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EXAMPLES OF BUCHSBAUM QUASI-GORENSTEIN RINGS

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ABSTRACT. The paper shows the existence of Buchsbaum quasi-Gorenstein rings of any admissible depth.

INTRODUCTION

Until now few examples of non-Cohen-Macaulay prime almost complete intersections were known. One possibility to find such examples is to go through linkages. According to an idea of Peskine and Szpiro [6], every prime ideal linked to a quasi-Gorenstein ideal is an almost complete intersection, cf. [9]. Recall that an ideal I of a ring A is called quasi-Gorenstein if the factor ring A/I is quasi-Gorenstein, i.e., the canonical module of A/I is isomorphic to A/I. However, to find non-Cohen-Macaulay quasi-Gorenstein ideals is usually also hard. For instance, Schenzel [8] used a result of Mumford on abelian varieties to give a class of non-Cohen-Macaulay quasi-Gorenstein rings that are Buchsbaum with depth 2.

The aim of this paper is to construct Buchsbaum quasi-Gorenstein rings of any admissible depth that are generated by monomials. Note that from the description of the local cohomology of the canonical module of a Buchsbaum ring [8] once can easily deduce that the depth t of a Buchsbaum quasi-Gorenstein ring A is either $\dim A$, i.e., A is a Cohen-Macaulay ring, or $2 \le t \le [(\dim A + 1)/2]$. Rings generated by monomials are, in other terms, affine semigroup rings whose structures can be described well by means of the underlying affine semigroups [11, 7]. For instance, there exist sufficient (and necessary) conditions for such rings to be Cohen-Macaulay, Gorenstein, or Buchsbaum. Using the theory of affine semigroup rings, we can translate the problem of constructing quasi-Gorenstein rings generated by monomials to one of finding certain kinds of systems of diophantine homogeneous linear equations. From this we then derive examples of (non-Cohen-Macaulay) Buchsbaum quasi-Gorenstein rings and, therefore, of Buchsbaum almost complete intersection rings of any admissible depth. Compared with Schenzel's results, our method

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has the advantage of being more explicit: one knows the parametric presentation of the given quasi-Gorenstein rings, and therefore one can compute their defining equations.

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1. Affine semigroup rings

In this section we collect some results on affine semigroup rings that will be used in the construction of Buchsbaum quasi-Gorenstein rings.

Let $\mathbb{N}$ denote the set of nonnegative integers. By an *affine semigroup* we mean a finitely generated submonoid S of the additive monoid $\mathbb{N}^n$, $n > 0$. Let $k[S]$ denote the semigroup ring of S over a field k. Then one can identify $k[S]$ with the subring of a polynomial ring $k[t_1, \dots, t_n]$ generated over k by the monomials $t^x = t_1^{x_1}, \dots, t_n^{x_n}$, $x = (x_1, \dots, x_n) \in S$. Of course, every ring generated over k by a finite set of monomials is an affine semigroup ring.

For further investigations we have to introduce some notation. Let A and B be two subsets of $\mathbb{Z}^n$. We denote by $G(A)$ the additive subgroup of $\mathbb{Z}^n$ generated by A, and by $A \pm B$ the set of all elements $a \pm b$, $a \in A$ and $b \in B$. Moreover, we denote by $k[A]$ the k-vector subspace of $k[G(A)]$ spanned by the elements of A. If $A + S \subseteq A$, we call A an *S-ideal*. In this case, $k[A]$ has a natural $\mathbb{Z}^n$-graded module structure over $k[S]$. If $A = B \setminus C$ for S-ideals $B \supseteq C$, we will identify $k[A]$ with the factor module $k[B]/k[C]$.

Let S be an arbitrary affine semigroup in $\mathbb{N}^n$ with $\operatorname{rank}_Z G(S) = r \geq 1$. Let $\mathscr{C}_S$ denote the convex polyhedral cone spanned by the elements of S in the space $\mathbb{Q}^n$. Then we call S a *standard* affine semigroup if $\mathscr{C}_S$ has exactly $n(r-1)$-dimensional faces lying on the hyperplanes $x_i = 0, i = 1, \dots, n$. According to Hochster [4, p. 323] (see also [11, §1]), every affine semigroup can be transformed isomorphically to a standard one.

