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ON THE STRONG PARROTT COMPLETION PROBLEM

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ABSTRACT. In this paper we parametrize all solutions of the Strong Parrott problem and obtain necessary and sufficient conditions for existence of isometric, coisometric, and unitary solutions.

1. INTRODUCTION

Let $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{K}_1$, and $\mathscr{K}_2$ be Hilbert spaces and

$$\begin{pmatrix} B_{11} & B_{12} \\ ? & B_{22} \end{pmatrix} : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{K}_1 \oplus \mathscr{K}_2 \tag{1.1}$$

be a partial matrix with B_{11}, B_{12}, and B_{22} known (bounded) linear operators and B_{21} unknown (indicated by "?"). Further, let also be given the Hilbert space $\mathscr{H}$ and the operators

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} : \mathscr{H} \to \mathscr{H}_1 \oplus \mathscr{H}_2, \qquad T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} : \mathscr{H} \to \mathscr{K}_1 \oplus \mathscr{K}_2. \tag{1.2}$$

We want to find the contractive completions of the problem

$$\begin{pmatrix} B_{11} & B_{12} \\ ? & B_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}, \tag{1.3}$$

i.e., we want to find the contractions $B = \left(\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{smallmatrix}\right)$ such that $BS = T$. The special case $S^*S = T^*T$ called the Strong Parrott problem was considered by Foiaş and Tannenbaum in [5]. The Strong Parrott problem arose out of questions in the theory of intertwining dilations (see [4]).

For problem (1.3) we establish necessary and sufficient conditions for the existence of a contractive solution. For this purpose, based on an observation due to Timotin [7], we reduce the problem to a positive semidefinite completion problem. In the case when a solution exists, we obtain a parametrization for the

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set of all solutions as well as necessary and sufficient conditions for the existence of an isometric, coisometric, and unitary solution.

2. The results

Theorem 2.1. *Problem* (1.3) *has a contractive solution if and only if the following conditions are satisfied*:

(i) $\|(B_{11}B_{12})\| \leq 1$,
(ii) $B_{11}S_1 + B_{12}S_2 = T_1$,
(iii) *the operator matrix*

$$\begin{pmatrix} I - B_{12}^*B_{12} - B_{22}^*B_{22} & S_2 - B_{12}^*T_1 - B_{22}^*T_2 \\ S_2^* - T_1^*B_{12} - T_2^*B_{22} & S_1^*S_1 + S_2^*S_2 - T_1^*T_1 - T_2^*T_2 \end{pmatrix} \tag{2.1}$$

is positive semidefinite.

Proof. Using a Schur complement argument, one easily sees that there exists a contractive solution of problem (1.3) if and only if there exists a positive semidefinite completion of the problem

$$\begin{pmatrix} I & 0 & S_1 & B_{11}^* & ? \\ 0 & I & S_2 & B_{12}^* & B_{22}^* \\ S_1^* & S_2^* & S^*S & T_1^* & T_2^* \\ B_{11} & B_{12} & T_1 & I & 0 \\ ? & B_{22} & T_2 & 0 & I \end{pmatrix}. \tag{2.2}$$

This was observed earlier by Timotin when $S^*S = T^*T$.

It is well known (see [2]) that the existence of a positive semidefinite completion of (2.2) is equivalent to the positive semidefiniteness of the two 4×4 principal submatrices of (2.2) formed with known entries. The positive semidefiniteness of the upper one can be reduced by a Schur complement argument to conditions (i) and (ii) while the positive semidefiniteness of the lower one is equivalent to (iii). □

In the case when $S^*S = T^*T$, condition (iii) reduces to $\|\binom{B_{12}}{B_{22}}\| \leq 1$ and $S_2 = B_{12}^*T_1 + B_{22}^*T_2$. In this way, we recover the result of [5, Theorem 2] (see also [7]).

Before stating our parametrization, we introduce some notation and recall some results. For a linear operator T we denote by $\overline{\mathscr{R}}(T)$ the closure of its range. For a contraction $T: \mathscr{F} \to \mathscr{G}$ we denote $D_T = (I - T^*T)^{1/2}$ and $\mathscr{D}_T = \overline{\mathscr{R}}(D_T)$.

