b_{k-1}(1+\varepsilon)\right\} \\ &\leq 2c_1(\alpha) \exp\left\{-\frac{1+\varepsilon}{\theta^2}(\log(k-1) + \log\log\theta)\right\} \\ &\leq 2c_1^*(\alpha) \exp\left\{-\frac{1+\varepsilon}{\theta^2} \log(k-1)\right\} \\ &\leq 2c_1^*(\alpha) \frac{1}{(k-1)^{(1+\varepsilon)/\theta^2}}, \end{aligned} \tag{2.9}$$

where $c_1^*(\alpha)$ is a constant depending on α, θ, ε.

Now choose θ to be so small that

$$\theta^2 < 1 + \varepsilon . \tag{2.10}$$

Then

$$\sum_{k=1}^{\infty} P(A_x(k)) < \infty . \tag{2.11}$$

Consequently, for $k > k_0(\omega)$

$$\mu_x(\alpha , b_k) \leq (1+\varepsilon) 2\alpha^2 b_{k-1}^2 \log\log b_{k-1}/c_0^2(\alpha) .$$

Let $b_{k-1} \leq b < b_k$. Then, obviously

$$\begin{aligned} \mu_x(\alpha , b) \leq \mu_x(\alpha , b_k) &< (1+\varepsilon)\frac{2\alpha^2 b_{k-1}^2 \log\log b_{k-1}}{c_0^2(\alpha)} \\ &\leq (1+\varepsilon)\frac{2\alpha^2 b^2 \log\log b}{c_0^2(\alpha)} , \end{aligned}$$

which proves the convergent part of the theorem.

Divergent part. Let $b_k = \theta^k$, where $\theta > 1$ will again be specified later on. For an arbitrary $\varepsilon > 0$, define

$$A_x(k) = \{\mu_x(\alpha , b_k) > u_k\} , \qquad k = 1 , 2 , \dots , \tag{2.12}$$

where

$$u_k = (1-\varepsilon) 2\alpha^2 b_k^2 \log\log b_k / c_0^2(\alpha) .$$

Moreover, let

$$\overline{A}_x(k+1) = \{\mu_x(\alpha , [b_k , b_{k+1})) > u_{k+1}\} \tag{2.13}$$

where

$$\mu_x(\alpha , [b_k , b_{k+1})) = \mu_x(s ; m_{b_k} \leq s < m_{b_{k+1}} , |W(s)| \leq \alpha b_{k+1}) . \tag{2.14}$$

Now choose $\theta > 1$ so large that $\alpha\theta > 1$. This implies that $\alpha b_{k+1} > b_k$. Then, as in the first part, x being fixed, for k big enough for $k > k_0$, $|x| < b_k$, and therefore

$$P(\overline{A}_x(k+1)) = P(\mu_x(\alpha , [b_k , b_{k+1})) > u_{k+1}) = P(\mu_{b_k}(\alpha , b_{k+1}) > u_{k+1}) . \tag{2.15}$$

Observe that in (2.15) the symmetry of $\mu_{b_i}(\alpha , b_j)$ plays an important role. Namely, $\mu_{b_i}(\alpha , b_j)$ and $\mu_{-b_i}(\alpha , b_j)$ have the same distribution. According to Lemma 3, if $0 < \alpha < 1$, we have (for k big enough)

(2.16)

$$\begin{aligned} P(\mu_{b_k}(\alpha , b_{k+1}) > u_{k+1}) &\geq \tfrac{1}{2} c_1(\alpha)(\cos(c_0(\alpha)))^{-1} \exp\{-(1-\varepsilon)\log\log b_{k+1}\} \\ &= \tfrac{1}{2} c_1(\alpha)(\cos(c_0(\alpha)))^{-1}/((k+1)\log\theta)^{1-\varepsilon} . \end{aligned}$$