Set $S_{(i)} = \{x \in S | x_i = 0\}$ and $S_i = S - S_{(i)}$, $i = 1, \dots, n$. Note that the sets S_i are S-ideals. Put

$$C_S = G(S) \setminus \bigcup_{i=1}^{n} S_i.$$

Let $H_M^i(k[S])$ denote the ith local cohomology module of $k[S]$ with respect to the maximal ideal $M = k[S \setminus (0)]$, $i = 0, \dots, r$. Note that $\dim k[S] = r$. From the fact that $\operatorname{Hom}_k(H_M^r(k[S]), k)$ is the canonical module of $k[S]$, we derive the following criterion for $k[S]_M$ to be a quasi-Gorenstein ring (which is implicitly contained in [2, 11]).

Lemma 1.1. *Let S be a standard affine semigroup. Then $k[S]_M$ is a quasi-Gorenstein ring if and only if there exists an element $y \in G(S)$ such that $C_s = y - S$.*

Proof. Obviously $k[C_S]$ has the structure of a factor module over $k[S]$. By [11, Corollary 3.8] we have $H_M^r(k[S]) \cong k[C_S]$. It is easily seen that

$$\operatorname{Hom}_k(k[C_S], k) \cong k[-C_S].$$

But $k[-C_S] \cong k[S]$ if and only if there exists an element $y \in G(S)$ (which corresponds to the shifting degree of the isomorphism) such that $y - C_S = S$ or equivalently $C_S = y - S$.

Unfortunately, one has been unable to find necessary and sufficient conditions for $k[S]_M$ to be a Buchsbaum ring in terms of S. However, there is a formula for the computation of the local cohomology modules of $k[S]$, and from the $\mathbb{Z}^n$-graded structure of the local cohomology modules of $k[S]$ one can sometimes decide that $k[S]_M$ is a Buchsbaum ring.

Let $[1, n]$ be the set of the integers $1, \dots, n$. For every subset J of the set $[1, n]$ we set

$$G_J = \bigcap_{i \notin J} S_i \setminus \bigcup_{j \in J} S_j$$

and denote by π_J the simplicial complex of all nonempty subsets I of J such that $\bigcap_{i\in I} S_{(i)} \neq \{0\}$. As usual, let $\widetilde{H}_q(\pi_J, k)$ denote the qth reduced homology group of π_J with coefficients in k.

Lemma 1.2 [11, Corollaries 3.4, 3.7]. *Let S be a standard affine semigroup such that $S = \bigcap_{i=1}^n S_i$. Then $H_M^1(k[S]) = 0$ and*

$$H_M^i(k[S]) \cong \bigoplus_{\substack{J \notin \pi_{[1,n]} \\ |J| \leq n-2}} k[G_J] \otimes_k \widetilde{H}_{i-2}(\pi_J; k)$$

for all $i = 2, \dots, r-1$.

In the following we will denote by E_x the x-graded part of a $\mathbb{Z}^n$-graded module over $k[S]$, $x \in \mathbb{Z}^n$.

Lemma 1.3 [7, Theorem 6.10]. *Set $V = \bigcup_{i=1}^{r-1} \{x \in \mathbb{Z}^n | [H_M^i(k[S])]_x \neq 0\}$. Suppose $x + y \notin V$ for all elements $x \in S \setminus (0)$ and $y \in V$. Then $k[S]_M$ is a Buchsbaum ring.*

2. Buchsbaum quasi-Gorenstein rings of any admissible depth

In this section we will apply the above criteria for quasi-Gorenstein and Buchsbaum affine semigroup rings to construct Buchsbaum quasi-Gorenstein rings of any admissible depth.

We will start with a subgroup G of $\mathbb{Z}^n$ and try to impose conditions on G so that there exists a submonoid S of $\mathbb{N}^n$ with $G(S) = G$ for which $k[S]_M$ is a quasi-Gorenstein ring.