Remark. Consider the Hilbert spaces $\mathscr{F}$ and $\mathscr{G}$, $\mathscr{F}_0$ a subspace of $\mathscr{F}$ and $T: \mathscr{F}_0 \to \mathscr{G}$ a contraction. It is known that (see for instance [1]) there exists a one-to-one correspondence between the set of all contractions $T': \mathscr{F} \to \mathscr{G}$ with $T'|\mathscr{F}_0 = T$ and the set of all contractions $G: \mathscr{F} \ominus \mathscr{F}_0 \to \mathscr{D}_{T^*}$ via

$$T' = (T \;\; D_{T^*}G): \mathscr{F}_0 \oplus (\mathscr{F} \ominus \mathscr{F}_0) \to \mathscr{G}. \tag{2.3}$$

Moreover, we have that

$$\dim \mathscr{D}_{T'} = \dim \mathscr{D}_T \oplus \mathscr{D}_G, \qquad \dim \mathscr{D}_{T'^*} = \dim \mathscr{D}_{G^*}. \tag{2.4}$$

Next consider matrix (1.1) with

$$\|(B_{11}\, B_{12})\| \le 1\,, \qquad \left\| \begin{pmatrix} B_{12} \\ B_{22} \end{pmatrix} \right\| \le 1\,. \tag{2.5}$$

The sufficiency of these two conditions for the existence of a contractive completion of (1.1) (i.e., without the additional requirement $BS = T$) was first proved in [6]. The contractivity of the operators in (2.5) imply by (2.3) the existence of contractions $G_1 : \mathscr{K}_1 \to \mathscr{D}_{B_{12}^*}$ and $G_2 : \mathscr{D}_{B_{12}} \to \mathscr{K}_2$ such that

$$B_{11} = D_{B_{12}^*} G_1\,, \qquad B_{22} = G_2 D_{B_{12}}\,. \tag{2.6}$$

It was proved in [1, 3] that there exists a one-to-one correspondence between the set of all contractive completions of (1.1) and the set of all contractions $\Gamma : \mathscr{D}_{G_1} \to \mathscr{D}_{G_2^*}$ given by

$$B_{21} = -G_2 B_{12}^* G_1 + D_{G_2^*} \Gamma D_{G_1}\,. \tag{2.7}$$

Moreover, in [1] it was proved that if $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ is the completion corresponding to the parameter Γ in (2.7) then

$$\dim \mathscr{D}_B = \dim \mathscr{D}_{G_2} \oplus \mathscr{D}_\Gamma \tag{2.8}$$

and

$$\dim \mathscr{D}_{B^*} = \dim \mathscr{D}_{G_1^*} \oplus \mathscr{D}_{\Gamma^*}\,. \tag{2.9}$$

Let us return to problem (1.3) and assume that the conditions of Theorem 2.1 are satisfied, so the problem admits a contractive solution. Any solution of problem (1.3) is, in particular, a solution for the unconstrained problem (1.1) and, therefore, B_{21} must be of form (2.7). The equation $B_{11}S_1 + B_{12}S_2 = T_2$ implies that if Γ is the parameter in (2.7) of a solution B then

$$\Gamma | \overline{\mathscr{R}}(D_{G_1} S_1) = \Gamma_0 \tag{2.10}$$

where $\Gamma_0 : \overline{\mathscr{R}}(D_{G_1} S_1) \to \mathscr{D}_{G_2^*}$ is uniquely determined by

$$D_{G_2^*} \Gamma_0 D_{G_1} S_1 := T_2 - B_{22} S_2 + G_2 B_{12}^* G_1 S_1\,. \tag{2.11}$$

We are ready to state our parametrization results.

Theorem 2.2. *Consider the linearly constrained completion problem* (1.3) *where the data satisfies the conditions of Theorem* 2.1. *Then there exists a one-to-one correspondence between the set of all contractive solutions of the problem and the set of all contractions*

$$G\colon \ker(S_1^* D_{G_1} | \mathscr{D}_{G_1}) \to \mathscr{D}_{\Gamma_0^*} \tag{2.12}$$

where G_1 *is given by* (2.6) *and* Γ_0 *is defined by* (2.11).

Proof. Following the discussion preceding this theorem, there exists a one-to-one correspondence between the set of all contractive solutions of problem (1.3) and the set of all contractive extensions of Γ_0 to $\mathscr{D}_{G_1}$. The contractivity of Γ_0 is ensured by the existence of a solution for problem (1.3). By the remark with the particular data $\mathscr{F} = \mathscr{D}_{G_1}$, $\mathscr{G} = \mathscr{D}_{G_2^*}$, $\mathscr{F}_0 = \overline{\mathscr{R}}(D_{G_1} S_1)$, $\mathscr{F} \ominus \mathscr{F}_0 = \ker(S_1^* D_{G_1} | \mathscr{D}_{G_1})$, and $T_0 = \Gamma_0$, we obtain a one-to-one correspondence between

all contractive extensions of Γ_0 to $\mathscr{D}_{G_1}$ and all contractions of the form (2.12), and so our result follows. □

Corollary 2.3. *In the hypothesis of Theorem* 2.2, *the linearly constrained completion problem* (1.3) *has a unique solution if and only if* $\mathscr{D}_{G_1} \subseteq \overline{\mathscr{R}}(S_1)$ *or* Γ_0 *is a coisometry.*

Proof. Following Theorem 2.2, there exists a unique contractive completion if and only if one of the spaces involved in (2.12) is the zero space. It is easy to see that $\ker(S_1^* D_{G_1}|\mathscr{D}_{G_1}) = \{0\}$ is equivalent with $\mathscr{D}_{G_1} \subseteq \overline{\mathscr{R}}(S_1)$ while $\mathscr{D}_{\Gamma_0^*} = \{0\}$ is equivalent to Γ_0 being a coisometry. □

Theorem 2.4. *Consider the linearly constrained completion problem* (1.3) *where the data satisfies the necessary and sufficient conditions of Theorem* 2.1.