In the case that $\alpha = 1$, (2.16) is also valid except that for the constant term before the exponential. Observe that the events $\overline{A}_x(k)$ are independent for $k > k_0$. (2.16) implies, via the Borel-Cantelli lemma, that $P(\overline{A}_x(k) \text{ i.o.}) = 1$. But obviously

$$\overline{A}_x(k) \subset A_x(k) = \{\mu_x(\alpha , b_k) > u_k\} ,$$

which implies the second half of our theorem. □

Proof of Theorem 2. Let $W(t)$ be a standard Wiener process with $W(0) = x$, and recall that

$$m_b = \inf\{t \geq 0,\ W(t) \notin [-b, b]\}. \tag{2.17}$$

First we prove that

$$\liminf_{b\to\infty} \frac{2\mu_x(\alpha, b)\log\log b}{\alpha^2 b^2} \geq 1 \quad \text{a.s.} \tag{2.18}$$

Observe that $\mu_x(\alpha, b) \geq m_{\alpha\cdot b}$ (if $b \geq x$, which we might assume). Hence it is enough to prove (as $0 < \alpha < 1$ is fixed, and $b \to \infty$) that

$$\liminf_{b\to\infty} \frac{2m_b \log\log b}{b^2} \geq 1. \tag{2.19}$$

We will show that (2.19) follows from the law of iterated logarithm. Denote $M(t) = \sup_{0\leq s\leq t} |W(s)|$. Then, for any $\varepsilon > 0$ and t large enough

$$M(t) \leq ((2+\varepsilon)t\log\log t)^{1/2}. \tag{2.20}$$

This clearly implies that again for t large enough

$$M^2(t)/\log\log M(t) \leq (2+2\varepsilon)t. \tag{2.21}$$

Now apply (2.21) for $t = m_b$ to get

$$b^2/\log\log b \leq (2+2\varepsilon)m_b. \tag{2.22}$$

$\varepsilon > 0$ being arbitrary, this clearly implies (2.18).

Turning to the converse part of the theorem, first observe that for any $\varepsilon > 0$,

$$m_b \leq b^2/((2-\varepsilon)\log\log b) \tag{2.23}$$

if b is large enough. To get (2.23), start with the converse part of (2.20),

$$M(t) \geq ((2-\varepsilon)t\log\log t)^{1/2}, \tag{2.24}$$

and apply it for m_b. Now choose a $0 < \delta < 1-\alpha$. Clearly, one can find an increasing (random) sequence $\{c_n\}$ with the property $c_{n+1} \geq c_n/\alpha$ such that

$$m_{c_n} \leq c_n^2/(2-\varepsilon)\log\log c_n \tag{2.25}$$

holds and m_{c_n} is a stopping time for every n. Choose $b_n = c_n/(\alpha+\delta)$. Then we have

$$m_{(\alpha+\delta)b_n} \leq \frac{(\alpha+\delta)^2 b_n^2}{(2-2\varepsilon)\log\log b_n} \tag{2.26}$$

if n is large enough.

Define for $n = 1, 2, \ldots$ the following two sequences of events:

$$E_n = \left\{\mu_x(\alpha, b_n) \leq \frac{(\alpha+\delta)^2 b_n^2}{(2-2\varepsilon)\log\log b_n}\right\}$$

and

$$F_n = \{|W_x(s)| \geq \alpha b_n \text{ for every } s \in [m_{(\alpha+\delta)b_n}, m_{b_n}]\}.$$

Observe first that $F_n \subset E_n$, as if F_n occurs then

$$\mu_X(\alpha, b_n) \le \mu_X(\alpha, (\alpha+\delta)b_n) \le m_{(\alpha+\delta)b_n},$$

implying E_n by (2.26).

Moreover, $P(F_n) = \delta/(1-\alpha)$ independently of n. Finally, the condition $c_{n+1} \ge c_n/\alpha$ and the strong Markov property also imply the independence of the events $\{F_n\}$. Hence infinitely many of the events $\{F_n\}$ occur with probability 1 and the same is true for $\{E_n\}$. □

3. A SIMILAR OPEN PROBLEM

In this section we propose a problem of similar nature. Let us formulate the question at first for the simple symmetric random walk case, which seems to be more natural.