Theorem 2.1. *Let G be a subgroup of $\mathbb{Z}^n$ that satisfies the following conditions:*

(i) *For any $i = 1, \dots, n$ there is an element $x \in G$ with $x_i = 0$ and $x_j > 0$ for all $j \neq i$.*

(ii) *There exists an element $y \in G$ such that $y_i = 1$ or -1 for all $i = 1, \dots, n$.*

Let I be the set of the numbers i with $y_i = 1$ and set

$$S = \{x \in G \cap \mathbb{N}^n | x_i \neq 1 \text{ for all } i \in I\}.$$

Then $k[S]_M$ is a quasi-Gorenstein ring.

Proof. We will first show that S is a standard affine semigroup. Assumption (i) implies that there is an element $x \in S$ such that $x_i > 0$ for all $i = 1, \dots, n$.

Given any set of generators of G, one can add a sufficiently large multiple of x to the generators to get a new set of generators of G in S. From this it follows that $G(S) = G$. Let L and L_i denote the linear subspace of $\mathbb{Q}^n$ spanned by G and $S_{(i)}$, respectively, $i = 1, \dots, n$. Let H_i be the hyperplane $x_i = 0$ of $\mathbb{Q}^n$. Since $L_i = L \cap H_i$ and $L + H_i = \mathbb{Q}^n$,

$$\dim_{\mathbb{Q}} L_i = \dim_{\mathbb{Q}} L + \dim_{\mathbb{Q}} H_i - \dim_{\mathbb{Q}} \mathbb{Q}^n = \dim_{\mathbb{Q}} L - 1.$$

From this it follows that the faces of $\mathscr{C}_S$ lying on H_i have the maximal possible dimension. Moreover, assumption (i) also implies that $S_{(i)} \neq S_{(j)}$ for $i \neq j$. This means that these faces are different. Obviously, $\mathscr{C}_S$ has no other maximal faces. Hence S is a standard affine semigroup. Further, from the definition of S we deduce that

$$S_i = \begin{cases} \{x \in G | x_i \geq 0 \text{ and } x_i \neq 1\} & \text{if } i \in I, \\ \{x \in G | x_i \geq 0\} & \text{if } i \in I. \end{cases}$$

Since $C_S = G \setminus \bigcup_{i=1}^n S_i$, we can compute C_S and obtain

$$C_S = \{x \in G | x_i < 0 \text{ or } x_i = 1 \text{ for } i \in I \text{ and } x_i < 0 \text{ if } i \notin I\}.$$

Now it is an easy matter to check that $C_S = y - S$. Thus, $k[S]_M$ is a quasi-Gorenstein ring by Lemma 1.1.

A subgroup of $\mathbb{Z}^n$ can be given as the set of all solutions in $\mathbb{Z}^n$ of a system of homogeneous linear equations in n variables. Therefore, in order to construct quasi-Gorenstein rings we just need to find systems of homogeneous linear equations that have solutions x with $x_i = 0$ and $x_j > 0$, $j \neq i$, for $i = 1, \dots, n$ and a solution y with $y_j = \pm 1$.

Example (cf. [11, Example 4.2]). Let G be the set of all solutions in $\mathbb{Z}^n$ of the equation $x_1 + x_2 = x_3 + x_4$. This equation has the following solutions: $(0, 2, 1, 1)$, $(2, 0, 1, 1)$, $(1, 1, 0, 2)$, $(1, 1, 2, 0)$, $(1, -1, 1, -1)$. Therefore, if we set

$$S = \{x \in G \cap \mathbb{N}^n | x_1 \text{ and } x_3 \neq 1\},$$

then $A = k[S]_M$ is a quasi-Gorenstein ring with $\dim A = 3$. By [11, Example 4.2], A is a non-Cohen-Macaulay ring. Note that S is isomorphic to the affine semigroup

$$T = \{x \in \mathbb{N}^3 | x_1 + x_2 - x_3 \geq 0,\ x_1 \neq 1,\ x_3 \neq 1\}.$$

One can easily compute the generators of T and obtain

$$k[T] = k[t_2, t_1^2, t_1^3, t_1^2t_3^2, t_2^2t_3^2, t_1^3t_3^2, t_1^3t_3^3, t_1^2t_2t_3^3, t_2^3t_3^3, t_1^4t_3^3, t_1^3t_2t_3^4].$$

