ON THE STRONG PARROTT COMPLETION PROBLEM

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Abstract. In this paper we parametrize all solutions of the Strong Parrott problem and obtain necessary and sufficient conditions for existence of isometric, coisometric, and unitary solutions.

1. Introduction

Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be Hilbert spaces and

\begin{equation}
\begin{pmatrix}
B_{11} & B_{12} \\
? & B_{22}
\end{pmatrix}
: \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2
\end{equation}

be a partial matrix with $B_{11}$, $B_{12}$, and $B_{22}$ known (bounded) linear operators and $B_{21}$ unknown (indicated by "?"). Further, let also be given the Hilbert space $\mathcal{H}$ and the operators

\begin{equation}
S = \begin{pmatrix}
S_1 \\
S_2
\end{pmatrix}
: \mathcal{H} \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2, \quad T = \begin{pmatrix}
T_1 \\
T_2
\end{pmatrix}
: \mathcal{H} \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2.
\end{equation}

We want to find the contractive completions of the problem

\begin{equation}
\begin{pmatrix}
B_{11} & B_{12} \\
? & B_{22}
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2
\end{pmatrix}
= \begin{pmatrix}
T_1 \\
T_2
\end{pmatrix},
\end{equation}

i.e., we want to find the contractions $B = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}$ such that $BS = T$. The special case $S^*S = T^*T$ called the Strong Parrott problem was considered by Foiaş and Tannenbaum in [5]. The Strong Parrott problem arose out of questions in the theory of intertwining dilations (see [4]).

For problem (1.3) we establish necessary and sufficient conditions for the existence of a contractive solution. For this purpose, based on an observation due to Timotin [7], we reduce the problem to a positive semidefinite completion problem. In the case when a solution exists, we obtain a parametrization for the
set of all solutions as well as necessary and sufficient conditions for the existence of an isometric, coisometric, and unitary solution.

2. THE RESULTS

Theorem 2.1. Problem (1.3) has a contractive solution if and only if the following conditions are satisfied:

(i) \( \| (B_{11}B_{12}) \| \leq 1 \),
(ii) \( B_{11}S_1 + B_{12}S_2 = T_1 \),
(iii) the operator matrix

\[
\begin{pmatrix}
I - B_{12}^*B_{12} - B_{22}^*B_{22} & S_2 - B_{12}^*T_1 - B_{22}^*T_2 \\
S_2^* - T_1^*B_{12} - T_2^*B_{22} & S_1^*S_1 + S_2^*S_2 - T_1^*T_1 - T_2^*T_2
\end{pmatrix}
\]

is positive semidefinite.

Proof. Using a Schur complement argument, one easily sees that there exists a contractive solution of problem (1.3) if and only if there exists a positive semidefinite completion of the problem

\[
\begin{pmatrix}
I & 0 & S_1 & B_{11}^* \\
0 & I & B_{12}^* & B_{22} \\
S_1^* & S_2^* & S^*S & T_1^* & T_2^* \\
B_{11} & B_{12} & T_1 & I & 0
\end{pmatrix}
\]

This was observed earlier by Timotin when \( S^*S = T^*T \).

It is well known (see [2]) that the existence of a positive semidefinite completion of (2.2) is equivalent to the positive semidefiniteness of the two \( 4 \times 4 \) principal submatrices of (2.2) formed with known entries. The positive semidefiniteness of the upper one can be reduced by a Schur complement argument to conditions (i) and (ii) while the positive semidefiniteness of the lower one is equivalent to (iii). □

In the case when \( S^*S = T^*T \), condition (iii) reduces to \( \| (B_{12} B_{22}) \| \leq 1 \) and \( S_2 = B_{12}^*T_1 + B_{22}^*T_2 \). In this way, we recover the result of [5, Theorem 2] (see also [7]).

Before stating our parametrization, we introduce some notation and recall some results. For a linear operator \( T \) we denote by \( \overline{\mathcal{R}}(T) \) the closure of its range. For a contraction \( T: \mathcal{F} \to \mathcal{G} \) we denote \( D_T = (I - T^*T)^{1/2} \) and \( \mathcal{D}_T = \overline{\mathcal{R}}(D_T) \).