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of i.i.d. r.v. with $P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$. Define the simple symmetric random walk as $S_n = X_1 + X_2 + \cdots + X_n$, $n = 1, 2, \dots$. Moreover let

$$\begin{aligned} \rho_1 &= \inf\{n;\ n > 0,\ S_n = 0\} \\ &\ \vdots \\ \rho_k &= \inf\{n;\ n > \rho_{k-1},\ S_n = 0\} \\ &\ \vdots \end{aligned}$$

be the endpoints of the consecutive excursions of S_n.

What can we say about

$$\nu(\alpha, k) = \#\{n;\ 0 < n \le \rho_k,\ |S_n| \le \alpha k\} \tag{3.1}$$

where α might be a fixed constant or could go to infinity?

Thus we are asking the behaviour of the amount of time spent by the simple symmetric random walk in an appropriate tube until completing k excursions.

The corresponding problem for the Wiener case goes as follows:

Let us consider a standard Wiener process $W(t)$, $W(0) = 0$, and denote its local time at x by $L(x, t)$. Define

$$T_r = \inf\{s,\ s > 0,\ L(0, s) \ge r\}, \tag{3.2}$$

the so-called inverse local time process. Then we might be interested in the following problem: How much time is spent by the Wiener process in a symmetric tube of size αr in the (random) time interval $0 < t < T_r$. (Here $0 < \alpha < 1$, α fixed, $\alpha \to 1$, $\alpha \to \infty$ cases are all meaningful questions.)

Define

$$\gamma(r, \alpha) = \gamma\{y,\ 0 < y \le T_r,\ |W(y)| < \alpha r\} \tag{3.3}$$

where γ is the ordinary Lebesgue measure. Then

$$D(x) = \int_0^\infty e^{-\lambda r} E_x(e^{-s\gamma(r, \alpha)})\, dr \tag{3.4}$$

can be evaluated based on Kac's method just as in our original problem. After some tedious computations one arrives at

$$E_0\left(\exp\left\{-s\frac{\gamma(r,\alpha)}{r^2}\right\}\right)=\exp\{-\sqrt{2s}\tanh(\alpha\sqrt{2s})\}. \tag{3.5}$$

Now the investigation of the asymptotic behaviour of this Laplace transform would probably give us some insight into the behaviour of $\gamma(r,\alpha)$ and would lead to similar results to the ones discussed in this paper.

We propose to investigate the Wiener case problem in which (3.5) might be helpful.

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References

1. A. N. Borodin, *On the distribution of functionals of Brownian local time*, LOMI preprint E-4-35, U.S.S.R. Academy of Sciences, Steklov Mathematical Institute, Leningrad Department, 1985.
2. K. L. Chung, *On the maximum partial sums of sequences of independent random variables*, Trans. Amer. Math. Soc. **64** (1948), 205–233.
3. G. Doetsch, *Introduction to the theory and application of the Laplace transformation*, Springer-Verlag, New York, Heidelberg, and Berlin, 1970.
4. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, Academic Press, New York, 1980.
5. R. Z. Hasminskiĭ, *Probability distributions for functionals of the trajectories of diffusion type random processes*, Dokl. Akad. Nauk SSSR **100** (1985), 22–25. (Russian)
6. M. Kac, *On some connections between probability theory and differential and integral equations*, Proc. Second Berkeley Sympos. Math. Statist. and Prob. 1950, Univ. of California Press, Berkeley and Los Angeles, pp. 189–215.
7. F. B. Knight, *Essentials of Brownian motion and diffusion*, Amer. Math. Soc., Providence, RI, 1981.
8. A. Rényi, *Probability theory*, North-Holland, Amsterdam, 1970.

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