According to [8, Remark (3.2) and Theorem (3.3)], if K_A is the canonical module of a Buchsbaum local ring $(A, \mathfrak{m})$, then K_A satisfies Serre's condition S_2 and

$$H_{\mathfrak{m}}^i(K_A) \cong H_{\mathfrak{m}}^{d-i+1}(A)$$

for all $i = 2, \dots, d-1$, $d = \dim A$. Moreover, if A is a quasi-Gorenstein ring ($K_A \cong A$), then A is either a Cohen-Macaulay ring or $2 \leq \operatorname{depth} A \leq [(d+1)/2]$. In fact, the following result shows the existence of Buchsbaum quasi-Gorenstein rings of a given dimension d of any admissible depth $t = 2, \dots, [(d+1)/2]$.

Lemma 2.2. *Let G be the subgroup of the solutions in $\mathbb{Z}^n$ of the equation*

$$(t-1)X_1 + X_2 + \cdots + X_t = (n-t-1)X_{t+1} + X_{t+2} + \cdots + X_n,$$

where t is an integer with $2 \le t \le [n/2]$. Set

$$S = \{x \in G \cap \mathbb{N}^n | x_1 \neq 1,\ x_{t+1} \neq 1\}.$$

Then $A = k[S]_M$ is a Buchsbaum quasi-Gorenstein ring with $\dim A = n-1$, $\operatorname{depth} A = t$, and $H^i_{\mathfrak{m}}(A) = 0$ for $i \neq t, n-t, n-1$. Moreover, $H^t_{\mathfrak{m}}(A) \cong k \oplus k$ if $n = 2t$, and $H^t_{\mathfrak{m}}(A) \cong H^{n-t}_{\mathfrak{m}}(A) \cong k$ if $n \neq 2t$.

Proof. It is obvious that G satisfies the conditions of Theorem 2.1 with the element y having the coordinates $y_1 = y_{t+1}$ and $y_i = -1$ for $i \neq 1, t+1$. Hence A is a quasi-Gorenstein ring. We have also proven in the proof of Theorem 2.1 that S is a standard affine semigroup with $G(S) = G$ and

$$S_i = \begin{cases} \{x \in G | x_i \ge 0 \text{ and } x_i \neq 1\} & \text{if } i = 1, t+1, \\ \{x \in G | x_i \ge 0\} & \text{if } i \neq 1, t+1. \end{cases}$$

From this it follows that $\dim A = \operatorname{rank}_{\mathbb{Z}} G = n-1$ and $S = \bigcap_{i=1}^n S_i$. Hence we can use Lemma 1.2 to compute the local cohomology modules $H^i_{\mathfrak{m}}(A) = H^i_M(k[S])$, $i = 1, \dots, n-2$. First the form of the linear equation implies that

$$\pi_{[1,n]} = \{J \subset [1,n] | J \not\supseteq [1,t], [t+1,n]\},$$

where $[1,t]$ and $[t+1,n]$ denote the sets of the integers $1, \dots, t$ and $t+1, \dots, n$. For $J = [1,t]$, π_J is the simplicial complex of all proper subsets of J. The geometric realization $|\pi_J|$ of π_J is homeomorphic to a $(t-2)$-dimensional sphere. Hence

$$\widetilde{H}_{i-2}(\pi_J; k) = \begin{cases} 0 & \text{if } i \neq t, \\ k & \text{if } i = t. \end{cases}$$

Similarly, for $J = [t+1, n]$, we have

$$\widetilde{H}_{i-2}(\pi_J; k) = \begin{cases} 0 & \text{if } i \neq n-t, \\ k & \text{if } i = n-t. \end{cases}$$