Remark. Consider the Hilbert spaces \( \mathcal{F} \) and \( \mathcal{G} \), \( \mathcal{F}_0 \) a subspace of \( \mathcal{F} \) and \( T: \mathcal{F}_0 \to \mathcal{G} \) a contraction. It is known that (see for instance [1]) there exists a one-to-one correspondence between the set of all contractions \( T': \mathcal{F} \to \mathcal{G} \) with \( T'|\mathcal{F}_0 = T \) and the set of all contractions \( G: \mathcal{F} \oplus \mathcal{F}_0 \to \mathcal{D}_{T'} \) via

\[
T' = (T D_{T'}, G): \mathcal{F}_0 \oplus (\mathcal{F} \oplus \mathcal{F}_0) \to \mathcal{G}.
\]

Moreover, we have that

\[
dim \mathcal{D}_{T'} = \dim \mathcal{D}_T \oplus \mathcal{D}_G, \quad \dim \mathcal{D}_{T'}^\ast = \dim \mathcal{D}_G^\ast.
\]
Next consider matrix (1.1) with

$$\| (B_{11} B_{12}) \| \leq 1, \quad \| \left( \begin{array}{c} B_{12} \\ B_{22} \end{array} \right) \| \leq 1.$$  \hfill (2.5)

The sufficiency of these two conditions for the existence of a contractive completion of (1.1) (i.e., without the additional requirement $BS = T$) was first proved in [6]. The contractivity of the operators in (2.5) imply by (2.3) the existence of contractions $G_1 : \mathcal{H}_1 \to \mathcal{D}_{B_{12}}$ and $G_2 : \mathcal{D}_{B_{12}} \to \mathcal{H}_2$ such that

$$B_{11} = D_{B_{12}}^* G_1, \quad B_{22} = G_2 D_{B_{12}}.$$  \hfill (2.6)

It was proved in [1, 3] that there exists a one-to-one correspondence between the set of all contractive completions of (1.1) and the set of all contractions $\Gamma : \mathcal{D}_{G_1} \to \mathcal{D}_{G_2}$ given by

$$B_{21} = -G_2 B_{12}^* G_1 + D_{G_2}^* \Gamma D_{G_1}.$$  \hfill (2.7)

Moreover, in [1] it was proved that if $B = \left( \begin{array}{c} B_{11} \\ B_{21} \\ B_{22} \end{array} \right)$ is the completion corresponding to the parameter $\Gamma$ in (2.7) then

$$\dim \mathcal{D}_B = \dim \mathcal{D}_{G_2} \oplus \mathcal{D}_T$$  \hfill (2.8)

and

$$\dim \mathcal{D}_{B^*} = \dim \mathcal{D}_{G_1} \oplus \mathcal{D}_T.$$  \hfill (2.9)

Let us return to problem (1.3) and assume that the conditions of Theorem 2.1 are satisfied, so the problem admits a contractive solution. Any solution of problem (1.3) is, in particular, a solution for the unconstrained problem (1.1) and, therefore, $B_{21}$ must be of form (2.7). The equation $B_{11} S_1 + B_{12} S_2 = T_2$ implies that if $\Gamma$ is the parameter in (2.7) of a solution $B$ then

$$\Gamma(\mathcal{R}(D_{G_1} S_1)) = \Gamma_0$$  \hfill (2.10)

where $\Gamma_0 : \mathcal{R}(D_{G_1} S_1) \to \mathcal{D}_{G^*_2}$ is uniquely determined by

$$D_{G_2}^* \Gamma_0 D_{G_1}, S_1 := T_2 - B_{22} S_2 + G_2 B_{12}^* G_1 S_1.$$  \hfill (2.11)

We are ready to state our parametrization results.

**Theorem 2.2.** Consider the linearly constrained completion problem (1.3) where the data satisfies the conditions of Theorem 2.1. Then there exists a one-to-one correspondence between the set of all contractive solutions of the problem and the set of all contractions

$$G : \ker(S_1^* D_{G_1}) \to \mathcal{D}_{T_0^*}$$  \hfill (2.12)

where $G_1$ is given by (2.6) and $\Gamma_0$ is defined by (2.11).

**Proof.** Following the discussion preceding this theorem, there exists a one-to-one correspondence between the set of all contractive solutions of problem (1.3) and the set of all contractive extensions of $\Gamma_0$ to $\mathcal{D}_{G_1}$. The contractivity of $\Gamma_0$ is ensured by the existence of a solution for problem (1.3). By the remark with the particular data $\mathcal{F} = \mathcal{D}_{G_1}$, $\mathcal{F} = \mathcal{D}_{G^*_2}$, $\mathcal{R} = \mathcal{R}(D_{G_1} S_1)$, $\mathcal{F} \oplus \mathcal{I}_0 = \ker(S_1^* D_{G_1}) \mathcal{D}_{G_1}$, and $T_0 = \Gamma_0$, we obtain a one-to-one correspondence between
all contractive extensions of $\Gamma_0$ to $\mathcal{D}_{G_1}$ and all contractions of the form (2.12), and so our result follows. \hfill \Box

**Corollary 2.3.** In the hypothesis of Theorem 2.2, the linearly constrained completion problem (1.3) has a unique solution if and only if $\mathcal{D}_{G_1} \subseteq \mathcal{R}(S_1)$ or $\Gamma_0$ is a coisometry.