(i) *Problem* (1.3) *has a coisometric solution* B *if and only if* G_1 *is a coisometry and*

$$\dim \mathscr{D}_{\Gamma_0^*} \leq \dim(\ker G_1 \cap \ker S_1^*). \tag{2.13}$$

(ii) *Problem* (1.3) *has an isometric solution* B *if and only if* G_2 *is an isometry,* $S^*S = T^*T$, *and* $\dim \mathscr{D}_{G_1} \leq \dim \mathscr{D}_{G_2^*}$.

(iii) *Problem* (1.3) *has a unitary solution* B *if and only if* G_1^* *and* G_2 *are isometries,* $S^*S = T^*T$, *and* $\dim \mathscr{D}_{G_1} = \dim \mathscr{D}_{G_2^*}$.

Proof. Let $B = (B_{ij})_{i,j=1}^2$ be an arbitrary contractive completion of problem (1.3) and let G denote the corresponding parameter from Theorem 2.3. Combining the remark, Theorem 2.2, and the relations (2.4), (2.8), and (2.9), we obtain

$$\dim \mathscr{D}_B = \dim \mathscr{D}_{G_2} \oplus \mathscr{D}_{\Gamma_0} \oplus \mathscr{D}_G \tag{2.14}$$

and

$$\dim \mathscr{D}_{B^*} = \dim \mathscr{D}_{G_1^*} \oplus \mathscr{D}_{G^*}. \tag{2.15}$$

By (2.15) problem (1.3) has a coisometric solution if and only if G_1 is a coisometry and there is a coisometry $G\colon \ker(S_1^* D_{G_1}|\mathscr{D}_{G_1}) \to \mathscr{D}_{\Gamma_0^*}$, i.e., $\dim \mathscr{D}_{\Gamma_0^*} \leq \dim(\ker S_1^* D_{G_1}|\mathscr{D}_{G_1})$. Since G_1 is a coisometry, D_{G_1} is the projection onto the kernel of G_1, so part (i) follows.

It is clear that the conditions G_2 is an isometry and $S^*S = T^*T$ are necessary for the existence of an isometric solution. Now, assuming that these two conditions are satisfied, if B is a contractive solution of problem (1.3) corresponding to the parameter Γ in (2.7) satisfying (2.10), then we have that

$$S^*(I - B^*B)S = S^*S - T^*T = 0.$$

A straightforward computation shows that in this case

$$I - B^*B = \begin{pmatrix} D_{G_1} D_\Gamma^2 D_{G_1} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.16}$$

and thus $D_\Gamma D_{G_1} S_1 = 0$. This latter relation implies that Γ is an isometry on $\overline{\mathscr{R}}(D_{G_1} S_1)$, and thus (2.10) implies that Γ_0 is an isometry. Relation (2.14) implies that the existence of an isometric completion is equivalent with the conditions that G_2 and Γ_0 are isometric and there exists an isometry $G\colon \ker(S_1^* D_{G_1}|\mathscr{D}_{G_1}) \to \mathscr{D}_{\Gamma_0^*}$. So it remains to prove that $\dim \mathscr{D}_{\Gamma_0^*} \leq$

$\dim(\ker S_1^* D_{G_1}|\mathscr{D}_{G_1})$ is equivalent with $\dim \mathscr{D}_{G_1} \leq \dim \mathscr{D}_{G_2^*}$. Since Γ_0 is an isometry, $\dim \mathscr{D}_{\Gamma_0^*} = \dim \mathscr{D}_{G_2^*} - \dim \overline{\mathscr{R}}(D_{G_1} S_1)$. The latter equality together with

$$\dim \ker(S_1^* D_{G_1}|\mathscr{D}_{G_1}) = \dim \mathscr{D}_{G_1} - \dim \overline{\mathscr{R}}(D_{G_1} S_1)$$

imply that $\dim \ker(S_1^* D_{G_1}|\mathscr{D}_{G_1}) \leq \dim \mathscr{D}_{\Gamma_0^*}$ is equivalent with $\dim \mathscr{D}_{G_1} \leq \dim \mathscr{D}_{G_2}^*$ and (ii) follows.

The proof of (iii) is similar to that of (ii), but we must add the condition that G_1 is a coisometry and that there exists a unitary G: $\ker(S_1^* D_{G_1}|\mathscr{D}_{G_1}) \to \mathscr{D}_{\Gamma_0^*}$, which finally gives the condition $\dim \mathscr{D}_{G_1} = \dim \mathscr{D}_{G_2^*}$. □

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