If $|J| \le n-2$ and $[1,t]$ is a proper subset of J, there exist integers $u, v \in [t+1, n]$, $u \in J$, $v \notin J$, and we can find solutions x of the linear equation with $x_i = 0$ for all $i \neq u, v$, $x_u < 0$, and x_v sufficiently large. Such solutions belong to G_J. Hence the supremum of the components of the elements of G_J is infinite. By [11, Lemma 4.5], this implies that the simplicial complex π_J is acyclic, i.e., all the reduced homology groups of π_J vanish. Taking all these facts into consideration we obtain $H^i_M(k[S]) = 0$ for $i \neq t, n-t$ and

$$H^t_M(k[S]) \cong k[G_{[1,t]}] \oplus k[G_{[t+1,n]}]$$

if $t = n-t$, or

$$H^t_M(k[S]) \cong k[G_{[1,t]}],$$
$$H^{n-t}_M(k[S]) \cong k[G_{[t+1,n]}]$$

if $t \neq n-t$. Now we are going to compute $G_{[1,t]}$. Let $x \in G_{[1,t]}$ be arbitrary. By the definition of the sets of G_J, we have $x \in \bigcap_{i=t+1}^n S_i$. Hence $x_i \ge 0$ for $i = t+1, \dots, n$ and, therefore,

$$(n-t)x_{t+1} + x_{t+2} + \cdots + x_n \ge 0.$$

On the other hand, since $x \notin S_i$ for $i = 1, \dots, t$, from the formula for S_i we see that

$$tx_1 + x_2 + \cdots + x_{t-1} \le 0.$$

Hence both sides of the equation must be zero at x. But this happens only if $x_1 = 1$, $x_i = -1$ for $i = 2, \dots, t-1$, and $x_i = 0$ for $i = t+1, \dots, n$. That means $G_{[1,t]}$ consists of only one element x with these components. Similarly, we show that $G_{[t+1,n]}$ consists of only one element y with $y_i = 0$ for $i = 1, \dots, t$, $y_{t+1} = 1$, and $y_i = -1$ for $i = t+2, \dots, n$. So we obtain the asserted formulas for the local cohomology modules of A. Moreover, since

$$\bigcup_{i=1}^{n-2} \{x \in \mathbb{Z}^n | [H_M^i(k[S])]_x \neq 0\} = \{x, y\}$$

and since $x - y$ and $y - x$ do not belong to S, the assumption of Lemma 1.3 is satisfied. Hence A is a Buchsbaum ring.

Example. The affine semigroup S of the preceding example yields a Buchsbaum quasi-Gorenstein ring A with $\dim A = 3$, $\operatorname{depth} A = 2$, and $H_{\mathfrak{m}}^2(A) \cong k \oplus k$.

Remark. Schenzel has shown that if R is the graded ring $\bigoplus_{n \in \mathbb{N}} H^0(X, L^{\otimes n})$, where L is a very ample invertible sheaf of an abelian variety X with $\dim X = g > 0$ and if $M = \bigoplus_{n>0} H^0(X, L^{\otimes n})$, then R_M is a Buchsbaum quasi-Gorenstein ring with $\dim R_M = g + 1$, $\operatorname{depth} R_M = 2$, and

$$\dim_k H_M^i(R_M) = \binom{g}{i-1}$$

for $i = 2, \dots, g$ [8, Theorem (4.1)].

According to [9], a prime quasi-Gorenstein ideal P of a regular local ring (R, M) is always linked to an almost complete intersection Q of R. By generic link [5, Proposition 2.6], if we replace R by a local ring of the form S_{MS}, where S is a polynomial ring over R, we may assume that Q is a prime ideal. Moreover, one can also show that

$$H_M^{i-1}(R/Q) \cong H_M^i(R/P)$$

for all $i < d = \dim R/Q$ and $H_M^{d-1}(R/Q) = 0$ (see the proof of [9, Proposition 2]). As the Buchsbaum property is preserved by linkage [1], we conclude that a Buchsbaum almost complete intersection local domain R/Q is either a Cohen-Macaulay ring or $1 \le \operatorname{depth} R/Q \le (d-1)/2$. The existence of Buchsbaum almost complete intersection local domains of any admissible depth follows from Lemma 2.2 by the theory of linkages.

Corollary 2.3. *There exist Buchsbaum almost complete intersection local rings A with $\dim A = d$ and $\operatorname{depth} A = t$ for any integers d, t with $1 \le t \le (d-1)/2$.*

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