**Proof.** Following Theorem 2.2, there exists a unique contractive completion if and only if one of the spaces involved in (2.12) is the zero space. It is easy to see that $\ker(S^* D_{G_1} | \mathcal{D}_{G_1}) = \{0\}$ is equivalent with $\mathcal{D}_{G_1} \subseteq \mathcal{R}(S_1)$ while $\mathcal{D}_{T_0} = \{0\}$ is equivalent to $\Gamma_0$ being a coisometry. \hfill \Box

**Theorem 2.4.** Consider the linearly constrained completion problem (1.3) where the data satisfies the necessary and sufficient conditions of Theorem 2.1.

(i) Problem (1.3) has a coisometric solution $B$ if and only if $G_1$ is a coisometry and

$$\dim \mathcal{D}_{T_0} \leq \dim (\ker G_1 \cap \ker S_1^*) \tag{2.13}$$

(ii) Problem (1.3) has an isometric solution $B$ if and only if $G_2$ is an isometry, $S^* S = T^* T$, and $\dim \mathcal{D}_{G_1} \leq \dim \mathcal{D}_{G_2}^* \tag{2.14}$

(iii) Problem (1.3) has a unitary solution $B$ if and only if $G_1^*$ and $G_2$ are isometries, $S^* S = T^* T$, and $\dim \mathcal{D}_{G_1} = \dim \mathcal{D}_{G_2}^* \tag{2.15}$

**Proof.** Let $B = (B_{ij})_{i,j=1}^2$ be an arbitrary contractive completion of problem (1.3) and let $G$ denote the corresponding parameter from Theorem 2.3. Combining the remark, Theorem 2.2, and the relations (2.4), (2.8), and (2.9), we obtain

$$\dim \mathcal{B} = \dim \mathcal{D}_{G_2} + \mathcal{D}_{T_0} + \mathcal{D}_G \tag{2.16}$$

and

$$\dim \mathcal{B}^* = \dim \mathcal{D}_{G_1}^* + \mathcal{D}_G^* \tag{2.17}$$

By (2.15) problem (1.3) has a coisometric solution if and only if $G_1$ is a coisometry and there is a coisometry $G$: $\ker(S^* D_{G_1} | \mathcal{D}_{G_1}) \to \mathcal{D}_{T_0}^*$, i.e., $\dim \mathcal{D}_{T_0}^* \leq \dim (\ker S^* D_{G_1} | \mathcal{D}_{G_1})$. Since $G_1$ is a coisometry, $D_{G_1}$ is the projection onto the kernel of $G_1$, so part (i) follows.

It is clear that the conditions $G_2$ is an isometry and $S^* S = T^* T$ are necessary for the existence of an isometric solution. Now, assuming that these two conditions are satisfied, if $B$ is a contractive solution of problem (1.3) corresponding to the parameter $\Gamma$ in (2.7) satisfying (2.10), then we have that

$$S^* (I - B^* B) S = S^* S - T^* T = 0 \tag{2.18}$$

A straightforward computation shows that in this case

$$I - B^* B = \begin{pmatrix} D_{G_1} D_{G_1}^2 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.19}$$

and thus $D_T D_{G_1} S_1 = 0$. This latter relation implies that $\Gamma$ is an isometry on $\mathcal{R}(D_{G_1} S_1)$, and thus (2.10) implies that $\Gamma_0$ is an isometry. Relation (2.14) implies that the existence of an isometric completion is equivalent with the conditions that $G_2$ and $\Gamma_0$ are isometric and there exists an isometry $G$: $\ker(S^* D_{G_1} | \mathcal{D}_{G_1}) \to \mathcal{D}_{T_0}$. So it remains to prove that $\dim \mathcal{D}_{T_0}^* \leq$
dim(\ker S^*_1 D_{G_1} | D_{G_1}) \text{ is equivalent with } \dim D_{G_1} \leq \dim D_{G_1^*}. \text{ Since } \Gamma_0 \text{ is an isometry, } \dim \Gamma_0^* = \dim D_{G_1^*} - \dim \overline{\mathcal{R}}(D_{G_1} S_1). \text{ The latter equality together with } \dim \ker(S^*_1 D_{G_1} | D_{G_1}) = \dim D_{G_1} - \dim \overline{\mathcal{R}}(D_{G_1} S_1) \text{ imply that } \dim \ker(S^*_1 D_{G_1} | D_{G_1}) \leq \dim \Gamma_0^* \text{ is equivalent with } \dim D_{G_1} \leq \dim D_{G_1^*} \text{ and (ii) follows.}

The proof of (iii) is similar to that of (ii), but we must add the condition that \( G_1 \) is a coisometry and that there exists a unitary \( G : \ker(S^*_1 D_{G_1} | D_{G_1}) \to \mathcal{D}_{G_1^*} \), which finally gives the condition \( \dim D_{G_1} = \dim D_{G_1^*}. \) \( \square \)

REFERENCES


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