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# TWO THEOREMS OF JOSEFSON-NISSENZWEIG TYPE FOR FRÉCHET SPACES

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ABSTRACT. We characterize the Fréchet-Montel (respectively, Fréchet-Schwartz) spaces by sequences in their dual spaces.

It is well known that a Fréchet space $E$ is Montel (respectively, Schwartz) if and only if it is separable and every weak*-null sequence in $E'$ also converges strongly to zero (respectively, converges uniformly to zero on some zero-neighbourhood in $E$). Because of the Josefson-Nissenzweig theorem [3], Jarchow asked in [5] if these two results are true without the assumption of the separability condition on the Fréchet space $E$.

In this short note we give a positive answer to both of these questions. These results were discovered during the Conference of Function Theory on Infinite-Dimensional Spaces held in the Universidad Complutense de Madrid from December 11–16, 1989. The Schwartz case had already been discovered independently by Lindström and Schlumprecht in [7] and by Bonet in [1].

We start by recalling the following result from [7], which is based on a very deep result due to Schlumprecht [8].

**Theorem 1.** *Let $E$ be a Fréchet space such that every weak*-null sequence in $E'$ is also strongly null-convergent. Then $E$ is reflexive.*

Now we are ready to prove the announced results.

**Theorem 2.** *A Fréchet space $E$ is Montel (respectively, Schwartz) if and only if weak*-null sequence in $E'$ is also strongly null-convergent (respectively, converges uniformly to zero on some zero-neighbourhood in $E$).*

*Proof.* Only the Montel case needs a proof because every Fréchet-Montel space is separable.

If $E$ is Montel then the condition is clearly necessary.

Let us now assume that every weak*-null sequence in $E'$ is strongly null-convergent. By the theorem above [7] $E$ must then be reflexive. Now we present two different proofs to show that $E$ is Montel.

First, by [6, Remark 1] it is enough to show that $U^\circ$ is $\sigma(E', E)$-sequentially compact for every zero-neighbourhood $U$ in $E$. To do this observe that $E'_\beta$

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is a DF-space whose strong dual is $E$ by reflexivity. Hence $\sigma(E', (E'_\beta)') = \sigma(E', E)$. Now according to [2, Theorem 11, Example 1.2(C)], every DF-space is weakly angelic (see, e.g., [4]), and since $U^\circ$ is always $\sigma(E', E)$-compact, it is also $\sigma(E', E)$-sequentially compact.

To give the second proof assume that $E$ is not Montel. Since $E$ is reflexive by the theorem above [7], there exists a weakly compact separable subset $A$ of $E$ that is not compact. Let $F$ be the closed linear span of $A$. By a recent result of Valdivia in [9], given $F$ we can find a closed separable subspace $G$ of $E$ that contains $F$ and has a topological complement $L$ in $E$. Certainly $G$ is separable but not Montel. Hence there is a $\sigma(G', G)$-null sequence in $L^\perp = G'$ that is not $\beta(G', G)$-null convergent. Since $G$ is complemented in $E$, this contradicts the assumption on $E$.

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